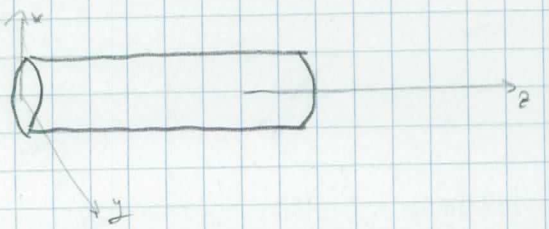
exercise 1 : coherence properties of a continuous atom laser 1



$$\varepsilon(k_z, n_x, n_y) = \hbar\omega_\perp (n_x + n_y + 1) + \frac{\hbar^2 k_z^2}{2m}$$

$$k_z = \frac{2\pi}{L} n_z \quad , \quad n_x, n_y \in \mathbb{N}, \quad n_z \in \mathbb{Z}$$

$$N(\beta,\mu) = \sum_{\substack{n_x, n_y \in \mathbb{N} \\ k_z}} \frac{1}{e^{\beta(\varepsilon(k_z, n_x, n_y) - \mu)} - 1} =$$

$$= \sum_{n_x, n_y \in \mathbb{N}} \cdot \int \frac{L}{2\pi} dk_z \frac{1}{e^{\beta(\hbar\omega_\perp(n_x+n_y+\cancel{1}) + \frac{\hbar^2}{2m}k_z^2 - \mu)}}$$

we forget it!

→ only divergence is in $n_x = n_y = 0$

→ other values → gapped by $\hbar\omega_\perp(n_x + n_y) > 0$.

$$\int \frac{L}{2\pi} dk_z \frac{1}{e^{\beta(\frac{\hbar^2}{2m}k_z^2 - \mu)} - 1} \overset{\mu=0}{\simeq} \int \frac{L}{2\pi} dk_z \frac{1}{e^{\beta\frac{\hbar^2}{2m}k_z^2} - 1} \simeq$$

$$\overset{k_z \to 0}{\simeq} \int \frac{L}{2\pi} dk_z \frac{1}{\beta\frac{\hbar^2}{2m}k_z^2} \quad \text{which diverges IR}$$

So: $N_{nc}^{max} = \infty$ ⟶ no BEC possible excited state population

$$N^{max}_{n_x,n_y} = \int \frac{L}{2\pi} dk_z \frac{1}{e^{\beta\left(\frac{\hbar^2}{2m}k_z^2 + \hbar\omega_\perp(n_x+n_y) - \mu\right)} - 1} =$$

$$= \frac{L}{\lambda_T} g_{1/2}\left(e^{-\hbar\omega_\perp(n_x+n_y)\beta}\right) < \infty$$

for $n_x, n_y \to \infty$ $\qquad N^{max}_{n_x,n_y} \sim e^{-\hbar\omega_\perp(n_x+n_y)\beta}$

$\Rightarrow \sum_{n_x,n_y \neq \{0,0\}}$ perfectly converges UV

* if $n_{1d} > n^c_{1d}$ extra particles go into $n_x = n_y = 0$ branch that can accomodate any number of particles for $\mu \to 0$

$$\langle \psi^\dagger(z)\,\psi(z')\rangle = \int \frac{L}{2\pi} dk_z \frac{1}{\sqrt{L}} e^{-ik_z z} \frac{1}{\sqrt{L}} e^{ik_z \cdot z'} \cdot N(k_z) =$$

$$= \frac{1}{2\pi}\int dk_z\, e^{ik_z(z-z')} \frac{1}{e^{\beta\left(\frac{\hbar^2}{2m}k_z^2 - \mu\right)} - 1} \overset{\text{low } k_z}{\simeq}$$

$$\simeq \frac{1}{2\pi}\int dk_z\, e^{ik_z(z-z')} \frac{1}{\beta\left(\frac{\hbar^2}{2m}k_z^2 - \mu\right)}$$

* integral converges UV (thanks to 1D geometry)

* macroscopically populated $n_x = n_y = 0$ $\Longrightarrow$ $\beta|\mu| \ll 1$

$$= \frac{2m k_B T}{2\pi \hbar^2} \int dk_z \; e^{i k_z (z-z')} \frac{1}{k_z^2 + \frac{2m|\mu|}{\hbar^2}} =$$

$$= \frac{m k_B T}{\pi \hbar^2} \int \frac{(l_c \, dk_z)}{l_c} \; e^{i(k_z l_c)(z-z')/l_c} \frac{l_c^2}{k_z^2 l_c^2 + 1}$$

$$= \frac{m k_B T}{\pi \hbar^2} \frac{l_c^2}{l_c} \int dk_z' \; e^{i k_z' z'} \frac{1}{k_z'^2 + 1}$$

where $l_c^2 = \frac{\hbar^2}{2m|\mu|}$ coherence length

$$= \frac{m k_B T}{\cancel{\pi} \hbar^2} \left(\frac{\hbar^2}{2m|\mu|} \right)^{1/2} \cancel{\pi} \exp\left(- |z-z'|/l_c\right) =$$

$$= \left(\frac{m (k_B T)^2}{2 \hbar^2 |\mu|} \right)^{1/2} \exp\left(- |z-z'|/l_c\right)$$

$$n_0 = \left(\frac{m (k_B T)^2}{2\hbar^2 |\mu|} \right)^{1/2} = \frac{1}{\lambda_T} \left(\frac{\pi k_B T}{|\mu|} \right)^{1/2} \Longrightarrow \quad |\mu| = \frac{m (k_B T)^2}{2\hbar^2 n^2}$$

$$l_c = \left(\frac{\hbar^2}{2m} \frac{2\hbar^2 n^2}{m (k_B T)^2} \right)^{1/2} = \frac{\hbar^2 n}{m \, k_B T} = \frac{n_0 \lambda_T^2}{2\pi} \gg \lambda_T$$

<u>case 2</u>: $\beta \hbar \omega_\perp \gg 1$.

$$N^{\text{max}}_{n_x, n_y} = \frac{L}{\lambda_T} g_{1/2}\left(e^{-\beta \hbar \omega_\perp (n_x + n_y)}\right) \simeq \frac{L}{\lambda_T} e^{-\beta \hbar \omega_\perp (n_x+n_y)}$$

$N^{\text{max}}_{\text{tot}}$ dominated by lowest terms $\{n_x = 1, n_y = 0\}$ $\{n_x = 0, n_y = 1\}$

* $N_0 = \frac{L}{\lambda_T} \left(\frac{\pi k_B T}{|\mu|} \right)^{1/2} \gg \frac{L}{\lambda_T} e^{-\beta \hbar \omega_\perp}$ ok in degenerate regime $\beta|\mu| \ll 1$

* $N_0 = \frac{L}{\lambda_T} g_{1/2}\left(e^{\beta\mu}\right) \gg \frac{L}{\lambda_T} e^{-\beta \hbar \omega_\perp}$ if $|\mu| \ll \hbar \omega_\perp$

$\rightarrow$ in this case, Transverse BEC is more due to freezing out excited transverse states than BEC effect

<u>case b</u> $\beta \hbar \omega_\perp \ll 1$

$$N^{max}_{n_x, n_y} = \frac{L}{\lambda_T} g_{1/2}\left(e^{-\beta \hbar \omega_\perp (n_x + n_y)}\right)$$

$$N^{max}_{tot} = \sum_{n_x, n_y \neq 0} \frac{L}{\lambda_T} g_{1/2}\left(e^{-\beta \hbar \omega_\perp (n_x + n_y)}\right) \simeq$$

$$= \frac{L}{\lambda_T} \int_0^\infty dn_x \, dn_y \; g_{1/2}\left(e^{-\beta \hbar \omega_\perp (n_x + n_y)}\right) =$$

$$= \frac{L}{\lambda_T} \int_0^\infty dn_T \cdot n_T \; g_{1/2}\left(e^{-\beta \hbar \omega_\perp n_T}\right) =$$

$$= \frac{L}{\lambda_T} \left(\frac{k_B T}{\hbar \omega_\perp}\right)^2 \int_0^\infty d\tilde{n} \; \tilde{n} \; g_{1/2}\left(e^{-\tilde{n}}\right)$$

as $\tilde{n}^{-1/2}$ for $\tilde{n} \to 0$ as $\tilde{n} e^{-\tilde{n}}$ for $\tilde{n} \to \infty$

$$\left[g_{1/2}(e^{-\mu}) \simeq \Gamma(1/2) \, \mu^{-1/2} \right]$$

$$= \frac{L}{\lambda_T} \left(\frac{k_B T}{\hbar \omega_\perp}\right)^2 \quad \#$$

$$N_0 = \frac{L}{\lambda_T} \left(\frac{\pi k_B T}{|\mu|}\right)^{1/2} \gg \frac{L}{\lambda_T} \left(\frac{k_B T}{\hbar \omega_\perp}\right)^2 \quad \#$$

if $|\mu| \ll \pi k_B T \left(\frac{\hbar \omega_\perp}{k_B T}\right)^4 = \pi \frac{(\hbar \omega_\perp)^4}{(k_B T)^3} \ll k_B T \quad \pi \left(\frac{\hbar \omega_\perp}{k_B T}\right)^4 \ll k_B T$

→ in This case Transverse BEC takes place despite many transverse modes being degenerate and highly occupied.