Exercise 6 : resonant scattering of light from 2 level atom

$$H = \frac{1}{2m}\left(P - \frac{q}{c}A\right)^2 + V(R) + H_{field}$$

$$\simeq \frac{P^2}{2m} + V(R) + H_{field} - \frac{q}{mc} P \cdot A(0) + \frac{q^2}{2mc^2} A(0)^2$$

at lowest order in q $\longrightarrow$ $P \cdot A(0)$ term.

Initial state : $|i\rangle = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} f(k,\varepsilon) \, a^{\dagger}_{k,\varepsilon} |vac\rangle$

with $f(k,\varepsilon) = \delta^3(k - k_i) \, \delta_{\varepsilon \varepsilon_i}$

* state not correctly normalized

* but gives finite photon density at atom.

↳ standard trick of scattering theory.

Approximation : atom's dipole only along $\hat{z}$: $\vec{d}_{eg} = \hat{z} \, d_{eg}$.

From theory of spontaneous emission : $\Gamma_{sp} = \frac{4}{3}\left(\frac{\omega_{eg}}{c}\right)^3 \frac{d_{eg}^2}{\hbar}$

Remind : $q\langle e|P|g\rangle = i\, \omega_{eg}\, m\, d_{eg}$

Final state : $\hat{a}^{\dagger}_{k_f,\varepsilon_f} |vac\rangle$

At 2nd order in perturbation theory :

$$M_{fi} = \frac{(i\,\omega_{eg}\, m\, d_{eg})^*}{mc} \cdot \sqrt{\frac{2\pi\hbar c}{k_f}}\, (\hat{\varepsilon}_f \cdot \hat{z}) \times \longrightarrow 1^{st} \text{ vertex P.A}$$

$$\times \frac{1}{\hbar c k_i - \hbar\omega_{eg} + i0^+} \times \longrightarrow \text{excited state propagator}$$

$$\times \frac{(i\,\omega_{eg}\, m\, d_{eg})}{mc} \sqrt{\frac{2\pi\hbar c}{k_i}}\, (\hat{\varepsilon}_i \cdot \hat{z}) \longrightarrow 2^{nd} \text{ vertex P.A}$$

Scattering rate :

$$\Gamma_{tot} = \frac{2\pi}{\hbar} \int \frac{d^3k_f}{(2\pi)^3} \sum_{\varepsilon_f} |M_{fi}|^2\, \delta(\hbar c k_f - \hbar c k_i)$$

$$= \frac{2\pi}{\hbar}\frac{1}{(2\pi)^3} \int k_f^2\, dk_f\, d\Omega_f \sum_{\varepsilon_f} \left(\frac{\omega_{eg}}{c}\right)^4 d_{eg}^4\, \frac{(2\pi)^2\hbar^2 c^2}{k_i k_f} \cdot (\hat{\varepsilon}_f \cdot \hat{z})^2 (\hat{\varepsilon}_i \cdot \hat{z})^2$$

$$\times \frac{1}{\hbar^2(\omega_i - \omega_{eg})^2 + 0^{+2}}\, \frac{1}{\hbar c}\, \delta(k_f - k_i)$$

$$= \frac{c}{\hbar^2} \int d\Omega_f \sum_{\varepsilon_f} \left(\frac{\omega_{eg}}{c}\right)^4 d_{eg}^4 \cdot (\hat{\varepsilon}_f \cdot \hat{z})^2 (\hat{\varepsilon}_i \cdot \hat{z})^2 \cdot \frac{1}{(\omega_i - \omega_{eg})^2 + 0^{+2}}$$

$$= c\, \frac{9}{4} \left(\frac{c}{\omega_{eg}}\right)^2 \frac{\Gamma_{sp}^2/4}{(\omega_i - \omega_{eg})^2 + 0^{+2}} \int d\Omega_f \sum_{\varepsilon_f} (\hat{\varepsilon}_f \cdot \hat{z})^2 (\hat{\varepsilon}_i \cdot \hat{z})^2 =$$

$$= \frac{9c}{16\pi^2}\lambda_{eg}^2 \frac{(\Gamma_{sp}/2)^2}{(\omega_i-\omega_{eg})^2+(0^+)^2}\cdot\int d\Omega_f \sum_{\varepsilon_f}(\varepsilon_f\cdot\hat{z})^2(\varepsilon_i\cdot\hat{z})^2 = \Gamma_{tot}$$

Rate vs. cross section:

$$\Gamma = n\cdot v\cdot\sigma$$

here: $n = 1$ (from the chosen normalization)

$v = c$

$$\Rightarrow \sigma = \frac{9}{16\pi^2}\lambda_{eg}^2\frac{(\Gamma_{sp}/2)^2}{(\omega_i-\omega_{eg})^2+(0^+)^2}\cdot\frac{8\pi}{3} = \frac{3}{2\pi}\lambda_{eg}^2\frac{(\Gamma_{sp}/2)^2}{(\omega_i-\omega_{eg})^2+(0^+)^2}$$

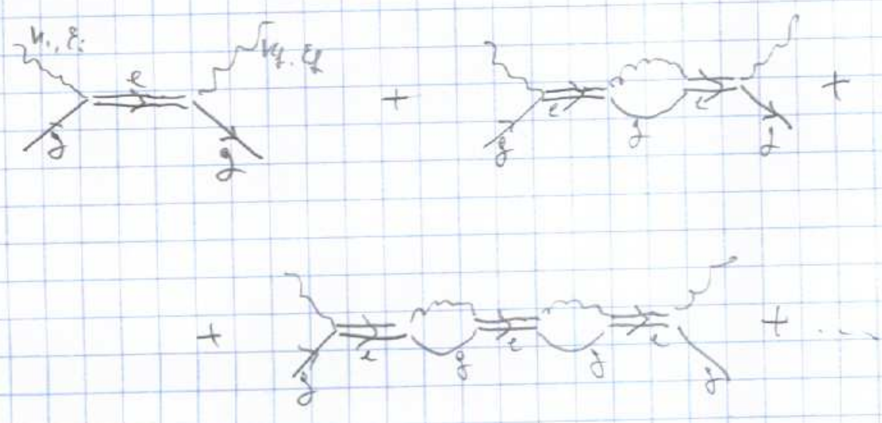
differential cross section:

$$d\sigma(k_i\,\varepsilon_i \to k_f\,\varepsilon_f) = \frac{9}{16\pi^2}\lambda_{eg}^2\frac{(\Gamma_{sp}/2)^2}{(\omega_i-\omega_{eg})^2+(0^+)^2}\,(\varepsilon_f\cdot\hat{z})^2(\varepsilon_i\cdot\hat{z})^2$$

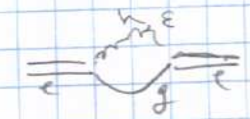
NOTE: cross section diverges as $\omega_i \to \omega_{eg}$.

lowest-order in perturbation theory no longer enough.

must resum all orders



Every order in perturbation theory introduces an extra factor:



$$\int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \left(\frac{i\omega_{eg} d_{eg}}{c}\right)^{*} \sqrt{\frac{2\pi\hbar c}{k}}\,(\hat{\varepsilon}\cdot\hat{z}) \times (1^{st}\text{ vertex})$$

$$\times \frac{1}{\hbar(ck_i - ck) + i0^+} \times (\text{propagator})$$

$$\times \underbrace{\frac{i\omega_{eg} d_{eg}}{c} \sqrt{\frac{2\pi\hbar c}{k}}\,(\hat{\varepsilon}\cdot\hat{z})}_{(2^{nd}\text{ vertex})} \times \frac{1}{\hbar(ck_i - \omega_{eg}) + i0^+} \quad (\text{external propag.})$$

$$= \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \frac{\omega_{eg}^2 d_{eg}^2}{c^2} \frac{2\pi\hbar c}{k} (\hat{\varepsilon}\cdot\hat{z})^2 \frac{1}{\hbar c(k_i - k + i0^+)}$$

$$= \frac{\omega_{eg}^2 d_{eg}^2}{(2\pi)^2 c^2} \int k^2\,dk\,d\Omega \sum_{\varepsilon} (\hat{\varepsilon}\cdot\hat{z}) \frac{1}{k}\left[PP\frac{1}{k_i - k} - i\pi\,\delta(k_i - k)\right] \cdot \frac{1}{\hbar(ck_i - \omega_{eg}) + i0^+}$$

$$= \Big[\underbrace{\hbar\Delta(k_i)}_{\text{slowly varying with } k_i} - i\,\frac{\omega_{eg}^2 d_{eg}^2}{4\pi c^2}\, k_i\, \frac{8\pi}{3}\Big] \cdot \frac{1}{\hbar(ck_i - \omega_{eg}) + i0^+}$$

$$\simeq \left[\Delta(k_i) - \frac{2i}{3} d_{eg}^2 \left(\frac{\omega_{eg}}{c}\right)^3\right] \frac{1}{\hbar(ck_i - \omega_{eg}) + i0^+}$$

$$= \left(\hbar\Delta - i\hbar\Gamma_{sp}/2\right) \frac{1}{\hbar(ck_i - \omega_{eg} + i0^+)}$$

Resum geometrical series:

$$\sum_{n\geq 0} \left(\frac{\hbar\Delta - i\hbar\Gamma_{sp}/2}{\hbar(ck_i - \omega_{eg} + i0^+)}\right)^n = \frac{1}{1 - \dfrac{\Delta - i\Gamma_{sp}/2}{ck_i - \omega_{eg} + i0^+}} = \frac{ck_i - \omega_{eg}}{ck_i - \underbrace{\omega_{eg} - \Delta}_{\text{Lamb-shifted } \bar{\omega}_{eg}} + i\Gamma_{sp}/2}$$

$$M_{fi} \to M_{fi} \times \frac{\hbar\omega_i - \omega_{eg}}{\hbar\omega_i - \bar{\omega}_{eg} + i\,\Gamma_{sp}/2}$$

smoothens out divergent amplitude on resonance

new resonance of width $\Gamma_{sp}/2$

$$\Rightarrow \quad d\sigma = \frac{9}{16\pi^2}\lambda_{eg}^2 \frac{(\Gamma_{sp}/2)^2}{(\omega_i - \bar{\omega}_{eg})^2 + (\Gamma_{sp}/2)^2}$$

resonant cross-section $\approx \lambda_{eg}^2$

slightly shifted resonance freq

radiative broadening of transition

$$\sigma = \frac{3}{2\pi}\lambda_{eg}^2 \frac{(\Gamma_{sp}/2)^2}{(\omega_i - \bar{\omega}_{eg})^2 + (\Gamma_{sp}/2)^2}$$

Full calculation of $\underline{\Delta}$ requires more technicalities.

<u>More details in</u> : C. Cohen-Tannoudji - J. Dupont-Roc - G. Grynberg "Photons and atoms", exercise 5.