Phase transitions in optics

1- optical bistability

cavity with intensity-dependent refractive index
coherent monochromatic pump.

$$i\dot{\alpha} = \omega_0 \alpha + \omega_{nl}|\alpha|^2\alpha - i\frac{\gamma}{2}\alpha + F e^{-i\omega t}$$

$$\omega_0 = \frac{\pi c}{n_0 L} \quad ; \quad \omega_{nl}|\alpha|^2 = -\frac{\pi c}{n_0^2 L}\cdot \delta n = -\frac{\pi c}{n_0^2 L}\, n_2 \cdot I .$$

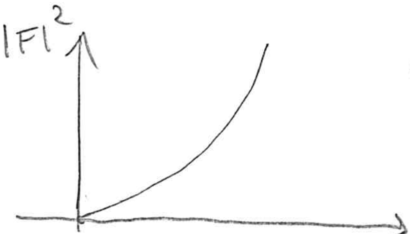
$$\longrightarrow \omega_{nl} = \dots$$

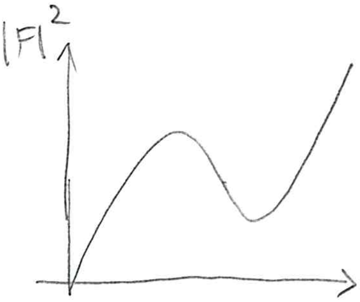
scales as $1/V$ mode photon confinement

Steady-state: $\left(\omega - \omega_0 - \omega_{nl}|\alpha|^2 + i\,\gamma/2\right)\alpha = F$

$$|\alpha|^2\left[(\omega-\omega_0-\omega_{nl}|\alpha|^2)^2 + \gamma^2/4\right] = |F|^2$$

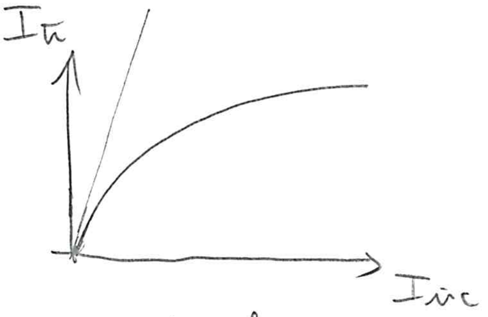
for $\omega_{nl} > 0$ $\qquad \omega < \omega_0$





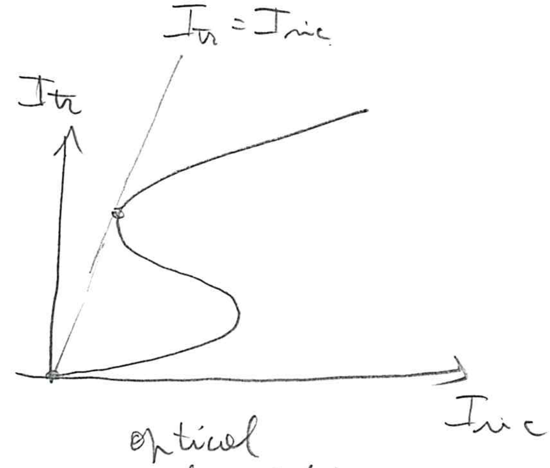
incident intensity $\propto |F|^2$

transmitted intensity $\propto |\alpha|^2$



optical limiter

$\omega < \omega_0$



optical bistability

$\omega > \omega_0$

Dynamical stability : $\alpha = (\underbrace{\bar{\alpha}}_{\text{steady state}} + \overbrace{\delta\alpha}^{\text{linearized fluctuation}}) e^{-i\omega t}$

$$i \frac{d}{dt} \delta\alpha = (\omega_0 - \omega)(\bar{\alpha} + \delta\alpha) + \omega_{nl} |\bar{\alpha} + \delta\alpha|^2 (\bar{\alpha} + \delta\alpha) + - i \frac{\gamma}{2} (\bar{\alpha} + \delta\alpha) + F$$

$$= \underbrace{\cancel{(\omega_0 - \omega)\bar{\alpha} + \omega_{nl} |\bar{\alpha}|^2 \bar{\alpha} - i \frac{\gamma}{2} \bar{\alpha} + F}}_{\to\ 0 \text{ by definition of } \bar{\alpha}} +$$

$$+ (\omega_0 - \omega)\delta\alpha + 2\omega_{nl} |\bar{\alpha}|^2 \delta\alpha + \omega_{nl} \bar{\alpha}^2 \delta\alpha^* - i \frac{\gamma}{2} \delta\alpha$$

$$+ O(\delta\alpha^2)$$

$$i \frac{d}{dt} \begin{pmatrix} \delta\alpha \\ \delta\alpha^* \end{pmatrix} = \begin{pmatrix} \omega_0 - \omega + 2\omega_{nl} |\alpha|^2 - i \frac{\gamma}{2} & \omega_{nl} \bar{\alpha}^2 \\ -\omega_{nl} \bar{\alpha}^{*2} & -(\omega_0 - \omega + 2\omega_{nl} |\alpha|^2) - i\gamma/2 \end{pmatrix} \cdot \begin{pmatrix} \delta\alpha \\ \delta\alpha^* \end{pmatrix}$$

stability $\rightarrow$ $\mathrm{Im}\,\lambda < 0$ for all λ's.

$$\left(\omega_0 - \omega + 2\omega_{nl}|\alpha|^2 - (\lambda + i\gamma/2)\right)\left[-(\omega_0 - \omega + 2\omega_{nl}|\alpha|^2) + - (\lambda + i\gamma/2)\right] + \omega_{nl}^2 |\alpha|^4 = 0 .$$

$$(\lambda + i\gamma/2)^2 - \left[(\omega_0 - \omega + 2\omega_{nl}|\alpha|^2)^2 - \omega_{nl}^2|\alpha|^4\right] = 0$$

$$\mathrm{Im}\,\lambda = -\gamma/2 \pm \mathrm{Im}\sqrt{(\omega_0 - \omega + 2\omega_{nl}|\alpha|^2)^2 - \omega_{nl}^2|\alpha|^4}$$

$$= -\frac{\gamma}{2} \pm \mathrm{Im}\sqrt{\underbrace{(\omega_0 + \omega_{nl}|\alpha|^2 - \omega)}_{\substack{\text{detuning from nonlinear mode} \\ @\ \omega_0 + \omega_{nl}|\alpha|^2}}(\omega_0 + \omega_{nl}|\alpha|^2 - \omega + 2\omega_{nl}|\alpha|^2)}$$

Instability requires that argument of $\sqrt{}$ be negative.

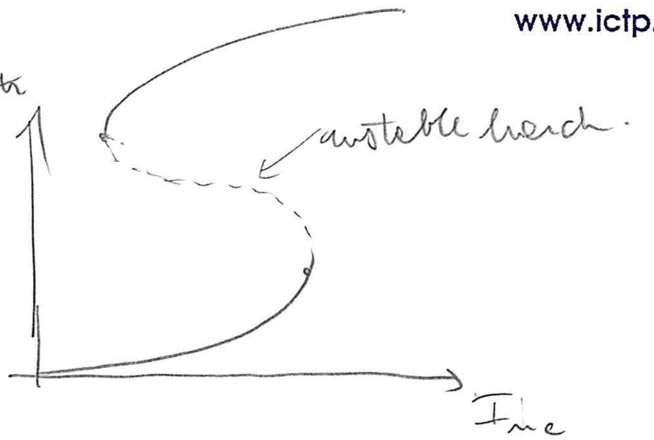
$$\Longleftrightarrow \quad -2\omega_{nl}|\alpha|^2 < (\omega_0 + \omega_{nl}|\alpha|^2 - \omega) < 0 .$$

$\rightarrow$ impossible in optical limiter $\omega_0 - \omega > 0$.

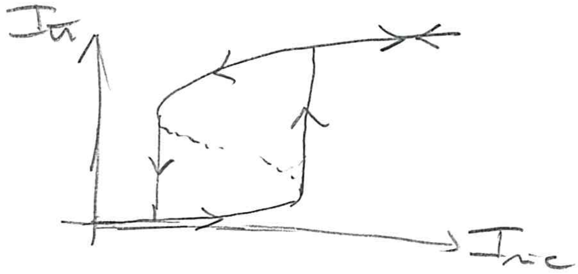
$\rightarrow$ in optical bistability:

$$\frac{d|F|^2}{d|\alpha|^2} = (\omega - \omega_0 - \omega_{nl}|\alpha|^2)^2 + \frac{\gamma^2}{4} + $$
$$+ |\alpha|^2 \cdot 2\,(\omega - \omega_0 - \omega_{nl}|\alpha|^2)\cdot(-\omega_{nl})$$
$$\text{roughly} \simeq (\omega_0 + \omega_{nl}|\alpha|^2 - \omega)(\omega_0 + \omega_{nl}|\alpha|^2 - \omega + 2\omega_{nl}|\alpha|^2)$$

dynamical instability $\Longleftrightarrow$ $\dfrac{d|F|^2}{d|\alpha|^2} < 0$.



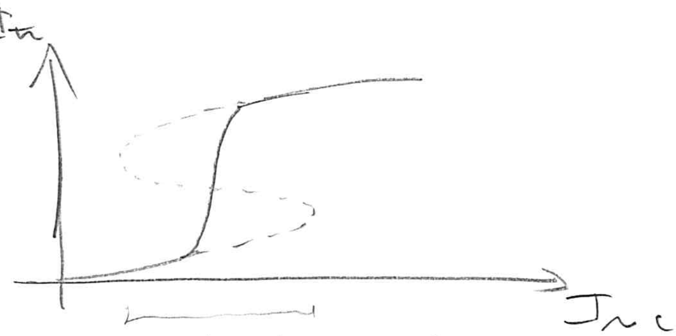
Hysteresis →



So far: mean-field theory with $\alpha \in \mathbb{C}$

what happens upon quantization?

quantum mechanics $\approx$ fluctuations.



quantum jumps between solutions
relative weight jumps @ 1st order phase trans.

see: DRUMMOND & WALLS, J. PHYS A 13, 725 ('80)

WALLS - MILBURN "QUANTUM OPTICS"

RISKEN et al. PRA 35, 1729 ('87); 38, 1349 ('88)

RODRIGUEZ et al PRL 118, 247402 ('17)

Physics most interesting in planar case $\alpha(r)$, $F(r) = F_0$

ROSCHER et al., ARXIV 1803.08514

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2- Parametric down conversion

$$i\dot{\alpha} = \omega_a \alpha - i\frac{\gamma_a}{2}\alpha + g\beta^2 + Fe^{-i\omega t}$$
$$i\dot{\beta} = \omega_b \beta - i\frac{\gamma_b}{2}\beta + 2g\alpha\beta^*$$

↳ can be derived from
$H_{int} = g(a^\dagger b^2 + a\, b^{\dagger 2})$
where $g \propto \chi^{(2)}$ of material.

simplest case $\omega = \omega_a = 2\omega_b$; $\gamma_a = \gamma_b = \gamma$

$$\begin{cases} i\dot{\tilde{\alpha}} = F + g\tilde{\beta}^2 - i\frac{\gamma}{2}\tilde{\alpha} \\ i\dot{\tilde{\beta}} = 2g\tilde{\alpha}\tilde{\beta}^* - i\frac{\gamma}{2}\tilde{\beta} \end{cases}$$

for slowly varying amplitudes
$\alpha(t) = \tilde{\alpha}\, e^{-i\omega t}$
$\beta(t) = \tilde{\beta}\, e^{-i\omega t/2}$

↳ from now on I forget ~'s.

Steady state:

$$2g\alpha\beta^* = i\frac{\gamma}{2}\beta \implies 2g\alpha = i\frac{\gamma}{2}\frac{\beta^2}{|\beta|^2} \quad \text{or } \beta = 0.$$

$$i\frac{\gamma}{2}\alpha = F + g\beta^2$$

a) $\beta = 0$; $\alpha = -\frac{2iF}{\gamma}$ "trivial solution"

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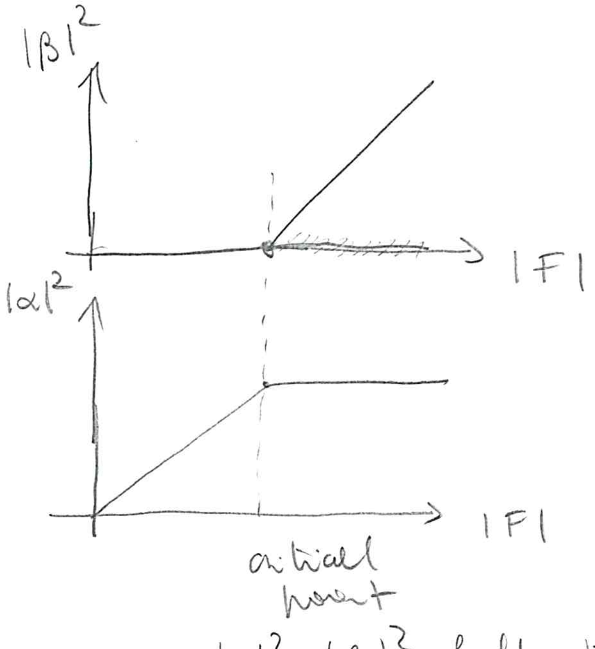
b) $i\frac{\gamma}{2}\beta^2 = 2g\alpha|\beta|^2 \quad \Rightarrow \quad \alpha = \frac{i\gamma}{4g}\frac{\beta^2}{|\beta|^2} = \frac{i\gamma}{4g}e^{2i\phi_\beta}$

$$i\frac{\gamma}{2}\alpha = F + g\,\frac{2g\alpha|\beta|^2}{i\gamma/2}$$

$$i\left(\frac{\gamma}{2} + \frac{4g^2|\beta|^2}{\gamma}\right)\alpha = F \quad \Rightarrow \quad \alpha = \frac{-iF}{\frac{\gamma}{2} + \frac{4g^2|\beta|^2}{\gamma}}$$

$$|\alpha|^2 = \frac{\gamma^2}{16g^2}$$

$$\left(\frac{\gamma}{2} + \frac{4g^2|\beta|^2}{\gamma}\right)\frac{\gamma}{4g} = |F| \quad \Rightarrow \quad |\beta|^2 = \underbrace{\left(\frac{4g}{\gamma}|F| - \frac{\gamma}{2}\right)}_{\text{only exists for } |F| > \frac{\gamma^2}{8g}}\cdot\frac{\gamma}{4g^2}$$



$|\alpha|^2, |\beta|^2$ fully determined.

Phase of α also. Phase of β up to ± 1.

↳ Ising SSB.

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Dynamical stability of trivial $\beta = 0$ solution.

$$i \frac{d}{dt} \begin{pmatrix} \delta\alpha \\ \delta\alpha^* \\ \delta\beta \\ \delta\beta^* \end{pmatrix} = \left(\begin{array}{cc|cc} -i\gamma/2 & 0 & 0 & 0 \\ 0 & -i\gamma/2 & 0 & 0 \\ \hline 0 & 0 & -i\gamma/2 & 2g\alpha \\ 0 & 0 & -2g\alpha^* & -i\gamma/2 \end{array} \right) \begin{pmatrix} \delta\alpha \\ \delta\alpha^* \\ \delta\beta \\ \delta\beta^* \end{pmatrix}$$

$$(\lambda + i\gamma/2)^2 + 4g^2|\alpha|^2 = 0$$

$$\lambda = -i\gamma/2 \pm i\,2g|\alpha|$$

stable if $\frac{\gamma}{2} > 2g|\alpha|$

i.e. $|\alpha| < \frac{\gamma}{4g}$

Since $|\alpha| = \frac{2|F|}{\gamma} \Rightarrow \frac{2|F|}{\gamma} < \frac{\gamma}{4g} \Rightarrow |F| < \frac{\gamma^2}{8g}$.

as expected.