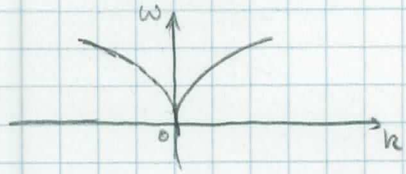
Exercise n°5 : (not so) quantum ducks

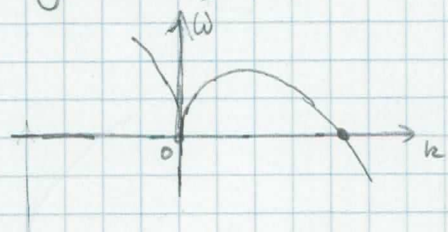
Dispersion of surface waves on a (deep) lake :

$$\omega(k) = \sqrt{g|k|}$$



In the reference frame of duck (moving at v) :

$$\omega(k) = \sqrt{g|k|} - k_x \cdot v$$

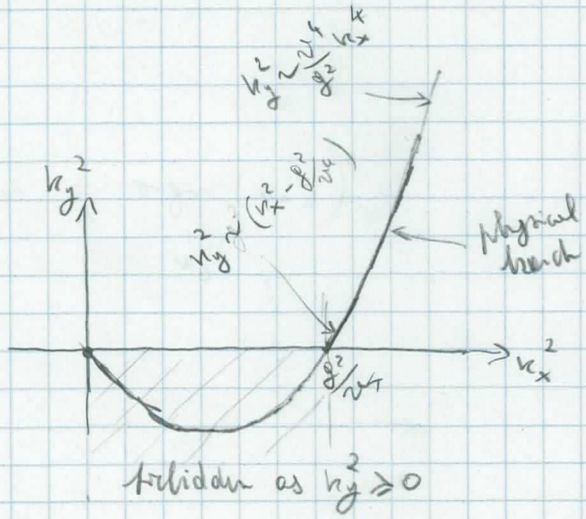


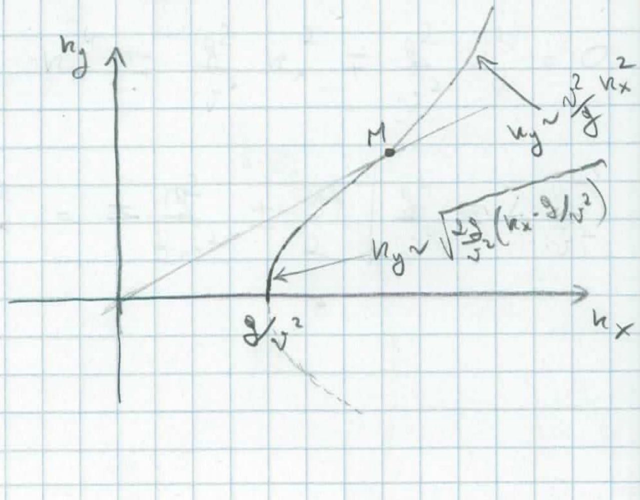
Stationary state → modes at $\omega(k) = 0$ excited, i.e.

$$\sqrt{g\,(k_x^2 + k_y^2)^{1/2}} = k_x \cdot v$$

$$g^2 (k_x^2 + k_y^2) = + k_x^4 v^4$$

$$k_x^4 v^4 - g^2 k_x^2 = g^2 k_y^2$$





→ single length scale g/v^2

→ shape independent of v

Group velocity of modes is parallel to $\nabla_k \omega$.
i.e. orthogonal to iso-lines of ω. (to the Right)

Iso-line at $\omega = 0$ is $k_x^4 v^4 - g^2 k_x^2 = g^2 k_y^2$
that has been plotted:

- vertical slope for $k_x \to \left(g/v^2\right)^+$ is vertical
- minimum value of slope at intermediate $k_x = \bar{k}_x$
- slope tend to vertical again as $k_x \to \infty$.

tangent vector is $\left(1, \frac{dk_y}{dk_x}\right) = \left(1, \frac{d}{dk_x}\sqrt{\frac{v^4}{g^2}k_x^4 - k_x^2}\right) =$

$$= \left(1, \frac{4\frac{k_x^3}{k_c^2} - 2k_x}{2\sqrt{\frac{k_x^4}{k_c^2} - k_x^2}}\right) =$$

$$= \left(1, \frac{2\,k_x^2/k_c^2 - 1}{\sqrt{\frac{k_x^2}{k_c^2} - 1}}\right) \qquad (\text{for } k_y > 0)$$

normal vector is $\left(\frac{1 - 2k_x^2/k_c^2}{\sqrt{k_x^2/k_c^2 - 1}}, 1\right)$

$$\tan\phi = \frac{\sqrt{k_x^2/k_c^2 - 1}}{1 - 2k_x^2/k_c^2}$$

$$\tan\phi = \begin{cases} \longrightarrow 0 & \text{for } k_x \to \infty \\ = -\sqrt{1/8} & \text{at } \frac{k_x^2}{k_c^2} = 3/2 \quad (\text{minimum val.}) \\ \longrightarrow 0 & \text{for } k_x \to k_c^+ \end{cases}$$

$$\frac{d}{dx}\,\frac{\sqrt{x-1}}{1-2x} = \frac{1}{1-2x}\,\frac{1}{2\sqrt{x-1}} - \frac{\sqrt{x-1}}{(1-2x)^2}\,(-2) =$$

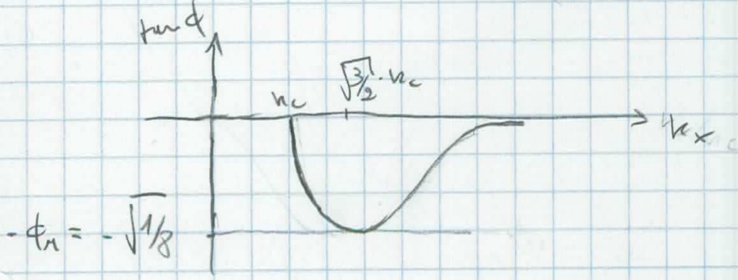
$$= \sqrt{x-1}\left[\frac{1}{2(1-2x)(x-1)} + \frac{2}{(1-2x)^2}\right] =$$

$$= \frac{\sqrt{x-1}}{2(1-2x)}\left[\frac{1}{x-1} + \frac{4}{1-2x}\right] = \frac{\sqrt{x-1}}{2(1-2x)}\,\frac{1-2x+4x-4}{(x-1)(1-2x)}$$

$$= \frac{1}{2\sqrt{x-1}\,(1-2x)^2}\,(2x-3) = 0$$

for $x = 3/2$.

$$\min(\tan\phi) = \frac{\sqrt{3/2-1}}{1-2\cdot 3/2} = \frac{\sqrt{1/2}}{-2} = -\sqrt{1/8}$$

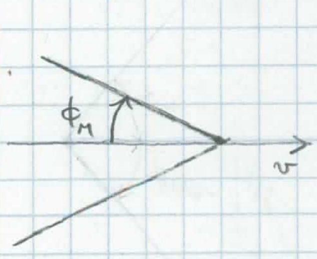


- only values $|\tan\phi| \leq 1/8$ possible

  $\hookrightarrow$ aperture of cone is $\phi_M = \operatorname{atan} \frac{1}{\sqrt{8}} \simeq 19°$

- for $|\phi| < \phi_M \longrightarrow$ 2 values of $\underline{k}$

  $\hookrightarrow$ two interpenetrating patterns

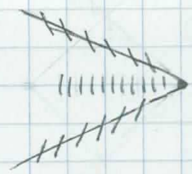


- limit $\phi \to 0$: $k_x \to \infty$, $k_y \to k_x^2/k_c$ (1)

  $k_x \to k_c$, $k_y \to 0$ (2)

① → very fast pattern ⟶ not observable (see later)

② → observable pattern right behind duck

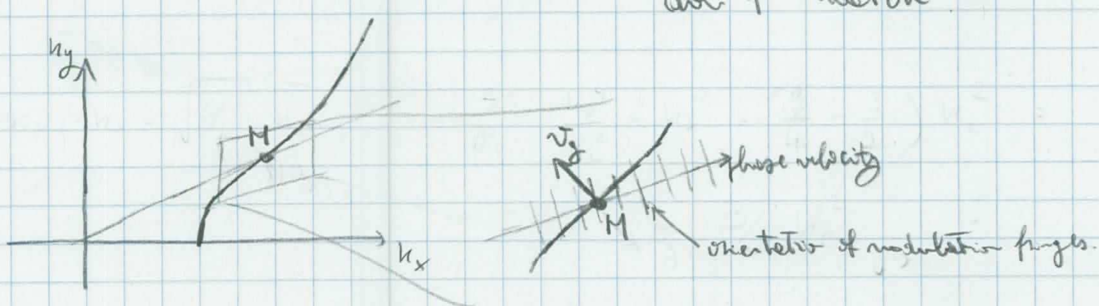


+ modulation $\perp$ $\underline{k}$

$$\Rightarrow \lambda = \frac{2\pi}{g/v^2} = \frac{2\pi v^2}{g}$$

$v = 1\,m/s \Rightarrow \lambda \simeq 0{,}6\,m$

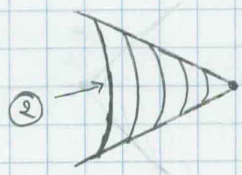
On the surface of cone $\phi = \phi_M$: group and phase velocities don't match



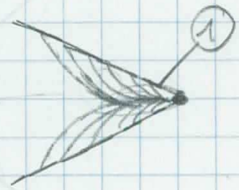
→ modulation is orthogonal to $\underline{k}$ (iso-phase $\perp$ $\underline{k}$)

→ not parallel to cone sides.

1st pattern



2nd pattern



Effect of duck size D :

minimum wave vector excited $|\underline{k}| = \left|\left(g/v^2, 0\right)\right| = g/v^2$

if $D \gg 1/(g/v^2)$ no excitation possible

i.e. $D \gg v^2/g$

$\Rightarrow$ critical speed : $v_c = \sqrt{g \cdot D}$

duck $D \simeq 20\,cm$ $\Rightarrow v_c = \sqrt{10\,m/s^2\ 0{,}2\,m} \sim 1{,}4\,ms^{-1}$.

Bibliography

G. B. Whitham, "Linear and nonlinear waves", Wiley 1974.

Čerenkov patterns analogous to the duck's wake are expected to appear for Čerenkov radiation in left-handed media, see e.g. :

– J. Lu, T. Grzegorczyk, Y. Zhang, J. Pacheco Jr, B. Wu, J. Kong and M. Chen, Opt. Expr. 11, 723 (2003)

but full calculation remains to be done ...