Lecture 4 : MF theory of BEC's , dynamics

Small fluctuations around stationary BEC

$$i\hbar \frac{\partial}{\partial t}\phi = -\frac{\hbar^2}{2m}\nabla^2\phi + V_{ext}(r)\phi + g|\phi|^2\phi$$

simplest case: homogeneous BEC $\quad \phi(r) = (\phi_0 + \delta\phi(r))\, e^{-i\mu_0 t/\hbar}$

$\phi_0 \to$ uniform, $\quad e^{-i\mu_0 t/\hbar} \to \mu = g|\phi_0|^2$

$$i\hbar\frac{\partial}{\partial t}(\cancel{\phi_0} + \delta\phi(r)) + \mu_0(\phi_0 + \delta\phi(r)) = -\frac{\hbar^2}{2m}\nabla^2(\cancel{\phi_0} + \delta\phi(r)) + g|\phi_0 + \delta\phi(r)|^2(\phi_0 + \delta\phi(r))$$

$$i\hbar\frac{\partial}{\partial t}\delta\phi + \cancel{\mu_0\phi_0} + \mu_0\,\delta\phi(r) = -\frac{\hbar^2}{2m}\nabla^2\delta\phi + \cancel{g|\phi_0|^2\phi_0} + 2g|\phi_0|^2\delta\phi +$$

$$+ g\phi_0^2\,\delta\phi^* + \underbrace{O(\delta\phi^2)}_{\to\ \text{neglect}}$$

$$i\hbar\frac{\partial}{\partial t}\delta\phi = -\frac{\hbar^2}{2m}\nabla^2\delta\phi + (2g|\phi_0|^2 - \mu)\,\delta\phi + g\,\phi_0^2\,\delta\phi^*$$

Phase of ϕ_0 arbitrary $\longrightarrow$ fix it to real, positive.

$$i\hbar\frac{\partial}{\partial t}\delta\phi = -\frac{\hbar^2}{2m}\nabla^2\delta\phi + \mu\,\delta\phi + \mu\,\delta\phi^*$$

Ansatz: $\quad \delta\phi(r,t) = U e^{ikx} e^{-i\omega t} + V^* e^{-ikx} e^{i\omega t}$

$$\hbar\left(\omega U e^{ikx} e^{-i\omega t} - \omega V^* e^{-ikx} e^{i\omega t}\right) =$$

$$= \left(\frac{\hbar^2 k^2}{2m} + \mu\right)\left(U e^{ikx} e^{-i\omega t} + V^* e^{-ikx} e^{i\omega t}\right) +$$

$$+ \mu\left(V e^{ikx} e^{-i\omega t} + U^* e^{-ikx} e^{i\omega t}\right)$$

Grouping $e^{ikx} e^{-i\omega t}$ and $e^{-ikx} e^{i\omega t}$ terms:

$$\begin{cases} \omega U = \left(\frac{\hbar^2 k^2}{2m} + \mu\right) U + \mu V \\ -\omega V^* = \left(\frac{\hbar^2 k^2}{2m} + \mu\right) V^* + \mu U^* \end{cases}$$

$$\mathcal{L} = \begin{pmatrix} \frac{\hbar^2 k^2}{2m} + \mu & \mu \\ -\mu & -\left(\frac{\hbar^2 k^2}{2m} + \mu\right) \end{pmatrix} \Longrightarrow \mathcal{L} \begin{pmatrix} U \\ V \end{pmatrix} = \hbar\omega \begin{pmatrix} U \\ V \end{pmatrix}$$

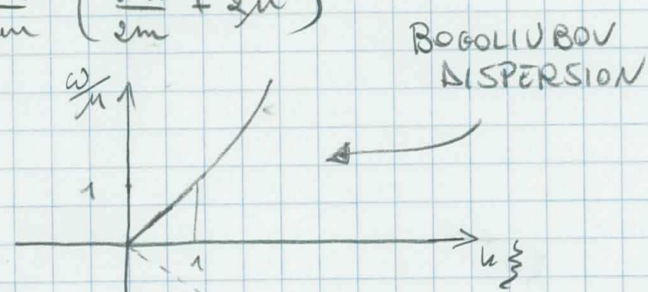
eigenvalue equation.

$$\left(\frac{\hbar^2 k^2}{2m} + \mu - \hbar\omega\right)\left(-\left(\frac{\hbar^2 k^2}{2m} + \mu\right) - \hbar\omega\right) + \mu^2 = 0$$

$$-\left(\frac{\hbar^2 k^2}{2m} + \mu\right)^2 + (\hbar\omega)^2 + \mu^2 = 0$$

$$(\hbar\omega)^2 = \left(\frac{\hbar^2 k^2}{2m} + \mu\right)^2 - \mu^2 = \frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + 2\mu\right)$$

$$\hbar\omega_\pm = \pm\sqrt{\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + 2\mu\right)}$$



$\xi = \left(\frac{\hbar^2}{m\mu}\right)^{1/2}$ healing length

a) $k\xi \ll 1 \longrightarrow \omega_k \simeq \underbrace{\left(\frac{\mu}{m}\right)^{1/2}}_{\text{sound speed}} k$ sound-like propagation

b) $k\xi \gg 1 \longrightarrow \omega_k \simeq \frac{\hbar^2 k^2}{2m} + \underbrace{\mu}_{\text{Hartree-shift}}$ single-particle mode.

$$U + V = \left(\frac{\frac{\hbar^2 k^2}{2m}}{\omega_k}\right)^{1/2}, \quad U - V = \left(\frac{\omega_k}{\frac{\hbar^2 k^2}{2m}}\right)^{1/2} \quad \text{both real}$$

$$\phi(x,t) = \phi_0 + \underbrace{(U+V)}_{\text{density}} \cos(kx - \omega t) + i \cdot \underbrace{(U-V)}_{\text{phase}} \sin(kx - \omega t)$$

limit $k\xi \ll 1$ $\Longrightarrow$ $U+V \to 0$ density perturb. small
$U-V \to \infty$ phase perturb. large $\Big\}$ phase $\downarrow$ Goldstone excitation
both $U, V \to \pm\infty$.

limit $k\xi \gg 1$ $\Longrightarrow$ $U \pm V \longrightarrow 1$

i.e $\Longrightarrow$ $U \to 1$, $V \to 0$

NOTE. Negative energy mode has $\omega_- = -\omega_+$, U and V exchanged,
$\Longrightarrow$ same effect on $\delta\phi$ (which is the physical object)

case of attractive interactions : $g<0$

$$\omega = \pm\sqrt{\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + 2gn\right)}$$ is imaginary for $\frac{\hbar^2 k^2}{2m} < 2|g|n$.

$\to$ a perturbation exponentially grows in time

$\boxed{\text{DYNAMICAL INSTABILITY}}$

* density modulation $\to$ creates potential wells $\to$ particle fall into wells.

NOTE : to get a D.I either $g<0$ or $m<0$

$\hookrightarrow$ effective mass in a periodic potential $\to$ can be <0.

Landau criterion for superfluidity

Moving condensate at v_0 : $\phi_0(r,t) = \bar{\Phi}_0 \, e^{ik_0 x} \, e^{-i\mu_0 t/\hbar}$

$$\mu_0 = \frac{\hbar^2 k_0^2}{2m} + g|\phi_0|^2, \quad \frac{\hbar k_0}{m} = v_0$$

$$\phi(r,t) = \left(\bar{\Phi}_0 + \delta\phi(r,t)\right) e^{ik_0 x} \, e^{-i\mu_0 t/\hbar}$$

$$i\hbar \frac{\partial}{\partial t}\delta\phi + \mu_0 \delta\phi + \cancel{\mu_0 \bar{\Phi}_0} = -\frac{\hbar^2}{2m}\nabla^2 \delta\phi - \frac{i\hbar^2 k_0}{m}\nabla \delta\phi +$$
$$+ \frac{\hbar^2}{2m} k_0^2 \left(\cancel{\bar{\Phi}_0} + \delta\phi\right) + \cancel{g|\bar{\Phi}_0|^2 \bar{\Phi}_0} +$$
$$+ 2g|\bar{\Phi}_0|^2 \delta\phi + g\bar{\Phi}_0^2 \, \delta\phi^*$$

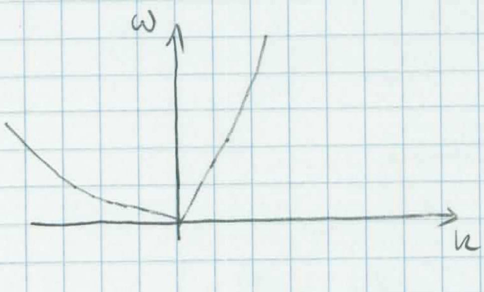
$$\delta\phi(r,t) = U e^{ikx} e^{-i\omega t} + V^* e^{-ikx} e^{i\omega t}$$

$$\hbar\omega U + \cancel{\mu_0 U} = \frac{\hbar^2}{2m}k^2 U + \frac{\hbar^2 k_0 k}{m} U + \cancel{\frac{\hbar^2}{2m}k_0^2 U} +$$
$$+ \cancel{2}g|\bar{\Phi}_0|^2 U + g\phi_0^2 V$$

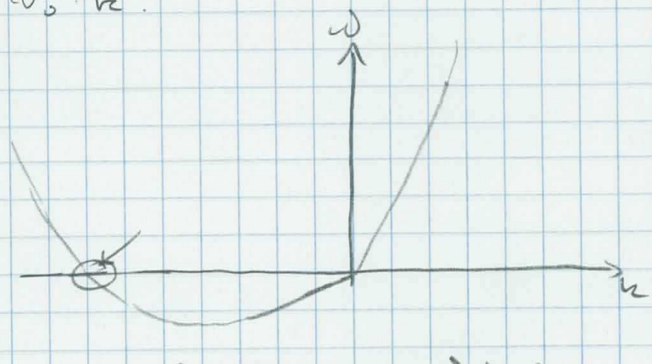
$$\hbar\omega U = \frac{\hbar^2 k^2}{2m} U + \frac{\hbar^2 k_0 k}{m} U + g|\bar{\Phi}_0|^2 U + g\phi_0^2 V$$

$\Rightarrow$ same Bogoliubov dispersion but for Doppler shift $\delta\omega = v_0 \cdot k$

$$\hbar\omega = \sqrt{\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + 2g|\bar{\Phi}_0|^2\right)} + \hbar v_0 \cdot k$$



sub-sonic flow $v_0 < c_s$



super-sonic $v_0 > c_s$

Small perturbation in moving BEC : $V_{ext}(r) = V_0\,\delta(r)$

V_0 small

$$i\hbar\frac{\partial}{\partial t}\delta\phi = -\frac{\hbar^2}{2m}\nabla^2\delta\phi - \frac{i\hbar^2 k_0}{m}\nabla\delta\phi + g|\Phi_0|^2\delta\phi + g\Phi_0^2\,\delta\phi^*$$

$$+\ \underbrace{V_0\,\delta(r)\,\Phi_0}$$

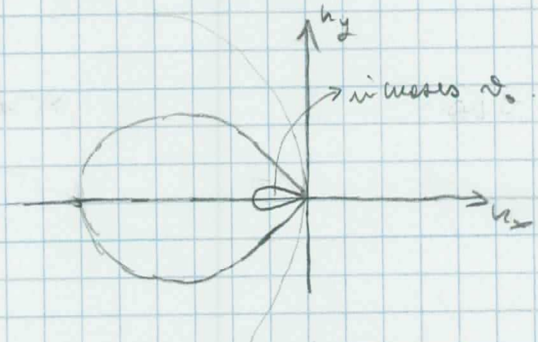
→ static perturbation

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\delta\phi\\ \delta\phi^*\end{pmatrix} = \mathcal{L}\begin{pmatrix}\delta\phi\\ \delta\phi^*\end{pmatrix} + V_0\,\delta(r)\,\Phi_0\begin{pmatrix}1\\ -1\end{pmatrix}$$

source term
- static in time → couples only to $\omega=0$ modes (resonance condition)
- $\delta(r)$ shape → couples to all k modes

sub-sonic flow: → $\omega_k = 0$ has no solution

super-sonic flow → $\omega_k = 0$ has locus of solutions

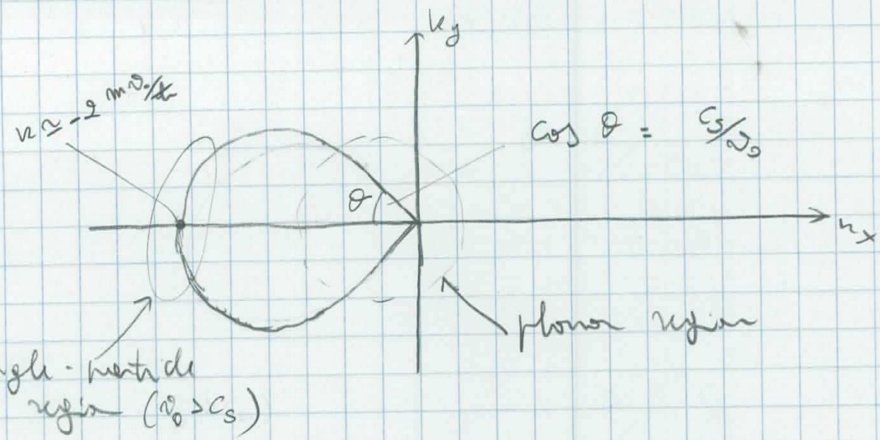


* defect creates perturbation in fluid
* in turn this induces force on defect

Resonance condition $\omega_k = 0$ satisfied for some k only if $v_0 > c_s$

critical speed $v_{cr} = \min_k \frac{\omega_k^{v=0}(k)}{k}$ → Landau critical speed

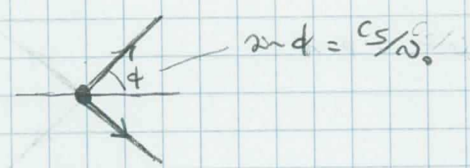
Qualitative features of induced perturbation:



Every k component propagates at $\underline{v_g = \nabla_k \omega_k}$, which is orthogonal to locus ($\omega_k = 0$) from the defect position

Every k has a <u>carrier</u> that oscillates in space at k

Phonon region:

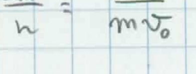


$$\begin{cases} v_0 \gtrsim c_s \longrightarrow \phi \lesssim \pi/2 \\ v_0 \gg c_s \longrightarrow \phi \to 0 \end{cases}$$

Čerenkov physics

Superposition of carriers $\longrightarrow$ Transverse width of wave

Single-particle region:



$$\left.\begin{array}{l} \text{propagates upstream} \\ \text{period of oscillations } \frac{2\pi}{k} = \frac{\hbar\pi}{m v_0} \end{array}\right\} \text{reflection of de Broglie wave}$$

Bibliography

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[soliton physics]