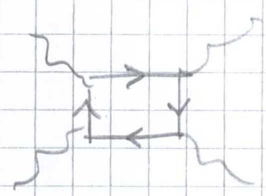
# Lecture 1

## Light as a fluid ?

| PROs | CONs |
|---|---|
| consists of many elementary photons | photons can be created / destroyed |
| black-body: thermal eq. in box | photons travel from source to absorber |
| photons move fast in box. | photons do not interact, each travels independently |
| | ↳ no collective behaviour |

## What if photons interacted ?

electronic polarizability of QED vacuum →

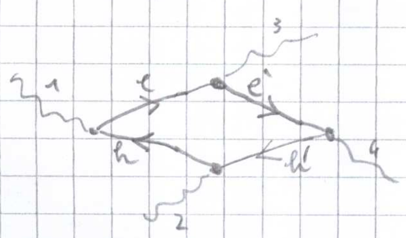


$$\sigma \simeq \alpha^4 \left(\frac{\hbar}{mc}\right)^2 \left(\frac{\hbar\omega}{mc^2}\right)^6$$

$\alpha \approx 1/137$, $\frac{\hbar}{mc} \approx 0{,}4$ pm

↳ ridiculously small !

in material media $\chi^{(3)}$ optical nonlinearity

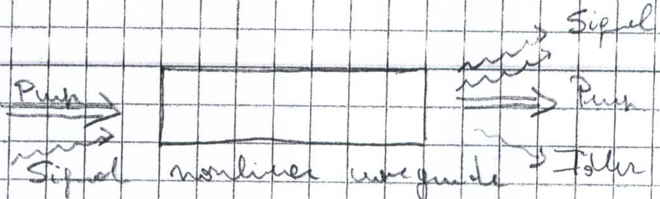



e/h pairs formed in semiconductor

or


two-photon absorption process

photon-photon collision → "four-wave mixing" in nonlinear optics




(Degenerate) four wave mixing

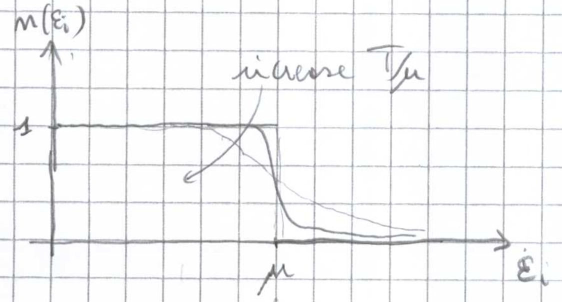
↳ PHOTON - PHOTON INTERACTIONS DO EXIST !

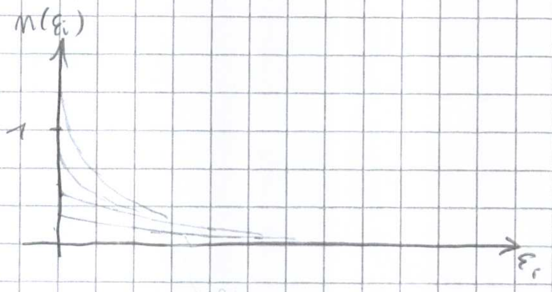
## Basics of quantum statistics :

| Fermions | Bosons |
|---|---|
| wf totally antisymmetric | wf totally symmetric |
| $\psi(r_1 \ldots r_i \ldots r_j \ldots r_N) =$ | $\psi(r_1 \ldots r_i \ldots r_j \ldots r_N) =$ |
| $= -\psi(r_1 \ldots r_j \ldots r_i \ldots r_N)$ | $= \psi(r_1 \ldots r_j \ldots r_i \ldots r_N)$ |
| max 1 particle per orbital | any number particles in same orbital |
| spin half-integer | spin integer |

(spin - statistics theorem)

| | |
|---|---|
| $n_i = \dfrac{1}{e^{\beta(\varepsilon_i - \mu)} + 1}$ | $n_i = \dfrac{1}{e^{\beta(\varepsilon_i - \mu)} - 1}$ |
| (Fermi - Dirac statistics) | (Bose - statistics) |




at low $T \ll \mu$:

$$n(\varepsilon_i) = \Theta(\mu - \varepsilon_i)$$

for fixed $N_T$: $\mu \xrightarrow{T\to 0} \varepsilon_F(N_T)$

"Fermi-energy"

finite compressibility $\frac{\partial \mu}{\partial N_T}$ even at $T \to 0$

$\mu < \min(\varepsilon_i)$ always

* non-degenerate gas $\frac{|\mu|}{T} \gg 1$

↳ $n(\varepsilon_i) \simeq e^{-|\mu|/T} \cdot e^{-\varepsilon_i/T}$

Boltzmann law

* degenerate gas $\frac{|\mu|}{T} \ll 1$

↳ $n(\varepsilon_i) \simeq \frac{1}{T}(\varepsilon_i - \mu)$

Black-body described by $\mu = 0$ case with $\varepsilon(k) = c|k|$

Massive particles in 3D box: $k = \frac{2\pi}{L} i_{x,y,z}$, $i \in \mathbb{Z}$

$$N_T = \sum_{i_x i_y i_z} \frac{1}{e^{\beta(\varepsilon(k_i)-\mu)} - 1} \simeq \int \frac{L^3}{(2\pi)^3} d^3k \frac{1}{e^{\beta(\varepsilon(k)-\mu)} - 1}$$

with $\varepsilon(k) = \frac{\hbar^2 k^2}{2m}$

for $\mu \to 0$ $\quad N_T(\beta = \frac{1}{k_B T}, \mu) \longrightarrow N_{max}(\beta) < \infty$

What happens if $N > N_{max}$ and particle # conserved?

Approx $\sum \to \int$ not so valid for lowest modes if $\mu \to 0$ :

$$N = \underbrace{\frac{1}{e^{-\beta\mu} - 1}}_{\text{occupation of } k=0 \text{ mode}} + N_{max}(\beta) \qquad N_{max}(T) = \frac{V}{\lambda_T^3} \underbrace{g_{3/2}(1)}_{2{,}612} \propto T$$

for $\mu \to 0$ diverges

finite value

occupation of $k=0$ mode

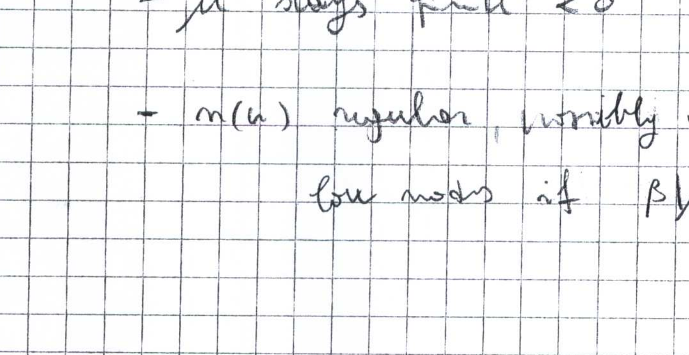
→ the Bose-Einstein condensate.

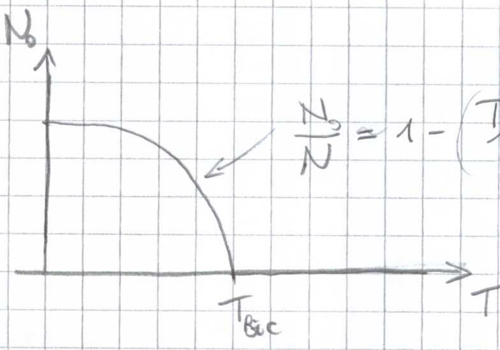
* if $N < N_{max}(\beta)$ :

- $\mu$ stays finite $< 0$
- $n(k)$ regular, possibly with Bose bunching into low modes if $\beta|\mu| < 1$.

* if $N > N_{max}(\beta)$ .

- $\mu$ reaches maximum value $\mu = 0$
- extra particles into BEC at $k=0$



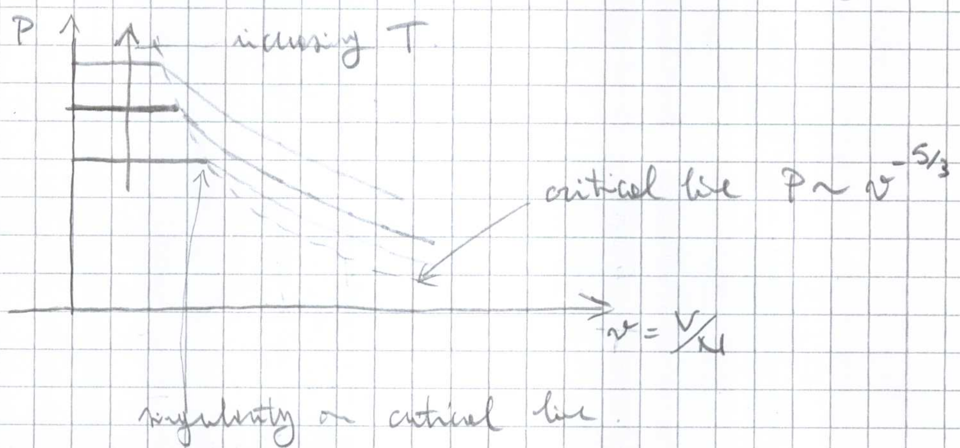



$$\frac{N_0}{N} = 1 - \left(T/T_{BEC}\right)^{3/2}$$ macroscopic fraction of particles into BEC $N_0 = O(N)$

(normally $n(k) = O(1)$)

$\rightarrow N_{max}(T_{BEC}) = N$

BEC is the (ONLY) phase transition that occurs in systems of NON-INTERACTING particles (at least to my knowledge...)




Correlation functions:

$$\langle \psi^\dagger(r)\, \psi(r') \rangle = ?$$

$$\rightarrow \int dr_2 \dots dr_N \sum_n p_n\, \psi_n(r', r_2 \dots r_N)\, \psi_n^*(r, r_2 \dots r_N) =$$

$$= \rho^{(1)}(r, r')$$ one-body density matrix

$$= \sum_i n_i \cdot \phi_i^*(r)\, \phi_i(r')$$

a) $|\beta\mu| \gg 1$ non-degenerate gas

$$\rho^{(1)}(r,r') = n \cdot \exp\left[-\pi r^2/\lambda_T^2\right]$$

with $\lambda_T$ thermal de Broglie wavelength

$$\lambda_T = \sqrt{\frac{2\pi\hbar^2}{m k_B T}}$$

b) $0 < |\beta\mu| \ll 1$ degenerate, but not BEC gas.

$$\rho^{(1)}(r,r') \simeq \frac{e^{-|r-r'|/\ell_c}}{|r-r'|} \quad \text{for } |r-r'| \to \infty.$$

$$\text{with } \ell_c = \frac{\lambda_T}{\sqrt{4\pi|\beta\mu|}} \gg \lambda_T$$

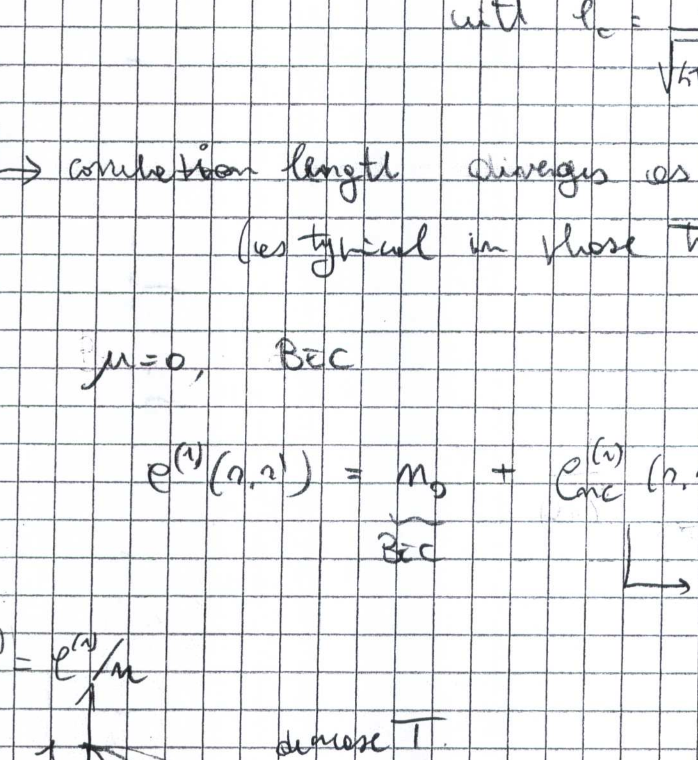
↳ correlation length diverges as $\mu \to 0$ towards BEC (as typical in phase transitions)

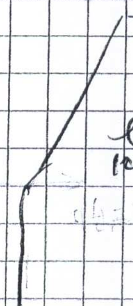
c) $\mu = 0$, BEC

$$\rho^{(1)}(r,r') = \underbrace{n_0}_{BEC} + \rho^{(1)}_{nc}(r,r')$$

↳ $\simeq \frac{1}{\lambda_T^2 r}$ power law with $\infty$ correlation length

$g^{(1)} = \rho^{(1)}/n$



**Penrose-Onsager criterion for BEC**

$$\lim_{|r-r'|\to\infty} g^{(1)}(r,r') = \frac{n_0}{n} \neq 0$$

↳ matter field $\psi$ is order parameter of BEC phase transition

Spontaneous symmetry breaking phenomenon $\psi \to \psi e^{i\phi}$, $\phi \in \mathbb{R}$.

## Bibliography


- lecture notes "new trends in BEC" @ www.science.unitn.it/~carusott

- Huang "Statistical Mechanics"

- Pitaevskii and Stringari "BEC"

- Cohen-Tannoudji's lectures @ collège de France www.phys.ens.fr/cours/college-de-france

- Gunton - Buckingham, Phys. Rev. 166, 152 ('68)