The phase operator in quantum mechanics

1. **The phase operator.** One of the most common definitions of the phase operator for a single-mode optical field in quantum mechanics reads

$$e^{i\phi} = (a^\dagger a + 1)^{-1/2} a. \tag{1}$$

   (a) Write the matrix elements of $e^{i\phi}$ in the basis of Fock states $|n\rangle$.

   (b) Evaluate the commutator of $e^{i\phi}$ with its Hermitian conjugate $(e^{i\phi})^\dagger$ and with the number operator $n = a^\dagger a$. Is the phase operator $e^{i\phi}$ unitary?

   (c) Define the operators $\sin\phi$ and $\cos\phi$. Are they Hermitian? Evaluate their commutator with the number operator n and give a physical interpretation of the result.

   (d) Determine the eigenvectors $|e^{i\phi}\rangle$ of the operator $e^{i\phi}$. Choosing the standard normalization $\langle 0|e^{i\phi}\rangle = 1$, prove the completeness relation:

$$\int d\phi \, |e^{i\phi}\rangle\langle e^{i\phi}| = 2\pi. \tag{2}$$

   Use this result to prove that the state of the system is univocally determined by the "wave function" $\psi(\phi) = \langle e^{i\phi}|\psi\rangle$. Characterize the Hilbert space of the physically legitimate wavefunctions $\psi(\phi)$ and find a basis of it in terms of plane waves.

   (e) Determine an analytical expression for $\psi(\phi)$ for a large amplitude coherent state $|\alpha\rangle$ with $|\alpha| \gg 1$ (coherent states are defined as usual as $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{1}{\sqrt{n!}} \alpha^n |n\rangle$) and determine its main qualitative properties (center of mass, width, etc.).

2. **Collapse and revival of the phase**. Consider the Hamiltonian:

$$\mathcal{H} = \frac{\hbar\chi}{2} n^2. \tag{3}$$

   (a) Write the Schrödinger for the temporal evolution of $\psi(\phi)$ under this Hamiltonian. Determine its eigenvalues and eigenvectors. Provide a physical justification to the result.

   (b) Determine qualitatively the temporal evolution of $\psi(\phi, t)$ starting from a coherent state $|\alpha\rangle$ with $|\alpha| \gg 1$. Determine the characteristic time t_{br} for the spreading of the initial wave packet (phase *collapse*). Determine the time t_{res} at which the wavepacket recovers its initial shape modulo a global translation (phase *revival*) ? What is the state of the system at the time $t = t_{\text{res}}/2$?

This exercise is inspired to a Travail Dirigé of *Advanced Quantum Mechanics* at the DEA de Physique Quantique, year 2003/4.