Lecture 9 : Non-equilibrium BEC

Equilibrium world:

* classical systems :
  - equiprobability postulate : all microscopic states have same probability
  - can be demonstrated in some simple cases (billiards). In other cases: assumed.
  - Liouville's theorem : phase-space density is constant along H-orbits.

  $\Rightarrow$ MICROCANONICAL ensemble.

  - system (weakly) coupled to reservoir of energy
  - probability distribution $p_i \propto \exp(-E_i/k_B T)$

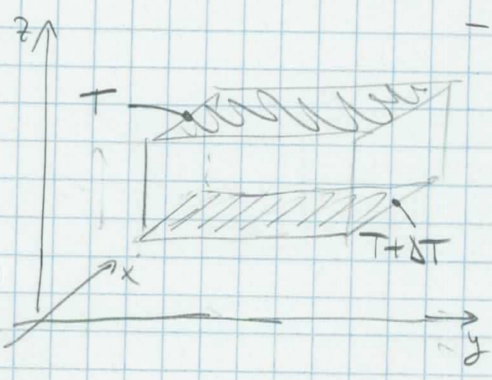
  $\Rightarrow$ CANONICAL ensemble.

* quantum systems
  - equilibrium density matrix $\hat{\rho} \propto e^{-\hat{H}/k_B T}$
  - eigenstates of $\hat{H}$ occupied according to a Boltzmann law $\exp(-E_i/k_B T)$

Non-equilibrium world

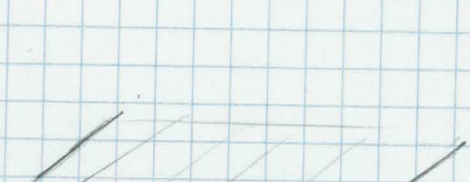
- no simple form of probability distribution known for generic systems.
- case by case analysis of system under consideration. Dynamics crucial
- non-equilibrium steady-state

Non-equilibrium phase transitions



- convection cells in fluid heated from below

for $\Delta T < \Delta T_c$:
- heat transported by conduction
- system homogeneous in xy
- temperature gradient along z



for $\Delta T > \Delta T_c$:
- homogeneous state dynamically unstable
- spontaneously breaks translational invariance
- pattern of convection cells appear in fluid
- symmetry fixed by boundary conditions (cylindrical rolls, hexagonal cells ...)

examples:

- geological activity in Earth's mantle. Tectonic plates
- Belousov - Zhabotinski reactions where continuous supply of reagents
- model of traffic flow.
- driven - diffusive lattice gases (electrolytes, some solid conductors ..)

Phase transitions

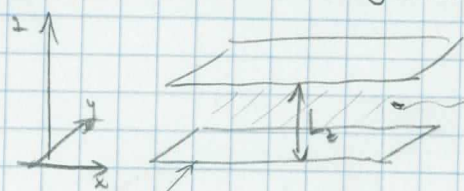
equilibrium systems

- singularities in thermodynamic functions
- order of transition: jump in $(n-1)^{th}$ derivative.
- second - order transitions $\longrightarrow$ spontaneous symmetry breaking, appearance of order parameter, long - range order, macroscopic diffusion time.
- example BEC: breaks $U(1)$ of $\hat{\psi}$ phase
  order parameter $\psi = \langle \hat{\psi} \rangle$ with given phase
  LRO: $\lim_{|r-r'|} \langle \hat{\psi}^{\dagger}(r) \hat{\psi}(r') \rangle > 0$, $\langle \hat{\psi}^{\dagger}(t+\tau) \psi(t) \rangle$ decays in time $\tau_c \to \infty$ for large system limit

* definition based on order parameter still valid for non-equilibrium steady-states

* singularities in observables as a f. external parameters.
  ↳ both require large system limit !

Planar laser cavity (VCSEL):



laser medium $\varepsilon = \varepsilon_r + i\,\varepsilon_i$

↳ amplification

effective index

mirrors, reflectivity $R \simeq 1$

i) $R = 1$ ideal cavity.

electric field $E(x,y,z;t) = \underbrace{E_0\, e^{ik_x x}\, e^{ik_y y}}_{\substack{\text{plane wave}\\ \text{along } x,y\\ \text{(translat. symmetry)}}} \cdot \sin(\pi z / L_z) \cdot e^{-i\omega t}$

time evolution

wf. along z fixed by boundaries (λ cavity)

Fresnel equation:

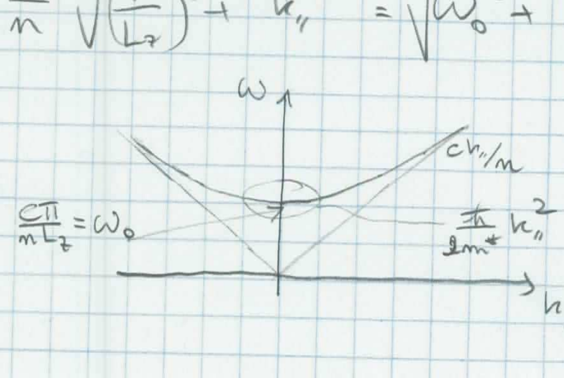
$$\varepsilon\, \omega^2 = k_x^2 + k_y^2 + \left(\pi / L_z\right)^2$$

→ quantization along z.

$$\text{if } \varepsilon_i = 0,\ n = \sqrt{\varepsilon} \ \therefore\ \omega = \frac{c}{n}\sqrt{\left(\frac{\pi}{L_z}\right)^2 + k_{\parallel}^2} = \sqrt{\omega_0^2 + \frac{c^2 k_{\parallel}^2}{n^2}}$$

effective photon mass:

$$m^* = \frac{\hbar \omega_0}{c^2} \cdot n^2$$



$$\omega \simeq \omega_0 + \frac{\hbar}{2m^*} k_{\parallel}^2$$

Including ε_i, assuming $|\varepsilon_i| \ll \varepsilon_2$:

$$\omega = \frac{c}{\sqrt{\varepsilon_2 + i\varepsilon_i}} \sqrt{\left(\frac{\pi}{L_z}\right)^2 + k_{\parallel}^2} \simeq$$

$$\simeq \frac{c}{\sqrt{2}} \left(1 - i\frac{\varepsilon_i}{2\varepsilon_2}\right) \frac{\pi}{L_z} \left(1 + \frac{1}{2}\left(\frac{L_z k_{\parallel}}{\pi}\right)^2\right)$$

$$\simeq \left(\omega_0 + \frac{\hbar k_{\parallel}^2}{2m^*}\right) \left(1 - i\frac{\varepsilon_i}{2\varepsilon_2}\right) \simeq$$

$$\simeq \underbrace{\omega_0 + \frac{\hbar k_{\parallel}^2}{2m^*}}_{\text{dispersion}} - \underbrace{i\frac{\varepsilon_i}{2\varepsilon_2}\omega_0}_{\text{amplification / losses}}$$

$$\begin{cases} \varepsilon_i > 0 & \text{losses} \\ \varepsilon_i < 0 & \text{amplification} \end{cases}$$

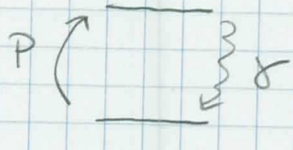
Example : $\longrightarrow$ 2-level system

$$\varepsilon(\omega) = 1 + \frac{f}{\omega_0 - \omega - i\Gamma/2} \qquad \begin{cases} \Gamma > 0 & \text{losses} \\ \Gamma < 0 & \text{amplification} \end{cases}$$

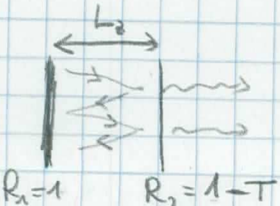
$*$ $\Gamma = P - \gamma$

pumping

decay



Radiative losses : non-perfectly reflecting mirrors.



cavity round trip:

- time $\tau = \frac{2n L_z}{c}$ (paraxial approx.)

- energy decreases by $1-T$ $\Longrightarrow$ $\gamma_{rad} = \frac{T}{\tau} = \frac{cT}{2nL_z}$

$\longrightarrow$ k-conservation during exit process : far-field reproduces in-cavity momentum distribution $\hat{E}_{ff}(k) = c^{te} \cdot \hat{a}_k$

lens system to reproduce near-field $\hat{E}_{fn}(x) = c^{te'} \cdot \hat{\psi}(x)$

$\hookrightarrow$ All properties of gas inferred from emitted light

Nonlinearity $\chi^{(3)}$
$\left\{\begin{array}{l} \text{main effect is on } \omega_0 = \frac{c\pi}{L_z(n+\delta n)} \\ \text{effect on other quantities is perturbative} \end{array}\right\}$ summarized in nonl. coupling g

External potential via spatial modulation of L_z : $V(x)$

Driving term : incident classical laser field incident on mirror.

Amplitude $F(x,t)$

System Hamiltonian

$$H = \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + V(x)\hat{\psi}^\dagger(x)\hat{\psi}(x) + \frac{g}{2}\hat{\psi}^\dagger(x)\hat{\psi}^\dagger(x)\hat{\psi}(x)\hat{\psi}(x) + F(x,t)\hat{\psi}^\dagger(x) + F^*(x,t)\hat{\psi}(x)$$

$\hat{a}_k \longrightarrow$ creates cavity photon at k, $\omega_k = \omega_0 + \frac{\hbar k^2}{2m^*} + \dots$

$\hat{\psi}(x) \longrightarrow$ field operator $= \mathcal{F}.T.\left[\hat{a}_k\right]$ $\to$ Does the "photon wf" exist?

Incoherent pump (i.e. amplification) and losses
have to be introduced at level of master equation

$$\frac{d}{dt}\rho = -i\,[H,\rho] + \ldots$$

Equation of motion at M.F. level: $\psi = \langle\hat{\psi}\rangle$

$$\frac{\partial\psi}{\partial t} = \underbrace{-i\left[\omega_0 - \frac{\hbar\nabla^2}{2m^*}\right]\psi}_{\text{free-field evolution}} + \underbrace{(P-\gamma)\psi}_{\text{incoherent pump + losses}} +$$

$$-i\underbrace{F(x,t)}_{\text{coherent drive}} \; \underbrace{-i\,g|\psi|^2\psi}_{\text{non-linearity}} \; -i\underbrace{V(x)\,\psi(x)}_{\text{external potential}}$$

$\longrightarrow$ generalization of GPE to non-equilibrium, interacting photon system

BEC in homogeneous system: $V(x) = 0$

only incoherent pump: eq. motion symmetric under $\psi \to \psi e^{i\theta}$

ansatz: $\psi(x,t) = \bar{\psi}\, e^{-i\bar{\omega}t}$

$$\bar{\omega}\bar{\psi} = \omega_0\bar{\psi} - i(P-\gamma)\bar{\psi} + g|\bar{\psi}|^2\bar{\psi}$$

$\bar{\psi} = 0$ is solution:

* stable for $P<\gamma$
* unstable for $P>\gamma$ $\to$ spontaneously break phase symmetry

To stabilise dynamics above threshold

$\longrightarrow$ gain saturation term

$$P\psi \longrightarrow (P_0 - P_1|\psi|^2)\psi \quad , \quad P_1 > 0 \text{ fixed,}$$

P_0 depends on pump intens

$$\bar{\omega}\bar{\psi} = \omega_0\bar{\psi} + g|\bar{\psi}|^2\bar{\psi} - i\left(P_0 - P_1|\bar{\psi}|^2 - \gamma\right)\bar{\psi}$$

* for $P_0 < \gamma \longrightarrow \bar{\psi} = 0$

* for $P_0 > \gamma \longrightarrow (P_0 - \gamma) = P_1|\bar{\psi}|^2$

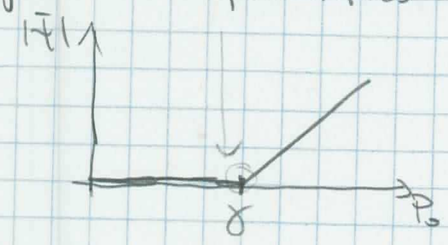
fixes $|\bar{\psi}|^2$, not its phase

$\Longrightarrow$ spontaneous symmetry breaking

Does this correspond to a phase transition ?

* singularity in "thermodynamical" quantities ?

$\longrightarrow$ YES



* long-range order , $\langle\psi^\dagger(x)\,\psi(x')\rangle \xrightarrow{|x-x'|\to\infty}$ finite ?

$\longrightarrow$ YES at MF. What happens with fluctuations not completely known ...

Condensate wavefunction. What determines it?

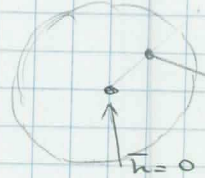
NO : energy minimization condition
NO : time-reversal symmetry
} ONLY dynamical stability

→ solution $\psi(x,t) = \bar{\psi}\, e^{i\bar{k}x}\, e^{-i\bar{\omega}t}$

is OK at MF, $\bar{\omega} = \omega_0 + \frac{\hbar \bar{k}^2}{2m^*} + g|\bar{\psi}|^2$

→ specific value of k to be selected by boundary conditions.

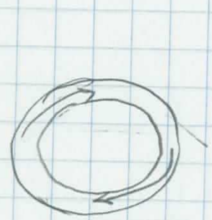
example : P(x) has cylindrical symmetry, apply LDA



→ solution with single $\bar{\omega}$ ⇒ relates $\bar{k}$ to $|\bar{\psi}|^2$
this latter determined by P(r).

→ May explain experimental observation of BEC into ring of modes at finite k.

Other example : ring lasers



→ emission in modes at different values of k.
Choice depends on frequency dependence of gain.
Many modes can be stable but for mode jumps.

<u>Parity symmetry</u> : modes $\pm k$ equivalent

two possibilities $\nearrow$ both lase simultaneously
$\searrow$ parity spontaneously broken

$\longleftrightarrow$ what happens depends on microscopic details of laser set-up

Simultaneous lasing of $\pm k$ interesting for applications to gyrometry

system at rest : $\omega^0_{\pm n} = \frac{c}{nR} N$, N integer

system rotating at Ω : $\omega_{\pm n} = \omega^0_{\pm n} \pm N \cdot \Omega$

$\hookrightarrow$ infer Ω from <u>beat note</u>.

$\rightarrow$ but one needs simultaneous lasing of <u>two distinct modes</u> at $\omega_{\pm n} = \tilde{\omega}_{\pm n} \pm N\Omega$ and <u>not</u> on a single mode which contains superposition of two plane waves at a single frequency.