Lecture 8 : Response of a BEC to weak perturbations. Superfluidity properties

* on the density:

$$H = H_{Bog} + \underbrace{\int d^3r\, \delta V(r)\, \left(\hat\psi^\dagger(r)\, \hat\psi(r)\right)}_{\delta\hat H}$$

$$\delta\hat H = \int d^3r\, \delta V(r)\, \Big[|\phi_0(r)|^2 \big(\overset{\text{no effect.}}{\cancel{N}} - \int d^3s\, \hat\Lambda^\dagger(s)\hat\Lambda(s)\big)$$

$$+\sqrt{N}\, \Lambda^\dagger(r)\, \phi_0(r) + h.c. + \qquad (1)$$

$$+ \Lambda^\dagger(r)\, \Lambda(r) \Big] \qquad (2)$$

(1) $\sqrt{N}\left(u_k^*(r)\, b_k^\dagger + v_k(r)\, b_k + u_k(r)\, b_k + v_k^*(r)\, b_k^\dagger\right) =$

$= \sqrt{N}\left[\left(u_k^*(r) + v_k^*(r)\right) b_k^\dagger + \left(u_k(r) + v_k(r)\right) b_k\right]$

↳ excites Bogolinbov mode out of BEC

↳ classical excitation of Bogol. mode.

(2) $\Lambda^\dagger(r)\Lambda(r) = \left(u_k^*(r)\, b_k^\dagger + v_k(r)\, b_k\right)\left(u_{k'}(r)\, b_{k'} + v_{k'}^*(r)\, b_{k'}^\dagger\right)$

↳ if $v_k \simeq 0$ → scatters Bogol excitation $k' \to k$

↳ generally: also creates pairs of Bogol excitation via $u_k^*(r)\, v_{k'}^*(r')\, b_k^\dagger\, b_{k'}^\dagger$ term

Summing up (0) + (2) :

$$\delta H_{0,2} = \int d^3r \; \delta V(r) \cdot \left(\Lambda^\dagger(r) \Lambda(r) - |\phi_0(r)|^2 \int d^3s \; \Lambda^\dagger(s) \Lambda(s) \right)$$

* if $\delta V(r) = \delta V_0$ flat in space

$$\Longrightarrow \quad \delta H_{0,2} = 0$$

* if $|\phi_0(r)|^2$ flat in space

$$\Longrightarrow \quad \delta H_{0,2} = \int d^3r \; \delta V(r) \cdot \Lambda^\dagger(r) \Lambda(r) - \delta \bar{V} \cdot \int d^3r \; \Lambda^\dagger(r) \Lambda(r)$$

$$= \int d^3r \left(\delta V(r) - \delta \bar{V} \right) \Lambda^\dagger(r) \Lambda(r)$$

$$\delta H_1 = \sqrt{N} \sum_k \int d^3r \; \delta V(r) \Big[\phi_0(r) \left(u_k^*(r) + v_k^*(r) \right) \hat{b}_k^\dagger + $$

$$+ \phi_0^*(r) \left(u_k(r) + v_k(r) \right) \hat{b}_k \Big]$$

$\Longrightarrow$ emission of coherent phonons

* Matrix element

$$M_k = \sqrt{N} \int d^3r \; \delta V(r) \; \phi_0(r) \left(u_k^*(r) + v_k^*(r) \right) =$$

$$= \sqrt{n} \; \delta \tilde{V}(k) \; (U_k + V_k) = \qquad [\text{homogeneous BEC}]$$

$$= \sqrt{n} \; \delta \hat{V}(k) \sqrt{\frac{\hbar^2 k^2 / 2m}{\epsilon_k^{Bog}}} .$$

* $k\xi \gg 1 \quad \Longrightarrow \quad M_k = \sqrt{n} \; \delta \hat{V}(k)$: single particle scattering

* $k\xi \ll 1 \quad \Longrightarrow \quad M_k \simeq \sqrt{n} \; \delta \hat{V}(k) \sqrt{\frac{\hbar k}{2 m c_s}}$: phonon mode.

* on the interaction constant g:

$$H = H_{Bog} + \int d^3r \quad \frac{\delta g}{2} \psi^\dagger(r) \psi^\dagger(r) \psi(r) \psi(r)$$

→ modulation of a via Feshbach effect

→ modulation of transverse confinement in low-d

$$\delta H = \int d^3r \quad \frac{\delta g}{2} \cdot \Big[|\phi_0(r)|^4 \quad a^\dagger_{\phi_0} a^\dagger_{\phi_0} a_{\phi_0} a_{\phi_0} +$$

$$+ \phi_0^*(r) |\phi_0|^2 \quad a^\dagger_{\phi_0} a^\dagger_{\phi_0} a_{\phi_0} \cdot \delta\psi(r) \quad + h.c.$$

$$+ 4 |\phi_0(r)|^2 \quad a^\dagger_{\phi_0} a_{\phi_0} \quad \delta\psi^\dagger(r) \, \delta\psi(r) \quad +$$

$$+ \quad \phi_0^*(r)^2 \quad a^\dagger_{\phi_0} a^\dagger_{\phi_0} \quad \delta\psi(r) \, \delta\psi(r) \quad + \quad h.c.$$

$$+ \ldots \qquad \Big] =$$

$$= \frac{\delta g}{2} \int d^3r \quad \Big[|\phi_0(r)|^4 \left(N(N-1) - 2N \, \delta\hat{N} \right) + \emptyset +$$

↳ orthogonality $(\phi^* \delta\psi)$ +

+ homogeneity

$$+ 4 |\phi_0(r)|^2 \; (N - \cancel{\delta\hat{N}}) \cdot \delta\psi^\dagger(r) \, \delta\psi(r) \; +$$

higher order

$$+ \phi_0^*(r)^2 \cdot N \quad \Lambda^\dagger(r) \Lambda^\dagger(r) \quad + h.c. \Big] =$$

$$= c^{te} - \delta g \cdot n \cdot \delta\hat{N} + 2\,\delta g \cdot n\ \delta\hat{N} +$$

$$+ \frac{\delta g}{2} \cdot n \int d^3r\ \Lambda^\dagger(r)\,\Lambda^\dagger(r) + h.c.$$

$$= \delta g \cdot n\ \delta\hat{N} + \frac{\delta g \cdot n}{2} \int d^3r \left(\Lambda^\dagger(r)\,\Lambda^\dagger(r) + h.c.\right)$$

$$= \frac{\delta g \cdot n}{2} \int d^3r\ \left(\Lambda(r) + \Lambda^\dagger(r)\right)\left(\Lambda(r) + \Lambda^\dagger(r)\right) + c^{te}$$

$$= \frac{\delta g \cdot n}{2} \int d^3r \sum_{nn'} \left(\left(u_n(r) + v_n(r)\right) b_n + \left(u_n^*(r) + v_n^*(r)\right) b_n^\dagger \right) \cdot$$

$$\cdot \left(n \leftrightarrow n' \right) =$$

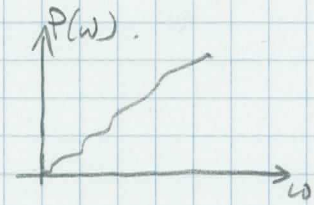
$$= \frac{\delta g \cdot n}{2} \cdot \sum_n \left(U_n + V_n\right)^2 \left(2\, b_n^\dagger b_n + b_n^\dagger b_{-n}^\dagger + b_n b_{-n}\right)$$

emits pairs of Bogol. particles

Modulation spectroscopy:

$\delta g(t) = \delta g \cdot \cos(\omega t) \longrightarrow$ emits pairs at $\pm n$ if $\omega_n^{Bog} + \omega_{-n}^{Bog} = \omega$

spectrum starts from $\omega = 0$ for BEC.



Superfluidity : several definitions possible.

1 – Landau criterion for drag force on defect

2 – response to transverse gauge field

$$H_{kin} = \frac{1}{2m}\left(\underline{P} - q\,\underline{A}(R,t)\right)^2$$

$$\underline{A}(R,t) = A_0\,\hat{e}_y\,e^{ik_x x}\,e^{-i\omega t} + c.c.$$

3 – BEC in moving frame $H = H_0 - \underline{P}\cdot\underline{v}$

4 – metastability of supercurrents

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(1) $v_c = \min_k \frac{\omega(k)}{k}$ if $v < v_c \longrightarrow$ no drag force on moving defect

(2) $\lim_{k\to 0}\lim_{\omega\to 0} \chi_t(k,\omega) \propto f_n \longrightarrow$ only normal part of fluid responds to $A_\perp$

$\lim_{k\to 0}\lim_{\omega\to 0} \chi_\ell(k,\omega) \longrightarrow$ sum rule due to gauge invariance

$\longrightarrow$ superfluid analog of Meissner effect of superconductors : $\vec{B}$ field is screened out by superconducting component.

(3) calculations for $H = H_0 - \underline{P}\cdot\underline{v}$ easier:

* ring geometry
* periodic boundary conditions
* response to rotation, i.e.

system at equilibrium in rotating frame

Bogoliubov calculation

$$H = E_0 + \sum_k \hbar\omega_k^{Bog}\, b_k^+ b_k - \underline{v}\cdot\sum_k \hbar\underline{k}\, b_k^+ b_k$$

* condensate at rest in $k=0$
* Bogol. particles $(\omega_k^{Bog}, \underline{k})$

$$= E_0 + \sum_k \hbar\left(\omega_k^{Bog} - \underline{k}\cdot\underline{v}\right) b_k^+ b_k$$

$f_n = \lim_{v\to 0} \frac{\langle P\rangle_v}{N\cdot m v}$: fraction of mass that is dragged

$$\langle P\rangle_v = \sum_k \hbar\underline{k}\, \langle b_k^+ b_k\rangle =$$

$$= \sum_k \frac{\hbar k}{e^{\beta(\hbar\omega_k^{Bog} - \underline{k}\cdot\underline{v})} - 1}$$

$$f_n = \frac{1}{Nmv}\lim_{v\to 0}\langle P\rangle_v \simeq \sum_k \frac{-\hbar k}{\left(e^{\beta\hbar\omega_k^{Bog}} - 1\right)^2}\cdot e^{\beta\hbar\omega_k^{Bog}} \cdot \frac{(-\hbar\beta\, \underline{k}\cdot\underline{v})}{N\cdot m\, v}$$

$$= \sum_k \frac{\hbar^2 k_x^2}{N m k_B T}\, \frac{e^{\beta\hbar\omega_k^{Bog}}}{\left(e^{\beta\hbar\omega_k^{Bog}} - 1\right)^2} =$$

$$= \sum_k \frac{\hbar^2 k_x^2}{N m \cancel{k_B T}} \cdot \frac{1}{\cancel{\beta \hbar}\, v_{gr}} \left[-\frac{\partial}{\partial k_x} \frac{1}{e^{\beta \hbar \omega_k} - 1} \right]$$

a) single particle approx: $k_B T \gg \mu$

$$\Longrightarrow v_{gr} \simeq \frac{\hbar k}{m}, \quad \omega_k \simeq \frac{\hbar k^2}{2m}$$

$$f_n = -\sum_k \frac{k_x}{N} \frac{\partial}{\partial k_x} \left(\frac{1}{e^{\beta \hbar \omega_k} - 1} \right) \simeq$$

[integration by parts]

$$\simeq -\frac{V}{N} \int \frac{d^3k}{(2\pi)^3} \left(\frac{\partial}{\partial k_x} \frac{k_x}{e^{\beta \hbar \omega_k} - 1} \right) - \frac{1}{e^{\beta \hbar \omega_k} - 1} =$$

$$= \frac{V}{N} \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta \hbar \omega_k} - 1} = \frac{N_{nc}}{N} = f_{nc}$$

b) phonon regime $\quad k_B T \ll \mu \quad \Longrightarrow v_{gr} \simeq c_s, \; \omega_k \simeq c_s \cdot |k|$

$$= \sum_k \frac{\hbar k_x^2}{N m \, c_s \cdot \frac{k_x}{|k|}} \left(-\frac{\partial}{\partial k_x} \frac{1}{e^{\beta \hbar \omega_k} - 1} \right) =$$

$$= \sum_k \frac{\hbar k_x \cdot |k|}{N m \, c_s} \left(-\frac{\partial}{\partial k_x} \frac{1}{e^{\beta \hbar \omega_k} - 1} \right) =$$

$$= -\frac{\hbar V}{N m c_s} \left[\underbrace{\cancel{\int \frac{d^3k}{(2\pi)^3} \frac{\partial}{\partial k_x} \left(\frac{k_x \cdot |k|}{e^{\beta \hbar k} - 1} \right)}}_{0} - \int \frac{d^3k}{(2\pi)^3} \left(\frac{\partial}{\partial k_x} (k_x \cdot |k|) \right) \frac{1}{e^{\beta \hbar c_s |k|} - 1} \right]$$

$$= \frac{\hbar}{N m c_s} \int \frac{d^3 k \cdot V}{(2\pi)^3} \frac{|k| + \frac{k_x^2}{|k|}}{e^{\beta \hbar c_s |k|} - 1} \simeq$$

$$\simeq \frac{\hbar}{N m c_s} V \frac{2\pi}{(2\pi)^3} \int dk \int_{-1}^{1} d\cos\theta \, (1 + \cos^2\theta) \frac{k^3}{e^{\beta \hbar c_s k} - 1}$$

$$= \# \frac{\hbar V}{N m c_s} \frac{1}{(\beta \hbar c_s)^4} = \frac{(k_B T)^4}{n m c_s^5 \hbar^3} \#$$

Performing all the calculations:

$$f_n = \frac{2\pi^2}{45} \frac{(k_B T)^4}{n m c_s^5 \hbar^3}$$

To be compared to:

$$f_{nc} = \frac{8}{3\sqrt{\pi}} (\rho a^3)^{1/2} \left[1 + \left(\frac{\pi k_B T}{2\mu} \right)^2 + \ldots \right]$$

in particular $f_n \ll f_{nc}$, as in ^4He.

NOTE: BEC assumed in $k = 0$

$$E_n^{BEC} = N \frac{\hbar^2 k^2}{2m} \simeq \frac{N \hbar^2}{2m} \left(\frac{2\pi}{L} j \right)^2 = \frac{N \hbar^2}{m L^2} (2\pi^2 j^2)$$

$\nearrow$ 3D: $N/L^2 = \frac{N}{L^3} \cdot L \to \infty$ in TD limit

$\longrightarrow$ 2D $\longrightarrow$ c^{te}.

$\longrightarrow$ 1D $\longrightarrow$ 0

So: in 3D BEC is always in $k=0$ in TD limit

in 1D BEC spread over many k values

$\hookrightarrow$ richer physics of winding numbers.

"Fast rotation": $v \gg \frac{\hbar}{m}\left(\frac{2\pi}{L}\right)$

GPE ground state has finite k such that

$$\frac{\hbar^2}{2m}k^2 - \frac{\hbar k}{m}v = \text{minimum, i.e.}$$

$$k_0 \equiv \frac{m}{\hbar}v \quad \left[2\pi/L\right]$$

$\longrightarrow$ BEC has finite current around ring: "supercurrent"

Provided $v \ll c_s$, this state is energetically stable

$$\omega_k^{Bog} = \left(\omega_k^{Bog} + \frac{\hbar k_0}{m}\cdot k\right) - k\cdot v \simeq \omega_k^{Bog}$$

(in rotating frame).

and $\omega_k^{Bog} = \omega_k^{Bog} + \frac{\hbar k_0}{m}\cdot \underline{k} > 0$ (in laboratory frame)

$\Longrightarrow$ supercurrent state is Landau-stable w/r to weak perturbations.

METASTABILITY of supercurrents.

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