Lecture 4/5. Quantum hydrodynamics and superfluidity

Small oscillations around steady state / ground state:

$\longrightarrow$ linearized GPE

for simplicity $\longrightarrow$ homogeneous system.

$$i\partial_t \phi = -\frac{\hbar}{2m}\nabla^2\phi + g|\phi|^2\phi$$

$$\phi(r,t) = \left[\phi_0 + \delta\phi(r,t)\right]e^{-i\mu t} \quad ; \quad \mu = g|\phi_0|^2$$

$$i\partial_t\phi = i\partial_t\left\{\left[\phi_0 + \delta\phi\right]e^{-i\mu t}\right\} = \left(\mu\phi_0 + \mu\,\delta\phi + i\partial_t\delta\phi\right)e^{-i\mu t}$$

$$-\frac{\hbar}{2m}\nabla^2\phi = -\frac{\hbar}{2m}\nabla^2\left\{\left[\phi_0+\delta\phi\right]e^{-i\mu t}\right\} = -\frac{\hbar}{2m}\nabla^2\delta\phi \; e^{-i\mu t}$$

$$g|\phi|^2\phi = \left[g|\phi_0|^2\phi_0 + 2g|\phi_0|^2\delta\phi + g\phi_0^2\,\delta\phi^* + \text{nonlinear terms}\right] \times e^{-i\mu t}$$

Combining all terms:

$$\cancel{\mu\phi_0} + \cancel{\mu\,\delta\phi} + i\partial_t\delta\phi = -\frac{\hbar}{2m}\nabla^2\delta\phi + \cancel{g|\phi_0|^2\phi_0} + \cancel{2}g|\phi_0|^2\delta\phi + g\phi_0^2\,\delta\phi^*$$

$$\left.\begin{array}{l} i\partial_t\delta\phi = -\frac{\hbar}{2m}\nabla^2\delta\phi + g|\phi_0|^2\delta\phi + g\,\phi_0^2\,\delta\phi^* \\ \text{assuming } \phi_0 \in \mathbb{R}^+ : \\ i\partial_t\delta\phi^* = \frac{\hbar}{2m}\nabla^2\delta\phi^* - g|\phi_0|^2\delta\phi^* - g\,\phi_0^2\,\delta\phi \end{array}\right\} \text{Bogoliubov equations}$$

$$i\partial_t \begin{pmatrix} \delta\phi \\ \delta\phi^* \end{pmatrix} = \mathcal{L} \begin{pmatrix} \delta\phi \\ \delta\phi^* \end{pmatrix}$$

$$\text{with } \mathcal{L} = \begin{pmatrix} -\frac{\hbar\nabla^2}{2m} + g|\phi_0|^2 & g\phi_0^2 \\ -g\phi_0^{*2} & \frac{\hbar\nabla^2}{2m} - g|\phi_0|^2 \end{pmatrix}$$

normal modes $\longrightarrow$ eigensolutions of $\mathcal{L}v = \varepsilon v$

invariance under translations $\longrightarrow$ look for solutions in the form

$$\begin{pmatrix} \delta\phi(r) \\ \delta\phi^*(r) \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} e^{ikr}$$

$$\mathcal{L}\left[\begin{pmatrix} U \\ V \end{pmatrix} e^{ikr}\right] = \begin{pmatrix} \frac{\hbar k^2}{2m} + g|\phi_0|^2 & g\phi_0^2 \\ -g\phi_0^{*2} & -\frac{\hbar k^2}{2m} - g|\phi_0|^2 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} e^{ikr}$$

$$\begin{pmatrix} \frac{\hbar k^2}{2m} + g|\phi_0|^2 - \varepsilon & g\phi_0^2 \\ -g\phi_0^{*2} & -\frac{\hbar k^2}{2m} - g|\phi_0|^2 - \varepsilon \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = 0.$$

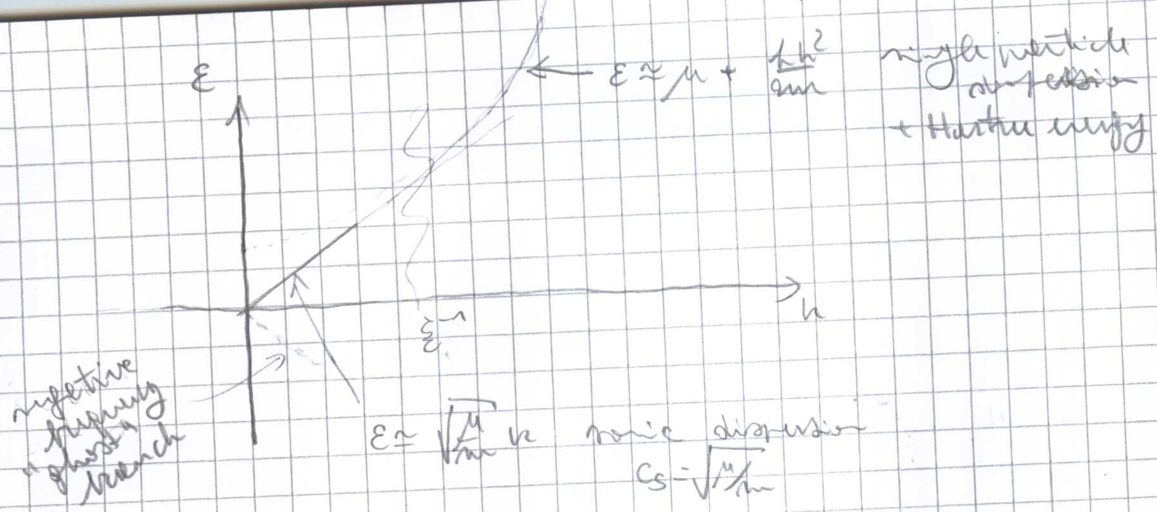
$$\left(\frac{\hbar k^2}{2m} + \mu - \varepsilon\right)\left(-\frac{\hbar k^2}{2m} - \mu - \varepsilon\right) + \mu^2 = 0$$

$$\left(\varepsilon - \left(\frac{\hbar k^2}{2m} + \mu\right)\right)\left(\varepsilon + \left(\frac{\hbar k^2}{2m} + \mu\right)\right) + \mu^2 = 0$$

$$\varepsilon^2 - \left(\frac{\hbar k^2}{2m} + \mu\right)^2 + \mu^2 = 0$$

$$\varepsilon^2 = \frac{\hbar k^2}{2m}\left(\frac{\hbar k^2}{2m} + 2\mu\right) \Longrightarrow \varepsilon = \pm\sqrt{\frac{\hbar k^2}{2m}\left(\frac{\hbar k^2}{2m} + 2\mu\right)}$$

Bogoliubov dispersion



transition for $k \approx \xi^{-1}$ (healing length)

$$\phi(r,t) = e^{-i\mu t}\Big[\phi_0 + (U+V)\cos(kx - \underbrace{\varepsilon}_{\in \mathbb{R}^+} t) + i\,(U-V)\sin(kx - \varepsilon t)\Big]$$

$$U+V = \left(\frac{\frac{\hbar k^2}{2m}}{\varepsilon_k}\right)^{1/2} = \begin{cases} \to 0 & \text{for } k\xi \to 0 \\ \to 1 & \text{for } k\xi \to \infty \end{cases}$$

$$U-V = \left(\frac{\varepsilon_k}{\frac{\hbar k^2}{2m}}\right)^{1/2} = \begin{cases} \to \infty & \text{for } k\xi \to 0 \\ \to 1 & \text{for } k\xi \to \infty \end{cases}$$

$\Big\{$ For $k\xi \to 0$ mode purely <u>phase-like</u> $\to$ Goldstone mode

For $k\xi \to \infty$ mode has single particle character

Landau criterion for superfluidity.

$$i\partial_t \phi = -\frac{\hbar\nabla^2}{2m}\phi + g|\phi|^2\phi + \underbrace{\delta(r-vt)\,V_0\,\phi}_{\text{moving defect at speed } v}$$

small $V_0 \longrightarrow$ linear expansion.

$$V_0\phi = V_0(\phi_0 + \delta\phi) \simeq V_0\phi_0$$

$$i\partial_t \begin{pmatrix} \delta\phi \\ \delta\phi^* \end{pmatrix} = \mathcal{L}\begin{pmatrix} \delta\phi \\ \delta\phi^* \end{pmatrix} + \underbrace{\delta(r-vt)\,V_0 \begin{pmatrix} \phi_0 \\ -\phi_0^* \end{pmatrix}}_{\text{source term in Bogoliubov eqs.}}$$

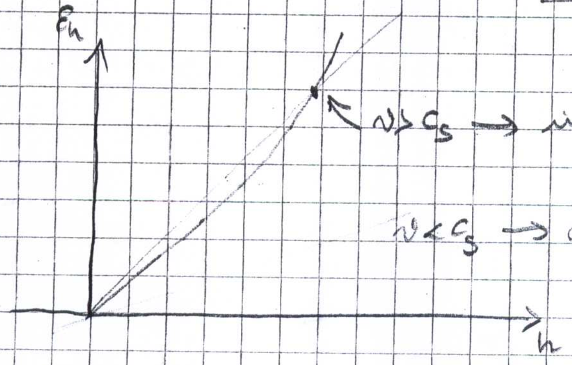
in Fourier space $\delta(r-vt) \longrightarrow \delta(\omega - k\cdot v)$

$$(\mathcal{L} - \omega - i0^+)\begin{pmatrix} \delta\phi(k,\omega) \\ \delta\phi^*(k,\omega) \end{pmatrix} = -V_0\,\delta(\omega - k\cdot v)\begin{pmatrix} \phi_0 \\ -\phi_0^* \end{pmatrix}$$

$$\begin{pmatrix} \delta\phi(k,\omega) \\ \delta\phi^*(k,\omega) \end{pmatrix} = -\frac{V_0\,\delta(\omega - k\cdot v)}{\mathcal{L}(k) - \omega - i0^+}\begin{pmatrix} \phi_0 \\ -\phi_0^* \end{pmatrix}$$

$\longrightarrow$ resonant denominator when $\varepsilon_k = k\cdot v$

Čerenkov condition for emission by uniformly moving source



$v > c_s \longrightarrow$ intersection exists

$v < c_s \longrightarrow$ no intersection

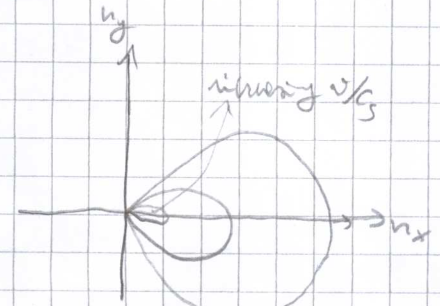
For $\underset{\sim}{v < c_s}$ $\longrightarrow$ no emission of Bogoliubov waves by moving defect. $\Rightarrow$ no friction force

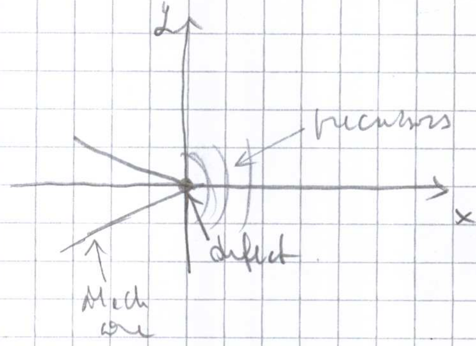
<u>SUPERFLUID</u> behavior

For $\underset{\sim}{v > c_s}$ $\longrightarrow$ emission of Bogoliubov waves:

Mach cone + precursors in real space

analogous to wave behind swimming duck creates friction force on object.





<u>Question</u>: what difference from swimming duck?

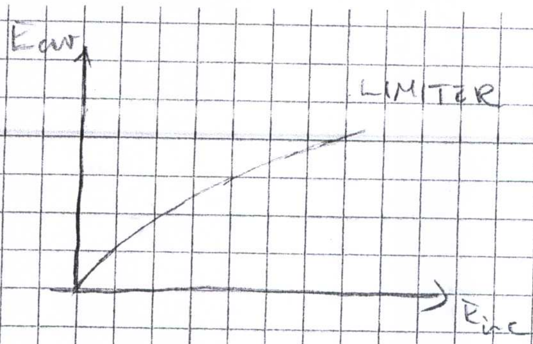
<u>Polariton case</u>: $E(r,t) = E_0 e^{-i\omega_{inc}t}$, $E_0 \in \mathbb{R}^+$

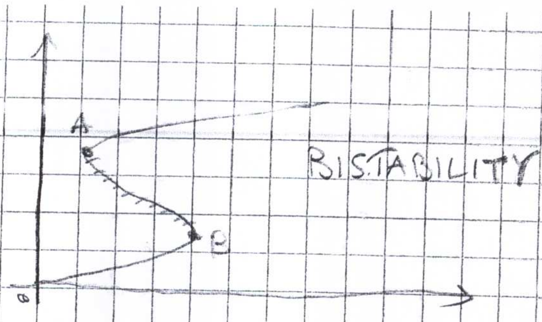
not determined by $\mu = g|E_0|^2$

$\Rightarrow$ more freedom of possible dispersions

$$\mathcal{L} = \begin{pmatrix} \frac{\hbar k^2}{2m} + 2g|E|^2 + \omega_0 - \omega_{inc} - \frac{i\gamma}{2}, & gE^2 \\ -gE^{*2} & -\frac{\hbar k^2}{2m} - 2g|E|^2 - \omega_0 + \omega_{inc} - \frac{i\gamma}{2} \end{pmatrix}$$

standard Bogoliubov $\longrightarrow$ special case $\omega_{inc} = g|E|^2$.





$\omega_0 + g|E|^2 > \omega_{inc}$

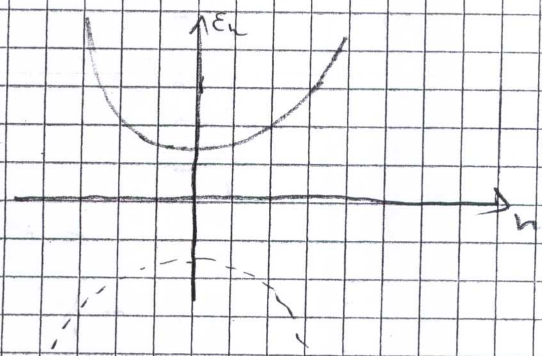
OB $\longrightarrow$ $\omega_0 + g|E|^2 < \omega_{inc}$

AB $\longrightarrow$ dynamically unstable

A∞ $\longrightarrow$ $\omega_0 + g|E|^2 > \omega_{inc}$

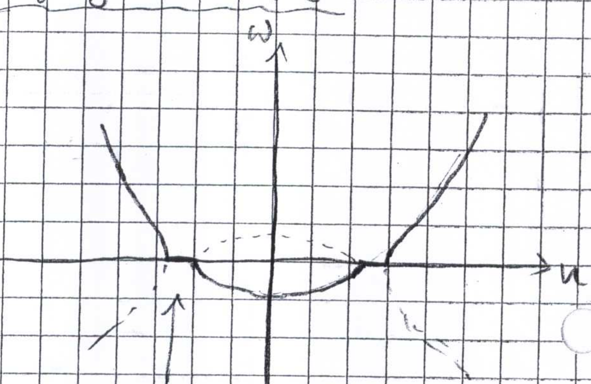
A $\longrightarrow$ $\omega_0 + g|E|^2 = \omega_{inc}$
standard Bogol.

$\omega_0 + g|E|^2 > \omega_{inc}$



gapped Bogoliubov dispersion

$\omega_0 + g|E|^2 < \omega_{inc}$



branch sticking region
↓
precursor of modulational instability

in particular, Goldstone mode no longer exists ($\omega \to 0$ for $k \to 0$)

Phase of "condensate" locked to the one of pump beam

↳ larger variety of superfluid/non superfluid behaviours

Saturable amplifier model

$$E(r,t) = \left[E_0 + \delta E(r,t)\right] e^{-i\omega t} \qquad \text{with} \quad \omega = g|E_0|^2 + \omega_0$$

$$|E_0|^2 = \frac{1}{\eta}\left[\frac{\Gamma_A}{\Gamma} - 1\right] \in \mathbb{R}^+$$

$$i\partial_t\left\{\left[E_0 + \delta E\right] e^{-i\omega t}\right\} = \left[\omega_0 - i\frac{\Gamma}{2} - \frac{\hbar}{2m^*}\nabla^2 + \frac{i}{2}\frac{\Gamma_A}{1+\eta|E_0+\delta E|^2} + g_{NL}|E_0+\delta E|^2\right] \cdot$$

$$\cdot \left(E_0 + \delta E\right) e^{-i\omega t}$$

$$i\partial_t\left(E_0 + \delta E\right) = \left[\omega_0 - \omega - i\frac{\Gamma}{2} - \frac{\hbar}{2m^*}\nabla^2\right]\left(E_0 + \delta E\right) +$$

$$+ \frac{i}{2}\frac{\Gamma_A}{1+\eta|E_0|^2}\left(E_0 + \delta E\right) + g_{NL}|E_0|^2\left(E_0 + \delta E\right)$$

$$- \frac{i}{2}\frac{\Gamma_A}{\left(1+\eta|E_0|^2\right)^2} \cdot \eta|E_0|^2\left(\delta E + \delta E^*\right) +$$

$$+ g_{NL}|E_0|^2\left(\delta E + \delta E^*\right)$$

$$i\partial_t\,\delta E = \left(\omega_0 - \omega - \frac{\hbar\nabla^2}{2m^*} + \cancel{2}g_{NL}|E_0|^2\right)\delta E + g_{NL}|E_0|^2\,\delta E^*$$

$$+ \left(-i\frac{\Gamma}{2} + \frac{i}{2}\frac{\Gamma_A}{1+\eta|E_0|^2}\right)\delta E - \frac{i}{2}\frac{\Gamma_A}{\left(1+\eta|E_0|^2\right)^2}\eta|E_0|^2\left(\delta E + \delta E^*\right)$$

$$\mathcal{L} = \begin{pmatrix} \frac{\hbar k^2}{2m} + \tilde{g}|E_0|^2 & , & \tilde{g}|E_0|^2 \\ -\tilde{g}^*|E_0|^2 & , & -\frac{\hbar k^2}{2m} - \tilde{g}^*|E_0|^2 \end{pmatrix} \qquad \text{with} \quad \tilde{g} = g_{NL} - \frac{i\Gamma_A}{2}\frac{\eta}{\left(1+\eta|E_0|^2\right)^2}$$

eigenfrequencies:

$$\left(\frac{\hbar k^2}{2m} + \tilde g |E_0|^2 - \varepsilon\right)\left(-\frac{\hbar k^2}{2m} - \tilde g^* |E_0|^2 - \varepsilon\right) + |\tilde g|^2 |E_0|^4 = 0$$

$$\left[\varepsilon - \left(\frac{\hbar k^2}{2m}+\tilde g|E_0|^2\right)\right]\left[\varepsilon + \left(\frac{\hbar k^2}{2m}+\tilde g^*|E_0|^2\right)\right] + |\tilde g|^2|E_0|^4 = 0$$

$$\varepsilon^2 + \varepsilon\left[\cancel{\frac{\hbar k^2}{2m}} + \tilde g^*|E_0|^2 - \cancel{\frac{\hbar k^2}{2m}} - \tilde g|E_0|^2\right] +$$

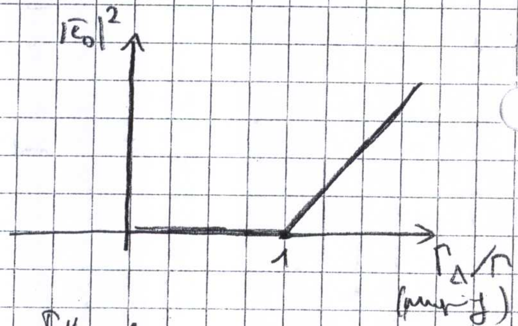
$$- \left(\frac{\hbar k^2}{2m} + g_{NL}|E_0|^2\right)^2 - \left(\frac{\Gamma_A \eta}{2}\cancel{\frac{|E_0|^2}{(1+\eta|E_0|^2)^2}}\right)^2 +$$

$$+\left(g_{NL}|E_0|^2\right)^2 + \cancel{\left(\frac{\Gamma_A\eta}{2}\frac{|E_0|^2}{(1+\eta|E_0|^2)^2}\right)^2} =$$

$$= \varepsilon^2 + \varepsilon\, i\Gamma_A \frac{\eta|E_0|^2}{(1+\eta|E_0|^2)^2} - \frac{\hbar k^2}{2m}\left(\frac{\hbar k^2}{2m} + 2g_{NL}|E_0|^2\right) = 0$$

$$\varepsilon = -i\frac{\Gamma_A}{2}\frac{\eta|E_0|^2}{(1+\eta|E_0|^2)^2} \pm \sqrt{-\left(\frac{\Gamma_A}{2}\frac{\eta|E_0|^2}{(1+\eta|E_0|^2)^2}\right)^2 + E^0_{Bog}(k)^2}$$

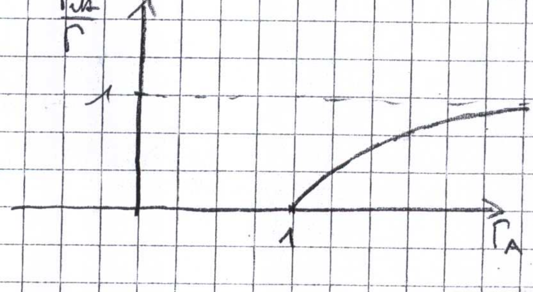
$$= -i\frac{\Gamma_{eff}}{2} \pm\sqrt{E^0_{Bog}(k)^2 - \frac{\Gamma_{eff}^2}{4}}$$



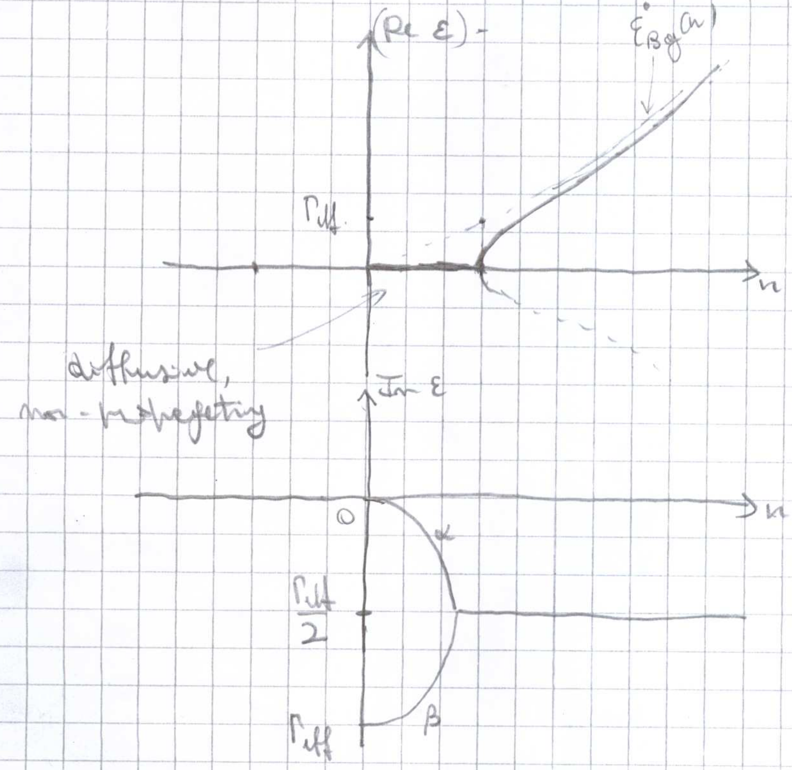
with $\Gamma_{eff} = \Gamma_A \dfrac{\eta|E_0|^2}{(1+\eta|E_0|^2)^2} =$

$$= \Gamma_A\frac{\Gamma_A/\Gamma - 1}{(\Gamma_A/\Gamma)^2}$$

$$= \Gamma\frac{\Gamma_A/\Gamma-1}{\Gamma_A/\Gamma}$$



Dispersion law



Goldstone mode is indeed present ($\varepsilon \to 0$ for $k \to 0$)

Characteristic damping rate Γ_{eff}

* $\Gamma_{eff} \to \Gamma$ for above critical point $\frac{\Gamma_A}{\Gamma} \gg 1$

* $\Gamma_{eff} \to 0$ at critical point $\Gamma_A/\Gamma = 1$

CRITICAL SLOWING DOWN for $\Gamma_A \to \Gamma^+$

β mode $\to$ intensity fluctuations decay fast

α mode $\to$ Goldstone mode: phase oscillations.

Do not propagate as sound waves but slowly decay to zero

Below critical point:

$$\varepsilon(k) = \frac{\hbar k^2}{2m^*} - \frac{\lambda}{2}(\Gamma - \Gamma_A)$$ again $\mathrm{Im}\,\varepsilon \to 0$ for $\Gamma_A \to \Gamma^-$

Can be measured as slow decay of system to pulsed perturbation or as slow dynamics of fluctuations

Consequences on superfluidity: Landau's $v_c = \min_k \frac{\mathrm{Re}[\omega(k)]}{k} = 0$

but numerical solution of GPE shows traces of a $\tilde{v}_c \lesssim c_s = \sqrt{\frac{g|\psi_0|^2}{m}}$

Why? new definition of $\tilde{v}_c$ required ... see PRL 105, 020602 (10)

<u>So far</u>: weak defect, described within perturbation theory

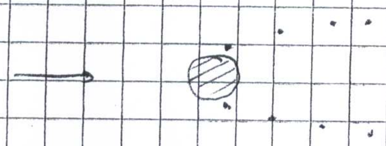
(GPE + linearized around stationary state $\Leftrightarrow$ Bogoliubov)

What about $\left\{\begin{array}{l}\text{large}\\ \text{strong}\end{array}\right\}$ defect? E.g. <u>impenetrable cylinder</u>

<u>Full solution of GPE needed</u> (for simplicity axisymmetric case)

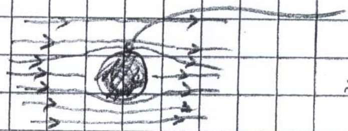
* $0 < v < 0{,}44\, c_s$ <u>superfluid flow</u>

* $0{,}44\, c_s < v < c_s$ periodic nucleation of <u>vortices</u> past the defect



<u>Physical explanation</u>:

v_∞



local velocity increased from v_∞

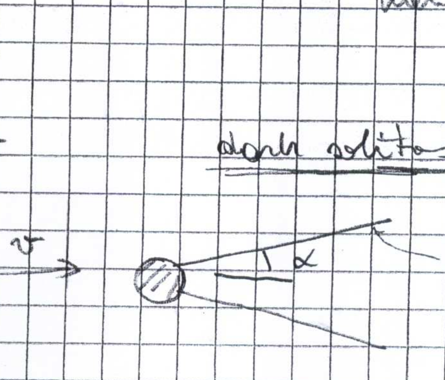
if $v_{\max} > c_s$ Landau "instability" of flow $\longrightarrow$ can nucleate vortices from surface of defect

Frisch - Pomeau - Rice PRL <u>69</u>, 1644 ('92)

<u>Alternative explanation</u> $\longrightarrow$ snake instability of dark soliton (see below)

Kamchatnov - Pitaevskii PRL <u>100</u>, 160402 ('08)

* $c_s < v$ <u>dark solitons</u> in the wake of defect



density minimum with <u>phase jump θ</u>

opening $\sin\alpha = \frac{c_s}{v}\cos\frac{\theta}{2} < \frac{c_s}{v}$ Mach cone

Polariton experiments under <u>coherent drive</u>:

- <u>phase</u> can not be locked to pump, otherwise no vortices/solitons
- coherent pump localized in half-space, then ballistic flow

$$i\frac{\partial}{\partial t}\phi = -\frac{\hbar\nabla^2}{2m}\phi + \omega_0\phi + g_{LL}|\phi|^2\phi - i\frac{\gamma}{2}\phi + i\,F_{ext}(r)$$

$\hookrightarrow F_{ext}(r) = f_0\, e^{ikr} e^{-i\omega t}\,\Theta(z)$

pump in region $z<0$ provides "boundary condition" to free dynamics in region $z>0$ that obeys standard GPE (+ loss term)

pump frequency ω_m fix "chemical potential" μ in $z>0$ region

$$\mu = \frac{\hbar k^2}{2m} + g_{LL}|\phi|^2$$

because of γ: $|\phi|^2$ decreases and k has to increase

$\hookrightarrow$ accelerated flow

<u>Experiments</u> with photon BEC's:

- Nat. Phys. 5, 805 ('09) Superfluid flow (theory in PRL 93, 166401 ('04))
- Science 332, 1167 ('11) Solitons (theory in PRB 83, 144513 ('11))
- Nat. Phys. 7, 635 ('11) Vortices
- Nature 443, 409 ('06) + PRB 77, 113303 ('08) BEC transition