Quantum theory of the beam splitter

Consider the model of beam-splitter that is sketched in the figure. Light is incident from the $a, b$ input arms and is transmitted/reflected into the $c, d$ output arms. In the classical theory, the beam-splitter operation is summarized by the scattering matrix $S(\omega)$ connecting the output fields to the input ones at a given frequency $\omega$:

$$
\begin{pmatrix}
E_c(\omega) \\
E_d(\omega)
\end{pmatrix} = S(\omega) \begin{pmatrix}
E_a(\omega) \\
E_b(\omega)
\end{pmatrix}.
$$

(1)

For a lossless beam-splitter, energy conservation arguments allow to write $S$ in the canonical form:

$$
S(\omega) = \begin{pmatrix}
t(\omega) & ir(\omega) \\
ir(\omega) & t(\omega)
\end{pmatrix}
$$

(2)

with $r, t$ real functions of $\omega$ and $t(\omega)^2 + r(\omega)^2 = 1$.

In a quantum description of light, the matrix $S$ has the meaning of scattering $S$-matrix connecting the amplitude of the single-photon eigenfunction of energy $\omega = ck$ on the $c, d$ arms to the amplitude of the same wavefunction on the $a, b$ arms:

$$
\begin{pmatrix}
\bar{\psi}_{c,\omega} \\
\bar{\psi}_{d,\omega}
\end{pmatrix} = S(\omega) \begin{pmatrix}
\bar{\psi}_{a,\omega} \\
\bar{\psi}_{b,\omega}
\end{pmatrix}.
$$

(3)

As usual, the eigenfunction has a plane wave spatial profile on each arm, e.g.

$$
\psi_{a,\omega}(x) = e^{ikx} \bar{\psi}_{a,\omega}.
$$

(4)

1. The states of the photon in the $a, b, c, d$ arms at a wavevector $k$ can be labelled as $|(a, b, c, d); k\rangle$. The $U$ operator describing the evolution of the photon from an early time $t_i$ to a late time $t_f$ sends the two-dimensional $|(a, b); k\rangle$ space into the $|(c, d); k\rangle$ space. Show that the matrix elements of the $U$ operator in this basis correspond to the elements of the $S$ matrix described above multiplied by an overall phase factor $\exp[-ick(t_f - t_i)]$ due to the photon frequency.
2. Show that the action of $\hat{U}$ on the creation operators $\hat{a}_k^\dagger$, $\hat{b}_k^\dagger$ for the $|(a, b); k\rangle$ states is:

\[
\hat{U} \hat{a}_k^\dagger \hat{U}^\dagger = t(k) \hat{c}_k^\dagger + ir(k) \hat{d}_k^\dagger \tag{5}
\]

\[
\hat{U} \hat{b}_k^\dagger \hat{U}^\dagger = ir(k) \hat{c}_k^\dagger + t(k) \hat{d}_k^\dagger, \tag{6}
\]

where $\hat{c}_k^\dagger$, $\hat{d}_k^\dagger$ are the creation operators for the $|(c, d); k\rangle$ states.

3. Write the corresponding equations for the destruction operators.

4. Write the late-time out-going destruction operators $\hat{U}^\dagger \hat{c}_k \hat{U}$, $\hat{U}^\dagger \hat{d}_k \hat{U}$ in terms of the early-time in-going ones $\hat{a}_k$, $\hat{b}_k$. Interpret these expression in the Heisenberg picture of quantum mechanics.

5. Show that a coherent state is obtained in the output for any generic input coherent state (neglect for simplicity the $k$ dependence):

\[
|\psi\rangle = \exp\left\{-|\alpha|^2 + |\beta|^2/2\right\} \exp(\alpha \hat{a}^\dagger + \beta \hat{b}^\dagger) |\text{vac}\rangle. \tag{7}
\]

Show that the outgoing coherent state fulfills the classical relations for the reflection/transmission amplitudes discussed above.

6. What state is obtained in the output if two photons are simultaneously incident on a perfectly 50/50 beam-splitter, one from arm $a$ and one from arm $b$? Can this effect be described in a classical picture?

7. (Optional) What happens if the two photons incide on the beam-splitter at different times? For this question, one has to explicitly take into account the $k$ dependence of field operators, etc.