Exercise 5

1) incident state $|\psi\rangle = \psi_a(k)\,|a;k\rangle + \psi_b(k)\,|b;k\rangle$

final state $\hat{O}|\psi\rangle = \psi_c(k)\,|c;k\rangle + \psi_d(k)\,|d;k\rangle$

with
$$\begin{pmatrix}\psi_c(k)\\ \psi_d(k)\end{pmatrix} = e^{-ick(t_f-t_i)}\, S(k) \begin{pmatrix}\psi_a(k)\\ \psi_b(k)\end{pmatrix}$$

$$\langle c;k|\hat{O}|\psi\rangle = \langle c;k|\hat{O}|a;k\rangle\, \psi_a(k) + $$
$$+ \langle c;k|\hat{O}|b;k\rangle\, \psi_b(k) = \psi_c(k)$$

$\Longrightarrow$ $\langle (c,d);k|\hat{O}|(a,b);k\rangle$ are matrix elements of $S(k)$.

2) $a_k^+ \dots d_k^+$ create a photon in some state.

B-S does not create/destroy photons

$\Longrightarrow$ a_k^+, b_k^+ go into c_k^+, d_k^+ under U

$$U a_k^+ U^+ |vac\rangle = U a_k^+ |vac\rangle \qquad \text{(vacuum does not evolve)}$$
$$\parallel$$
$$\left(\psi_c(k)\, c_k^+ + \psi_d(k)\, d_k^+\right)|vac\rangle$$
$$\parallel$$
$$\left(t(k)\, c_k^+ + i r(k)\, d_k^+\right)|vac\rangle$$

$$\Longrightarrow \begin{cases} U a_n^{+} U^{+} = \left[t(n)\, c_n^{+} + i r(n)\, d_n^{+}\right] e^{-in\omega(t_f-t_i)} \\ U b_n^{+} U^{+} = \left[t(n)\, d_n^{+} + i r(n)\, c_n^{+}\right] e^{-in\omega(t_f-t_i)} \end{cases}$$

3)

$$\begin{cases} U a_n U^{+} = \left[t(n)\, c_n - i r(n)\, d_n\right] e^{in\omega(t_f-t_i)} \\ U b_n U^{+} = \left[t(n)\, d_n - i r(n)\, c_n\right] e^{in\omega(t_f-t_i)} \end{cases}$$

$$a_n = U^{+} U a_n U^{+} U = \left[t(n)\, U^{+} c_n U - i r(n)\, U^{+} d_n U\right] e^{in\omega(t_f-t_i)}$$

$$b_n = U^{+} U b_n U^{+} U = \left[t(n)\, U^{+} d_n U - i r(k)\, U^{+} c_n U\right] e^{in\omega(t_f-t_i)}$$

4) Inverting these relations:

$$\begin{pmatrix} U^{+} c_n U \\ U^{+} d_n U \end{pmatrix} = \begin{pmatrix} t(n) & i r(n) \\ i r(n) & t(n) \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cdot e^{-in\omega(t_f-t_i)}$$

5)

$$|\psi\rangle = \exp\left[-(|\alpha|^2 + |\beta|^2)/2\right] \exp\left(\alpha a^{+} + \beta b^{+}\right) |vac\rangle$$

coherent state $a|\psi\rangle = \alpha|\psi\rangle$, $b|\psi\rangle = \beta|\psi\rangle$

$$c\, U|\psi\rangle = U U^{+} c\, U|\psi\rangle = U(t\, a + i r\, b)|\psi\rangle =$$

$$= U \cdot (t\alpha + i r\beta)|\psi\rangle = (t\alpha + i r\beta)\, U|\psi\rangle$$

$\Longrightarrow$ $U|\psi\rangle$ is a coherent state for c and d

Amplitude $\gamma = t\alpha + i r\beta$ as in classical scatt

6) $|\psi\rangle = a^\dagger b^\dagger |vac\rangle$

$$U|\psi\rangle = U a^\dagger b^\dagger |vac\rangle = U a^\dagger U^\dagger U b^\dagger U^\dagger U |vac\rangle$$

$$= (t c^\dagger + i r d^\dagger)(i r c^\dagger + t d^\dagger)|vac\rangle$$

$$= \left[irt\, c^{\dagger 2} + irt\, d^{\dagger 2} + (t^2 - r^2)\, c^\dagger d^\dagger \right] |vac\rangle$$

for 50/50 beam splitter $t^2 = r^2 = 1/2$

$$\Rightarrow U|\psi\rangle = \frac{i}{2}\left(c^{\dagger 2} + d^{\dagger 2}\right)|vac\rangle =$$

$$= \frac{i}{\sqrt{2}} \left(\frac{c^{\dagger 2}}{\sqrt{2}} + \frac{d^{\dagger 2}}{\sqrt{2}} \right) |vac\rangle$$

2 particles in c　　　2 particles in d

7)

$$|\psi\rangle = \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi}\, \phi(k)\, \phi(k')\, e^{-ikx_a}\, e^{-ik'x_b}\, a_k^\dagger\, b_{k'}^\dagger\, |vac\rangle$$

for ϕ real $\longrightarrow$ wavepackets centred at x_a, x_b

$$U|\psi\rangle = \ldots + \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi}\, \phi(k)\, \phi(k')\, e^{-ikx_a}\, e^{-ik'x_b} \cdot$$

$$\left[t(k)\, t(k')\, c_k^\dagger\, d_{k'}^\dagger - r(k)\, r(k')\, c_{k'}^\dagger\, d_k^\dagger \right] e^{-ic(k+k')(t_f - t_i)} \cdot |vac\rangle$$

Assume beam-splitter is frequency independent and 50/50

$$= \ldots + \frac{1}{2} \int \frac{dk}{2\pi} \frac{dk'}{2\pi} \left(e^{-ik\bar{x}_a} e^{-ik'\bar{x}_b} - e^{-ik\bar{x}_b} e^{-ik'\bar{x}_a} \right) \phi(k)\, \phi(k') \cdot c_k^\dagger d_{k'}^\dagger |vac\rangle$$

with $\bar{x}_a = x_a - ct$, $\bar{x}_b = x_b - ct$

$$= \dots + \frac{1}{2} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \, \phi(k) \, \phi(k') \cdot e^{-i(k+k')(x_a - ct + x_b - ct)/2}$$

$$\cdot \left(e^{-i(k-k')(x_a - x_b)/2} - e^{i(k-k')(x_a - x_b)/2} \right) \cdot$$

$$\cdot \, c_k^\dagger d_{k'}^\dagger |vac\rangle$$

if $|(k-k')(x_a-x_b)| \ll 1 \longrightarrow$ two terms cancel and photons only cc or dd

if $|(k-k')(x_a-x_b)| \gtrsim 1 \longrightarrow$ also cd states possible

* $|k-k'|$ at most of the order of Δk
* $\Delta k \sim 1/L$ with $L =$ wave packet length

$\Rightarrow \left|\frac{x_a - x_b}{L}\right| \ll 1$ is condition for <u>wavepacket overlap</u>

<u>So</u>: quantum interference forbids cd states if incident wavepackets overlap

Effect can <u>not</u> be observed with <u>classical fields</u>