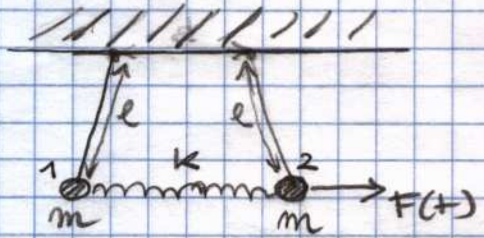
Exercise 3 : driven coupled harmonic oscillators



* pair of pendula
* linked by spring.
* spring rest length = distance betw. pendula

$$m\ddot{x}_1 = -\frac{gm}{l}x_1 + k(x_2 - x_1) - \gamma_1 \dot{x}_1$$

$$m\ddot{x}_2 = F_{ext} - \frac{mg}{l}x_2 - k(x_2 - x_1) - \gamma_2 \dot{x}_2$$

$\omega_0 = \sqrt{g/l}$ = pendulum "bare frequency"

all other terms small w/r ω_0

Switch to $\alpha_{1,2}$, $\alpha_{1,2}^*$ variables

$$\alpha_{1,2} = \sqrt{\frac{m\omega_0}{2\hbar}}\, x_{1,2} + \frac{i}{\sqrt{2m\hbar\omega_0}}\, p_{1,2}$$

$$\begin{cases} \dot{p}_1 = -\frac{gm}{l}x_1 + k(x_2 - x_1) - \gamma_1 \dot{x}_1 \\ \dot{p}_2 = F_{ext} - \frac{gm}{l}x_2 - k(x_2 - x_1) - \gamma_2 \dot{x}_2 \\ \dot{x}_1 = p_1/m \\ \dot{x}_2 = p_2/m \end{cases}$$

gives :

$$\begin{cases} \dot{\alpha}_1 = -i\omega_0 \alpha_1 + \dfrac{i k}{2m\omega_0}\left(\alpha_2 + \alpha_2^* - \alpha_1 - \alpha_1^*\right) + \\ \qquad + \dfrac{\gamma_1}{2}\left(\alpha_1^* - \alpha_1\right) \\ \dot{\alpha}_2 = -i\omega_0 \alpha_2 + \dfrac{i k}{2m\omega_0}\left(\alpha_1 + \alpha_1^* - \alpha_2 - \alpha_2^*\right) + \\ \qquad + \dfrac{\gamma_2}{2}\left(\alpha_2^* - \alpha_2\right) + \dfrac{i}{\sqrt{2m\hbar\omega_0}} F_{ext}(t) \end{cases}$$

+ equations for α_1^*, α_2^*.

Rotating wave approximation :

- neglect coupling of α's with α^*'s

- isolate component of F_{ext} around ω_0

$$F_{ext}(t) \simeq \bar{F}_{ext}(t)\, e^{-i\omega_0 t}$$

with $\bar{F}_{ext}(t)$ slowly varying w/r ω_0.

- define $\bar{\alpha}_{1,2}(t) = \alpha_{1,2}(t) \exp(i\omega_0 t)$

$$\begin{cases} \dot{\bar{\alpha}}_1 = \underbrace{\dfrac{i k}{2m\omega_0}}_{\Omega}\left(\bar{\alpha}_2 - \bar{\alpha}_1\right) - \dfrac{\gamma_1}{2}\bar{\alpha}_1 \\ \dot{\bar{\alpha}}_2 = \dfrac{i k}{2m\omega_0}\left(\bar{\alpha}_1 - \bar{\alpha}_2\right) - \dfrac{\gamma_2}{2}\bar{\alpha}_2 + \underbrace{\dfrac{i}{\sqrt{2m\hbar\omega_0}}\bar{F}_{ext}}_{f_{ext}} \end{cases}$$

Assume $\bar{F}_{ext}(t) = F^{0}_{ext} \exp(-i\underbrace{(\omega_L - \omega_0)}_{+\delta_L}t)$:

$$\begin{cases} -i\delta_L \bar{\alpha}_1 = +i\Omega(\bar{\alpha}_2 - \bar{\alpha}_1) - \frac{\gamma_1}{2}\bar{\alpha}_1 \\ -i\delta_L \bar{\alpha}_2 = +i\Omega(\bar{\alpha}_1 - \bar{\alpha}_2) - \frac{\gamma_2}{2}\bar{\alpha}_2 + i f^{0}_{ext} \end{cases}$$

$$i\left(-\delta_L + \Omega - i\frac{\gamma_1}{2}\right)\bar{\alpha}_1 = i\Omega\bar{\alpha}_2$$

$$i\left(-\delta_L + \Omega - i\frac{\gamma_2}{2}\right)\bar{\alpha}_2 = i\Omega\bar{\alpha}_1 + i f^{0}_{ext}$$

$$i\left[\underbrace{-(\delta_L - \Omega)}_{\bar{\delta}_L} - i\frac{\gamma_2}{2} - \Omega\frac{i\Omega}{i\left[\underbrace{-(\delta_L - \Omega)}_{\bar{\delta}_L} - i\gamma_1/2\right]}\right]\bar{\alpha}_2 = i f^{0}_{ext}$$

$$\left(\bar{\alpha}_2 = \frac{f^{0}_{ext}}{-\bar{\delta}_L - i\gamma_2/2 - \frac{\Omega^2}{-\bar{\delta}_L - i\gamma_1/2}}\right)$$

Poles of response at:

$$-\bar{\delta}_L - i\gamma_2/2 - \frac{\Omega^2}{-\bar{\delta}_L - i\gamma_1/2} = 0$$

i.e. $\left(-\bar{\delta}_L - i\gamma_2/2\right)\left(-\bar{\delta}_L - i\gamma_1/2\right) = \Omega^2$.

Correspond to eigenvalues of matrix:

$$\begin{pmatrix} i\gamma_1/2 & \Omega \\ \Omega & i\gamma_2/2 \end{pmatrix}$$

* hermitian part → coupling Ω.
  eigenvectors $(1, \pm 1)$

* anti hermitian part → losses $\gamma_{1,2}$
  eigenvectors $(1,0)$, $(0,1)$

which dominates depends on $\Omega / |\gamma_1 - \gamma_2|$

Characteristic equation:

$$\bar{\delta}_L^2 + i\frac{\gamma_1+\gamma_2}{2}\bar{\delta}_L - \left(\frac{\gamma_1\gamma_2}{4} + \Omega^2\right) = 0$$

$$\bar{\delta}_L^{1,2} = -\frac{i}{4}(\gamma_1+\gamma_2) \pm \frac{1}{2}\sqrt{-\left(\frac{\gamma_1+\gamma_2}{2}\right)^2 + 4\left(\frac{\gamma_1\gamma_2}{4} + \Omega^2\right)}$$

$$= -\frac{i}{4}(\gamma_1+\gamma_2) \pm \frac{1}{2}\sqrt{4\Omega^2 - \left(\frac{\gamma_1-\gamma_2}{2}\right)^2}$$

Two limits :

(i) $\Omega^2 >> (\gamma_1 - \gamma_2)^2$

$$\bar{\delta}_L^{1,2} = -\frac{i}{4}(\gamma_1+\gamma_2) \pm \Omega \left(1 - \frac{(\gamma_1-\gamma_2)^2}{16\Omega^2}\right)^{1/2} \simeq$$

$$\simeq \pm \Omega - \frac{i}{2}\left(\frac{\gamma_1+\gamma_2}{2}\right)$$

$\pm\Omega$ → Rabi splitting of two coupled modes

$\frac{\gamma_1+\gamma_2}{2}$ → effective linewidth = average of $\gamma_{1,2}$.

(ii) $\Omega^2 << (\gamma_1 - \gamma_2)^2$

$$\bar{\delta}_L^{1,2} = -\frac{i}{4}(\gamma_1+\gamma_2) \pm i\left(\frac{\gamma_1-\gamma_2}{4}\right)\left[1 - \frac{16\Omega^2}{(\gamma_1-\gamma_2)^2}\right]^{1/2}$$

$$\simeq -\frac{i}{2}\gamma_{1,2} \qquad \text{in general}$$

* most remarkable limit $\gamma_1 << \Omega << \gamma_2$:

$$\bar{\delta}_L^{1,2} \simeq -\frac{i}{2}\gamma_2 \,,\, -\frac{i}{2}\left(\gamma_1 + 4\frac{\Omega^2}{\gamma_2}\right)$$

→ both poles have vanishing frequency

→ very different damping rates

Residues

if $f(z)$ has a pole at z_0, then

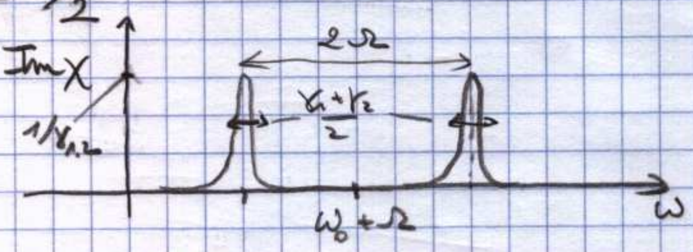
$$\frac{d}{dz}\frac{1}{f(z)}\bigg|_{z_0} = \frac{1}{w}, \quad w \text{ being its residue.}$$

In our case:

$$-\frac{d}{d\delta}\left(-\delta - i\frac{\gamma_2}{2} - \frac{\Omega^2}{-\delta - i\gamma_1/2}\right) =$$

$$= +1 + \frac{\Omega^2}{(\delta + i\gamma_1/2)^2} \implies w = \frac{+1}{1 + \left(\frac{\Omega}{\delta + i\gamma_1/2}\right)^2}$$

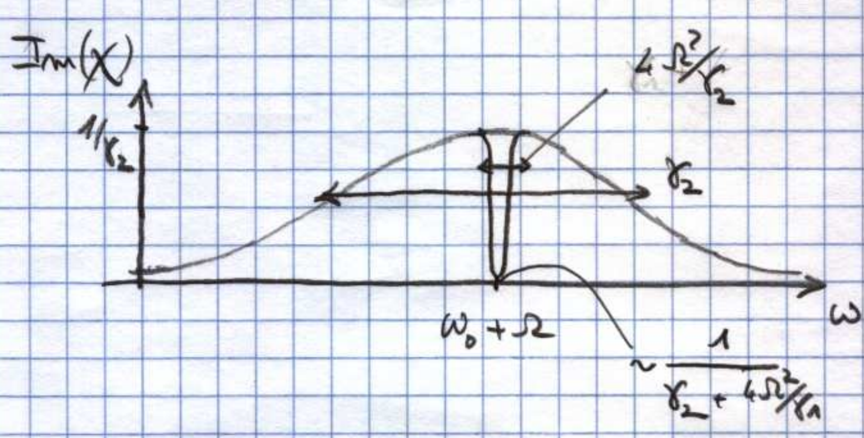
(i) $\bar{\delta}_L^{1,2} = \pm\Omega \implies w_{1,2} \simeq 1/2$



(ii) $$\bar{\delta}_L^{1,2} = \begin{cases} -i\,\gamma_2/2 & (1) \\ -\frac{i}{2}\left(\gamma_1 + 4\frac{\Omega^2}{\gamma_2}\right) & (2) \end{cases}$$

$$w_1 = \frac{1}{1 + \left(\frac{\Omega}{i(\gamma_1 - \gamma_2)/2}\right)^2} \simeq \frac{1}{1 - 4\Omega^2/\gamma_2^2} \simeq 1 + 4\frac{\Omega^2}{\gamma_2^2}$$

$$w_2 = \frac{1}{1 + \left(\frac{\Omega}{-i\gamma_1/2 - i\,4\Omega^2/2\gamma_2 + i\gamma_1/2}\right)^2} = \frac{1}{1 - \gamma_2^2/4\Omega^2} \simeq -4\Omega^2/\gamma_2^2 < 0$$



* negative narrow peak digged within broader positive line

* Im χ always positive

NOTE:

$$\bar{\alpha}_2 = \frac{f^0_{ext}}{-\bar{\delta}_L - i\gamma_2/2 - \frac{\Omega^2}{-\bar{\delta}_L - i\gamma_1/2}} = \chi(\bar{\delta}_L)\, f^0_{ext}$$

as usual: Im χ = absorbed energy from drive.

In summary

- case (i) : Autler-Townes doublet

- case (ii) : Alzetta-Gozzini Electromagnetically Induced Transparency effect.