The phase operator – sketch of solution

1-a)

$$e^{i\phi} = (a^{+}a+1)^{-1/2}\, a$$

$$\langle n| e^{i\phi} |n'\rangle = \langle n| (a^{+}a+1)^{-1/2}\, a |n'\rangle =$$

$$= \frac{1}{(n+1)^{1/2}} \langle n| a |n'\rangle = \frac{\sqrt{n+1}\;\delta_{n+1,n'}}{(n+1)^{1/2}} = \delta_{n+1,n'}$$

which means that $e^{i\phi}|n\rangle = |n-1\rangle$ if $n>0$

$= 0$ if $n=0$

$$\langle n|(e^{i\phi})^{+}|n'\rangle = \left(\langle n'| e^{i\phi} |n\rangle\right)^{*} = \delta_{n,n'+1} = \delta_{n-1,n'}$$

that is $(e^{i\phi})^{+}|n\rangle = |n+1\rangle$

$$\langle n|\left[e^{i\phi}, (e^{i\phi})^{+}\right]|n'\rangle = \langle n| e^{i\phi}(e^{i\phi})^{+} - (e^{i\phi})^{+}e^{i\phi} |n'\rangle$$

$$= \delta_{n',n} - (1-\delta_{n,0})\,\delta_{n',n} = \delta_{n0}\,\delta_{n'0}$$

that is $\left[e^{i\phi}, (e^{i\phi})^{+}\right] = |0\rangle\langle 0|$

A unitary operator is such that $UU^{+} = U^{+}U = \mathbb{1}$, so $e^{i\phi}$ is <u>not</u> unitary.

b)

$$\langle n|\left[e^{i\phi}, \hat{n}\right]|n'\rangle = \langle n| e^{i\phi}\hat{n} - \hat{n}\, e^{i\phi} |n'\rangle = \delta_{n+1,n'}\,(n'-n) =$$

$$= \delta_{n+1,n'}$$

$$\langle n|\left[(e^{i\phi})^{+}, n\right]|n'\rangle = -\delta_{n-1,n'} = -(e^{i\phi})^{+}$$

that is $\quad [e^{i\phi}, n] = e^{i\phi}$

c)

$$\cos\phi = \frac{1}{2}\left[e^{i\phi} + (e^{i\phi})^{+}\right]$$

$$\langle n|\cos\phi|n'\rangle = \frac{1}{2}\left(\delta_{n+1,n'} + \delta_{n-1,n'}\right)$$

$$\sin\phi = \frac{1}{2i}\left[e^{i\phi} - (e^{i\phi})^{+}\right] = \frac{1}{2i}\left(\delta_{n+1,n'} - \delta_{n-1,n'}\right)$$

are both hermitian as $\langle n|\cos\phi|n'\rangle = \langle n'|\cos\phi|n\rangle^*$
on a basis of states $|n\rangle$

$$[\cos\phi, n] = \frac{1}{2}\left[e^{i\phi} + (e^{i\phi})^{+}, n\right] = \frac{1}{2}\left(e^{i\phi} - (e^{i\phi})^{+}\right) =$$

$$= i \sin\phi$$

$$[\sin\phi, n] = \frac{1}{2i}\left[e^{i\phi} - (e^{i\phi})^{+}, n\right] = \frac{1}{2i}\left(e^{i\phi} + (e^{i\phi})^{+}\right) =$$

$$= -i\cos\phi$$

$\Rightarrow$ the operator n generates phase rotations,
since $\frac{d}{d\phi}\cos\phi = -\sin\phi$ and $\frac{d}{d\phi}\sin\phi = \cos\phi$

d)

$$\widehat{(e^{i\phi})}|e^{i\phi}\rangle = \lambda|e^{i\phi}\rangle \qquad \text{with } \lambda = e^{i\phi}$$

$$|e^{i\phi}\rangle = \sum_{n=0}^{\infty} a_n|n\rangle \qquad \widehat{(e^{i\phi})}|e^{i\phi}\rangle = \sum_{n=0}^{\infty} a_{n+1}|n\rangle$$

$$\Rightarrow a_{n+1} = \lambda a_n \Rightarrow a_n = \lambda^n$$

that is $|e^{i\phi}\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle$

$$\langle m|\int d\phi\, |e^{i\phi}\rangle\langle e^{i\phi}|\, m'\rangle = \langle m|\int d\phi \sum_{m=0}^{\infty} e^{-im\phi}|m\rangle$$

$$\sum_{m'=0}^{\infty} e^{im'\phi}\langle m'|m'\rangle = \int d\phi\; e^{i(m'-m)\phi} =$$

$$= 2\pi\,\delta_{m,m'}$$

$$|\psi\rangle = \int d\phi\, |e^{i\phi}\rangle\langle e^{i\phi}|\psi\rangle = \int d\phi\, \psi(\phi)\,|e^{i\phi}\rangle$$

$\psi_{|F:n\rangle}(\phi) = \langle e^{i\phi}|n\rangle = e^{-in\phi}$ are a basis of Hilbert space of plane waves of momentum $m = 0, -1, -2 \ldots$

e)

$$\psi_{|coh:\alpha\rangle}(\phi) = \langle e^{i\phi}|coh:\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{1}{\sqrt{n!}}\, e^{-in\phi}\alpha^n$$

$$\simeq e^{-|\alpha|^2/2}\int dx\, e^{-i\phi\cdot x}\,\frac{\alpha^x}{\sqrt{x!}} =$$

$$= e^{-|\alpha|^2/2}\int dx\, e^{-ix(\phi-\mathrm{Arg}(\alpha))}\,\frac{|\alpha|^x}{\sqrt{x!}}$$

$$\ell(x) = \log \frac{|\alpha|^x}{\sqrt{x!}} \simeq x \log|\alpha| - \frac{x}{2}\log x + \frac{x}{2}$$

max ℓ for $\log|\alpha| - \frac{\log x}{2} - \frac{1}{2} + \frac{1}{2} = 0$

i.e. $x = |\alpha|^2$

$$\frac{d^2\ell}{dx^2} = -\frac{1}{2x} = -\frac{1}{2|\alpha|^2}$$

$\frac{1}{2(2|\alpha|^2)}$

$$\Rightarrow \quad \ell(x) \simeq -\frac{1}{4|\alpha|^2}\left(x - |\alpha|^2\right)^2$$

$$\psi_{|\mathrm{coh}:\alpha\rangle}(\phi) \simeq C \int dx\, e^{-ix(\phi - \mathrm{Arg}\,\alpha)} e^{-\frac{1}{4|\alpha|^2}(x - |\alpha|^2)^2} \qquad (*)$$

$$= C' e^{-i|\alpha|^2(\phi - \mathrm{Arg}\,\alpha)} \; e^{-|\alpha|^2(\phi - \mathrm{Arg}\,\alpha)^2}$$

for coherent state $\Delta n \sim \sqrt{n}$ (cf variance of (*))

from wave function $\Delta\phi \sim \frac{1}{2|\alpha|}$; centred at $\phi = \mathrm{Arg}\,\alpha$

$$\Rightarrow \quad \Delta n \cdot \Delta\phi \sim 1/2$$

2-a)

$$H = \frac{\hbar\chi}{2}\hat{n}^2$$

$$i\hbar\frac{\partial\psi}{\partial t}(\phi,t) = i\hbar\frac{\partial}{\partial t}\langle e^{i\phi}|\psi(t)\rangle = \langle e^{i\phi}|i\hbar\frac{d\psi}{dt}\rangle$$

$$= \langle e^{i\phi}|H|\psi\rangle = \langle e^{i\phi}|\frac{\hbar\chi}{2}\hat{n}^2|\psi\rangle =$$

$$= \frac{\hbar\chi}{2}\langle e^{i\phi}|\hat{n}^2|\psi\rangle = \frac{\hbar\chi}{2}\sum_n e^{-in\phi}n^2\langle n|\psi\rangle$$

$$= -\frac{\hbar\chi}{2}\frac{d^2}{d\phi^2}\sum_n e^{-in\phi}\langle n|\psi\rangle =$$

$$= -\frac{\hbar\chi}{2}\frac{\partial^2}{\partial\phi^2}\psi(\phi) \qquad \text{Schrödinger eq.}$$

On Hilbert space $\mathcal{H}$ $\longrightarrow$ eigenvectors are plane waves corresponding to Fock states.

$$H|\phi_n\rangle = \lambda_n|\phi_n\rangle \quad \text{with} \quad \phi_n(\phi) = e^{in\phi}$$
$$\lambda_n = \frac{\hbar\chi}{2}n^2$$

b)

Gaussian state $\longrightarrow$ expands as usual q.m. particle of mass $\frac{\hbar\chi}{2} = \frac{\hbar^2}{2m^*} \Rightarrow m^* = \hbar/\chi$

$$\left.\frac{d\phi}{dt}\right|_{asympt.} = \frac{\hbar}{m^*}\cdot\frac{1}{\Delta\phi} = \frac{\hbar}{\hbar/\chi}\cdot\sqrt{2n} = \chi\sqrt{2n}$$

$$\text{characteristic } t_{1/2} = 2\pi\Big/\left.\frac{d\phi}{dt}\right|_{asympt} \sim \frac{2\pi}{\chi\sqrt{2n}}$$

$$|\psi(t)\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\frac{\chi}{2}n^2 t} |n\rangle$$

if $t = \frac{4\pi}{\chi} \Rightarrow |\psi(t)\rangle = |\psi(0)\rangle = |coh: \alpha\rangle$

if $t = \frac{2\pi}{\chi} \Rightarrow |\psi(t)\rangle = |coh: -\alpha\rangle$

if $t = \pi/\chi \Rightarrow$ more complex phase

$$\exp\left(-i\frac{\chi}{2}n^2\frac{\pi}{\chi}\right) = \exp\left(-i\frac{\pi}{2}n^2\right) =$$

$$= \{1, e^{-i\pi/2}, 1, e^{-i\pi/2}, \dots\}$$

$$= \{1, -i, 1, -i, \dots\} =$$

$$= \beta\{1, 1, 1 \dots\} + \beta'\{1, -1, 1, -1 \dots\}$$

with $\beta + \beta' = 1$, $\beta - \beta' = -i \Rightarrow \beta - (1-\beta) = -i$

$$\Rightarrow \beta = \frac{1}{2}(1-i)$$

that is $|\psi(t)\rangle = \frac{1-i}{2}|coh: \alpha\rangle + \frac{1+i}{2}|coh: -\alpha\rangle$

Schrödinger cat state! Analogous for all $t = \frac{2\pi}{q\chi}$, $q \in \mathbb{N}$

NOTE: mean-field limit $\chi n = c^{te}$, $n \to \infty$, $\chi \to 0$

$t_{tr} \propto \sqrt{n}$, $t_{rev} \propto n$