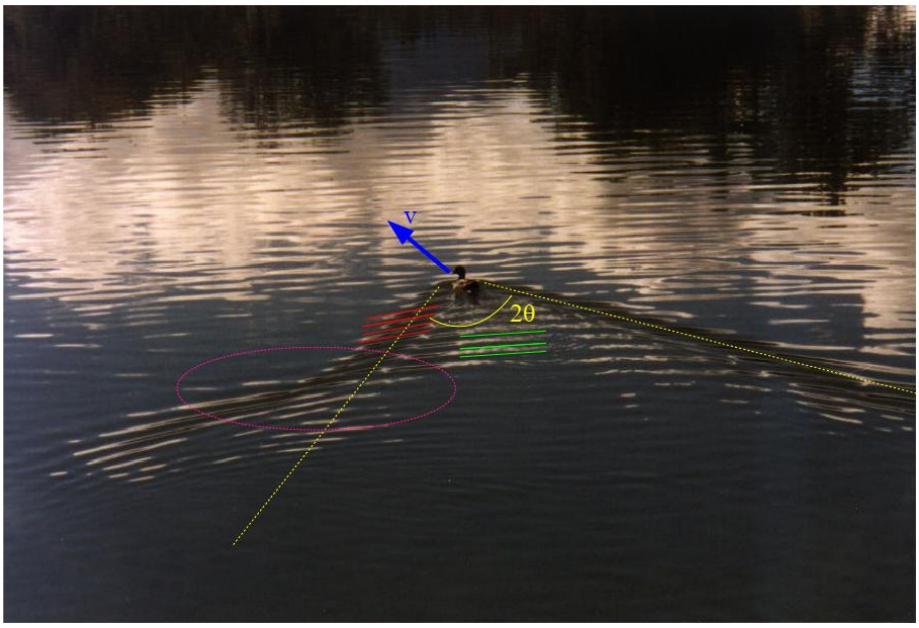
(Not so) quantum ducks

Consider a duck which is smoothly swimming on a quiet lake as shown in the picture. Using the concepts of the Čerenkov theory of the radiation emitted by an charge that is uniformly moving through a homogeneous medium, study the main qualitative properties of the wake that is observed on the surface of the lake past the duck.



1. We can model the perturbation of the water surface as the real part of a two-dimensional field $h(\mathbf{r},t)$ of dispersion relation $\omega(k)$ sourced by a force field that follows the shape of the duck. Justify the Fourier space evolution equation

$$\frac{d\tilde{h}(\mathbf{k})}{dt} = -i\,\omega(k)\,\tilde{h}(\mathbf{k}) + \tilde{f}(\mathbf{k},t) \tag{1}$$

   where $\tilde{f}(\mathbf{k},t)$ is the spatial Fourier transform of the force field $f(\mathbf{r},t)$ exerted by the duck on the water surface.

2. Determine the Čerenkov constraint imposed to the spatial and temporal Fourier transform $\tilde{f}(\mathbf{k},\omega)$ by the condition of uniform motion at speed $\mathbf{v}$. Interpret it in terms of energy and momentum conservation.

3. Using the specific form of the dispersion relation $\omega(k) = \sqrt{g\,k}$ of surface gravity waves (where g is the gravitational acceleration), determine the equation for the field modes that are resonantly excited under the point-like duck approximation.

4. Study the **k**-space geometrical shape of the locus of excited modes. Working in the duck reference frame, discuss the role of the phase and group velocities in the emission process and identify the features of the locus that correspond to the two velocities.

5. Based on the above concepts, try to explain the main features that one can observe in the picture. In particular:

   (a) calculate the aperture of the main conical wake (indicated as θ in the figure). How does this angle depend on the speed of the duck?

   (b) give an explanation for the modulation that is visible on the main conical wake (red lines in the figure).

   (c) what is the physical origin of the oscillations that are visible inside the conical wake (green lines in the figure)?

   (d) estimate from the figure the speed at which the duck is moving.

   (e) how can you interpret the sudden change in the wake that is visible on the bottom-left side of the picture (in the purple circle) ?

6. (Optional) Discuss the analogies and the differences compared to the Čerenkov cone of light emitted by a relativistically moving charge in a non-dispersive dielectrics of frequency-independent refractive index n.