Relation between chemical potential and GPE eigenvalue

chemical potential (at $T=0$) : $\mu = \frac{\partial E}{\partial N}$

Mean-field energy $E[\phi] = \int dr \; \frac{|\nabla\phi|^2}{2m} + V_{ext}|\phi|^2 + \frac{g}{2}|\phi|^4$

stationary GPE = stationary point of $E[\phi]$ functional at fixed $N = \int dx \, |\phi|^2$

$$\frac{\delta}{\delta\phi^*}\left(E[\phi] - \mu N[\phi]\right) = \int dz \left(\frac{-\nabla^2\phi}{2m} + V_{ext}\phi + g|\phi|^2\phi - \mu\phi\right)$$

Lagrange multiplier

$$\frac{\delta}{\delta\phi^*}(E - \mu N) = 0 \implies -\frac{\nabla^2\phi}{2m} + V_{ext}\phi + g|\phi|^2\phi = \mu\phi$$

GPE

For any N, define ϕ_0^N = lowest-energy solution of GPE.

$$\frac{d}{dN}E[\phi_0^N] = \frac{\delta E}{\delta\phi}\cdot\frac{d\phi_0^N}{dN} + \frac{\delta E}{\delta\phi^*}\left(\frac{d\phi_0^N}{dN}\right)^*$$

$$\frac{d\phi_0^N}{dN} = \frac{1}{N}\phi_N^0 \cdot \langle\phi_0^N|\frac{d\phi_0^N}{dN}\rangle + \nabla\phi_\perp$$

longitudinal. $\qquad \langle\nabla\phi_\perp|\phi_0^N\rangle = 0$

$$\frac{d}{dN}E[\phi_0^N] = \frac{\delta E}{\delta\phi}\left(\frac{1}{N}\phi_N^0 \cdot \langle\phi_0^N|\frac{d\phi_0^N}{dN}\rangle + \nabla\phi_\perp\right) +$$

$$+ \frac{\delta E}{\delta\phi^*}\left(\frac{1}{N}\phi_N^{0*}\langle\phi_0^N|\frac{d\phi_0^N}{dN}\rangle^* + \nabla\phi_\perp^*\right)$$

terms in $\nabla\phi_\perp$ vanish as $\frac{\delta E}{\delta\phi} = \mu\phi^*$ and $\langle\nabla\phi|\phi\rangle = 0$

$$\frac{d}{dN} E[\phi_0^N] = \frac{2}{N} \, Re \left[\frac{\delta E}{\delta \phi^*} \cdot \phi_N^{0*} \cdot \left< \frac{d\phi_0^N}{dN} \middle| \phi_0^N \right> \right]$$

$$= \frac{2}{N} \, Re \left[\int dr \left(\phi_N^{0*}(r) \cdot \mu \, \phi_N^0(r) \right) \cdot \left< \frac{d\phi_0^N}{dN} \middle| \phi_0^N \right> \right]$$

$$= \mu \left(\left< \frac{d\phi_0^N}{dN} \middle| \phi_0^N \right> + \left< \phi_0^N \middle| \frac{d\phi_0^N}{dN} \right> \right) =$$

$$= \mu \frac{d}{dN} \left< \phi_0^N \middle| \phi_0^N \right> = \mu .$$

Things are much simpler if system is <u>homogeneous</u>:

$$\phi_0^N(r) = \sqrt{\frac{N}{V}} \quad , \quad \frac{\delta}{\delta \phi^*} E = g|\phi|^2 \phi \quad , \quad \mu = g|\phi|^2 = g\frac{N}{V}$$

$$E[\phi_0^N] = \frac{\mu N}{2} = \frac{g N^2}{2V}$$

$$\frac{dE[\phi_0^N]}{dN} = g\frac{N}{V} = \mu .$$