Theoretical studies of thermal vortices in a 2D Bose gas

Iacopo Carusotto
BEC CNR-INFM and Università di Trento, Italy

in collaboration with:

Yvan Castin
Laboratoire Kastler Brossel de l'Ecole Normale Superieure, Paris, France
Motivation

Recent observations of thermal vortices in 2D Bose gases at finite $T$

- Calculate density and correlation of vortex positions
- Extract from expts some info on the state of the system
- Relation to BKT phase transition ?
- Other interesting observables ?

Our point of view: **semiclassical field theory**

- results quantitatively accurate for $k_B T > \mu$
- based on C-number wavefunctions : can locate vortices
- no need for UV cut-off as in classical theories

Array of Josephson junctions in an optical lattice. Expansion: phase defects $\rightarrow$ vortices

V. Schweikhard, S. Tung, E. A. Cornell, 2007
**The physical system**

\[ H = \sum_k \frac{\hbar^2 k^2}{2m} a_k^{\dagger} a_k + \frac{g_0}{2} \int dr \, \Psi^{\dagger}(r) \Psi(r) \overline{\Psi}(r) \overline{\Psi}(r) \]

Bose field Hamiltonian on a 2D lattice:

- **parabolic free particle dispersion**
- **interactions: discrete-\(\delta\) potential of strength** \(g_0 = \sqrt{8\pi} \frac{\hbar^2}{m} \frac{a_{3D}}{a_z}\)
- **models atoms in effectively 2D geometry**
- **homogeneous system, periodic boundary conditions**
The semiclassical field method

Density matrix $\rho$ written in terms of Glauber-P distribution:
$$\rho = \int d\psi \ P[\psi] \ |coh:\psi\rangle\langle coh:\psi|$$

Imaginary-time evolution: pseudo-Fokker-Planck eq. for $P[\psi]$,
$$\tau = 0 \rightarrow \beta = \frac{1}{k_B T}$$

$$\begin{align*}
\frac{\partial}{\partial \tau} P[\psi] &= -E[\psi] P[\psi] + \\
&- \partial_\psi (F[\psi] P[\psi]) + \\
&+ \frac{g_0}{4 \ dV} \partial^2_\psi (\psi^2 P[\psi])
\end{align*}$$

Classical field

Semi-classical

Exact

Cut-off needed to avoid blackbody catastrophe

Non-positive diffusion
$$D = -\frac{g_0}{4 \ dV} \begin{pmatrix}
0 & \psi^2 \\
\psi^*^2 & 0
\end{pmatrix}$$

No mapping on stoch. problem

Other methods necessary
IC, Y. Castin, J. Dalibard, PRA 2000
IC, Y. Castin, J. Phys B 2001

Mapping onto stochastic problem:
- random sampling of initial wavefunction
- weight: classical energy $E[\psi] = \int \psi^* (h_0 - \mu) \psi + \frac{g_0}{2} |\psi|^4$
- drift: imaginary-time GPE $F[\psi] = -\frac{1}{2} (h_0 - \mu + g_0 |\psi|^2) \psi$
- pseudo-diffusion negligible if: high temperature $k_B T > \mu$, weak interaction $g_0 \ll 1$
- similar method in Canonical ensemble
Validation of the semiclassical field method

Simplified 2D Bogoliubov Hamiltonian

\[ H = \sum_k \left( \frac{\hbar^2 k^2}{2m} + \mu \right) a_k^\dagger a_k + \frac{\mu}{2} (a_k^\dagger a_{-k}^\dagger + a_k a_{-k}) \]

Mean energy per mode
- dotted: classical (equipartition)
- dashed: semiclassical
- solid: exact

Pair correlation function \( g^{(2)}(r) \)
- dashed: semiclassical
- solid: exact

- Semiclassical method quantitatively excellent as long as \( T \gg \mu \)
- Only fails in short-distance effects due to quantum fluctuations (two-body scatt. function)
- No UV divergences in observables
Semiclassical MC results for the 2D Bose gas

Canonical ensemble, N=1000 atoms

Simulated snapshots of *in situ* density and vortex locations

How to understand them? What physics do they teach us?
1- Normal and Non-Condensed fractions

Normal fraction:
\[ f_n = \frac{\langle P_x^2 \rangle}{N m k_B T} \]

Non-condensed fraction:
\[ f_{nc} = 1 - N_0 / N \]

finite size
\[ \left\{ \begin{array}{l} \text{Bose-Einstein condensation} \\ \text{universal jump in superfluid fraction smeared out} \end{array} \right. \]

- \( T_{BKT} \) (roughly) estimated from jump condition
\[ T_{BKT} = \frac{\pi n}{2 m} f_s = \frac{1}{4} T_{\text{deg}} f_s \]

- Reasonable agreement with theory
\[ n_{BKT} = \frac{m T}{2 \pi} \log \frac{\xi}{m g_0}, \quad \xi \approx 380 \]
2- Density fluctuations

\[ g^{(2)}(x, x') = \frac{\langle n(x) n(x') \rangle}{\langle n(x) \rangle \langle n(x') \rangle} \]

Ideal thermal gas:
- HB-T effect \( g^{(2)}(x=x') = 2 \)

Density fluctuations reduced by:
- Bose-Einstein condensation effect (ideal gas \( g_0 = 0 \), canonical ensemble)
- Repulsive interactions (\( g_0 > 0 \)) (quasi-condensation phenomenon)

Density fluctuations generally significant around \( T_{\text{BKT}} \):
(can play important role in superfluidity breaking!!)
3- Current-Current correlations

Finite temperature fluctuating, zero-mean mass current

Current-current correlations: \( C_{ij}(x - x') = \langle J_i(x) J_j(x') \rangle \)

\[
\begin{align*}
\text{● longitudinal} & \quad C_L(q \rightarrow 0) \rightarrow m_n k_B T \\
\text{● transverse} & \quad C_T(q \rightarrow 0) \rightarrow m_n k_B T * f_n
\end{align*}
\]

(f-sum rule) (resp. to transv. gauge field)

correlations related to response functions via fluctuation-dissipation theorem

\[
C_{ij} (x - x') = \langle J_i(x) J_j(x') \rangle
\]
4- Density of thermal vortices

Canonical (C) ensemble:
- ideal and interacting: similar physics
- high $T$: linear increase $n_v \sim T/T_{\text{deg}}$
- low $T$: activation law $n_v \sim \exp(-\Delta / k_B T)$

Ideal gas, Grand-Canonical (GC):
- low $T$: quadratic $n_v \sim (T/T_{\text{deg}})^2$
- high $T$: linear increase $n_v \sim T/T_{\text{deg}}$

(old result: Berry, Halperin)
Vortex density: physical discussion (I)

Vortex density: probability of having a node $\psi(r)=0$ in classical field

Ideal gas, Grand-Canonical (GC):
- strong density fluctuations $g^{(2)}(r=0)=2$
  $$n_{v,+}^{GC} \simeq n \cdot \left( \frac{T}{T_{\text{deg}}} \right)^2 \quad \text{for } T \gg T_{\text{deg}}$$
  $$\left( \frac{T}{T_{\text{deg}}} \right)^{1/2} \quad \text{for } T \ll T_{\text{deg}}$$

Ideal gas, Canonical (C):
- condensate present
- density fluctuation suppressed by BEC
- activation law:
  $$n_{v,+} = n_{v,+}^{GC} \cdot \exp \left[ -\frac{N_0}{\delta N} \right]$$
- Condensate depleted in larger system: GC result valid down to lower $T$

\[ g_0 = 0 \]
\[ g_0 = 0.1 \]
\[ g_0 = 0.333 \]
Vortex density: physical discussion (II)

SC energy functional:

\[ U[\psi] = \sum_k |\alpha_k|^2 k_B T \left( e^{\beta E_k} - 1 \right) + \frac{g_0}{2} \int d\mathbf{r} |\psi(\mathbf{r})|^4 \]

- \( U[\psi] \) depends on \( T \) to include Bose statistics of high energy modes

Classical GP energy would give \( \Delta \to 0 \) for UV cut-off \( \to \infty \)

- Activation law

\[ n_v(T) = C(T) \cdot \exp(-\Delta/k_B T) \]

- Activation energy

\[ \Delta(T) = \min_{\text{node}} U[\psi] - \min_{\text{no node}} U[\psi] \]

Ideal gas \( g_0 = 0 \):

\[ \Delta \approx \frac{\pi \hbar^2 n}{m \log(L/\lambda_{th})} \to 0 \]

Interacting gas \( g_0 > 0 \):

\[ \Delta \leq \frac{2 \pi \hbar^2 n}{m} \frac{1 - 2 n g_0 / k_B T}{\log[k_B T / 2 n g_0]} \to k_B T_{BKT} \]

Increasing system size

\[ L/\lambda_{th} = 6, 12, 24, 48 \]

Logarithmically vanishing in TD limit

(a)
5- Vortex-Vortex correlations

\[ G_{v,+}^{(2)}(r) = \langle n_{v,+}(r) n_{v,-}(0) \rangle \]

- **High T** (still degenerate):
  - peak at \( r = 0 \): +/- attraction
  - not much dependence on \( g_0 \)
  - similar to GC.

- **Low T** (activation regime):
  - longer correlations for \( g_0 = 0 \)

\[ G_{v,+}^{(2)}(r) [n^2] \]

- **High T**
  - \( g_0 = 0 \)
  - \( g_0 = 0.1 \)
  - \( g_0 = 0.333 \)

- **Low T**
  - \( g_0 = 0 \)
  - \( g_0 = 0.1 \)
  - \( g_0 = 0.333 \)
Some snapshots of vortex locations

- Selected realizations with thermally activated bound pairs
- Wider pairs in ideal gas
- Vortex proliferation
- Hard to distinguish free vortices from bound pairs and clusters
How to experimentally observe all these features?

Look for minima in density snapshot

Hard to identify vortices:
- smooth transition from density fluctuation to vortex pair.
- both coexist in a 2D gas at finite T
- around $T_{BKT}$: density fluctuations are already large
- Easier in arrays of Josephson junctions (JILA): density fluctuations frozen
Alternative strategy: looking at mass current

Strong sensitivity to Doppler shift $\omega \rightarrow \omega - kv$, i.e.

\begin{align*}
\text{strong dispersion of dielectric constant } & \quad \omega \frac{d\epsilon}{d\omega} \approx \frac{c}{v_g} \gg 1
\end{align*}

Significant refraction index, but weak absorption

**Optical probe**

**Electromagnetically Induced Transparency** effect:

- 3-level atom in $\Lambda$ configuration
- black line in absorption (Alzetta et al., 1977)
- very small $v_g$, down to m/s regime (Hau 2000)

**Refraction index** experienced by probe:

- depends on local current: $\epsilon_p \approx 1 + \alpha \hat{J} \cdot (\hat{k}_c - \hat{k}_p)$
- accessible: phase-contrast imaging, diffraction
- non-destructive, *in situ* measurement
Simulated images for single (bent) vortex in 3D BEC

**Top view**

- $k_c$ along x, $k_p$ along z

**Side view**

- $k_c$ along z, $k_p$ along x

\[
\Delta \phi \approx \pi \left( \frac{\hbar |k_p|}{m v_g} \right)
\]

Images for $v_g = 1$ m/s


similar ideas: Leonhardt and Piwnicki, PRL 84, 822 (2000)
Diffraction on current fluctuations in 2D gas

Coupling beam $k_c$ along x on 2D plane, probe along z

Look at far-field diffraction pattern

\[ I_{sc}(\vec{q}) \propto C_{xx}(\vec{q}) = \langle J^+_x(\vec{q}) J_x(\vec{q}) \rangle \]

\[ C_{xx}(q_x \rightarrow 0, q_y = 0) = mn k_B T \]

\[ C_{xx}(q_x = 0, q_y \rightarrow 0) = mn k_B T f_n \]

\[ \left| \frac{\Delta I_{sc}}{I_{inc}} \right|_{q \rightarrow 0} \approx c^{te} T T_{deg} \frac{Y_e^2}{\Omega_C^4} [f_n] \Delta \theta_x \Delta \theta_y \]

Alternative (perhaps easier) experiment: stimulated Bragg scattering on current fluctuations

Method not limited to bosons: vortices and superfluidity in Fermi clouds??
In conclusion...

Semiclassical-MC study of weakly interacting 2D Bose gas:

- B-E condensation due to finite-size effect, superfluid jump smeared out by finite size
- Density of thermal vortices: activation law, correlations of vortex locations
- But density fluctuations significant: smooth crossover in T
  → Seems hard to extract more information on KT from vortices (at least continuous space without lattice)
- Current-current correlations: clear info on $f_n$
  → Hopefully experimentally accessible by slow-light imaging


and thanks to ...

Luca Giorgetti

Support from: