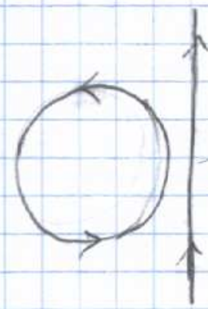
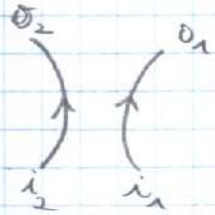
exercise 1 : filters and rings



kind of beamsplitter

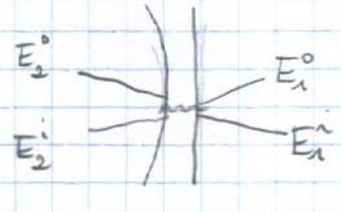


$$\begin{pmatrix} E_{o_1} \\ E_{o_2} \end{pmatrix} = M \begin{pmatrix} E_{i_1} \\ E_{i_2} \end{pmatrix}$$

Energy conservation $\longrightarrow$ $M^\dagger M = \mathbb{1}$

e.g. $M = \begin{pmatrix} t & r \\ -r & t \end{pmatrix}$, $|t|^2 + |r|^2 = 1$, $t, r \in \mathbb{R}$

Fiber Transmission



$$\left.\begin{aligned} E_2^o &= t E_2^i - r E_1^i \\ E_1^o &= t E_1^i + r E_2^i \end{aligned}\right\} \text{beam-splitter.}$$

$E_2^i = \exp(i k \cdot 2\pi R)\, E_2^o$ round-trip condition

$$E_2^o = t \cdot \exp(ik\, 2\pi R)\, E_2^o - r\, E_1^i$$

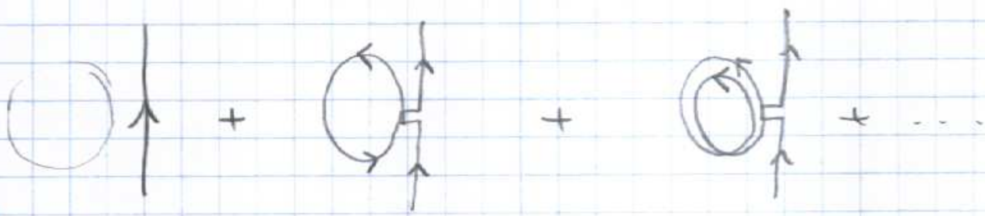
$$E_2^o = \frac{-r}{1 - t \exp(ik\, 2\pi R)}\, E_1^i$$

$$E_1^o = t E_1^i - r \exp(ik\, 2\pi R) \frac{r}{1 - t \exp(ik\, 2\pi R)}\, E_1^i =$$

$$= E_1^i \left[t - \frac{r^2 e^{2\pi i k R}}{1 - t e^{2\pi i k R}} \right]$$

$$\approx E_1^i\, t + E_1^i\, (-r)\, e^{2\pi i k R} \cdot r +$$

$$+ E_1^i\, (-r)\, e^{2\pi i k R}\, t\, e^{2\pi i k R} \cdot r + \dots$$



Non-absorbing limit: $k \in \mathbb{R}$, $|\exp(2\pi i k R)| = 1$

* resonant point $2\pi k R = 2\pi m$

$$E_1^o = E_1^i \left[t - \frac{1-t^2}{1-t} \right] = E_1^i \left[t - (1+t) \right] = - E_1^i$$

$$|E_1^o|^2 = |E_1^i|^2 \longrightarrow \text{perfect transmission}$$

* in general :

$$E_1^o = E_1^i \left[t - \frac{(1-t^2)\, e^{2\pi i k R}}{1 - t\, e^{2\pi i k R}} \right] =$$

$$= E_1^i \left[\frac{t - \cancel{t^2 e^{2\pi i k R}} - e^{2\pi i k R} + \cancel{t^2 e^{2\pi i k R}}}{1 - t\, e^{2\pi i k R}} \right]$$

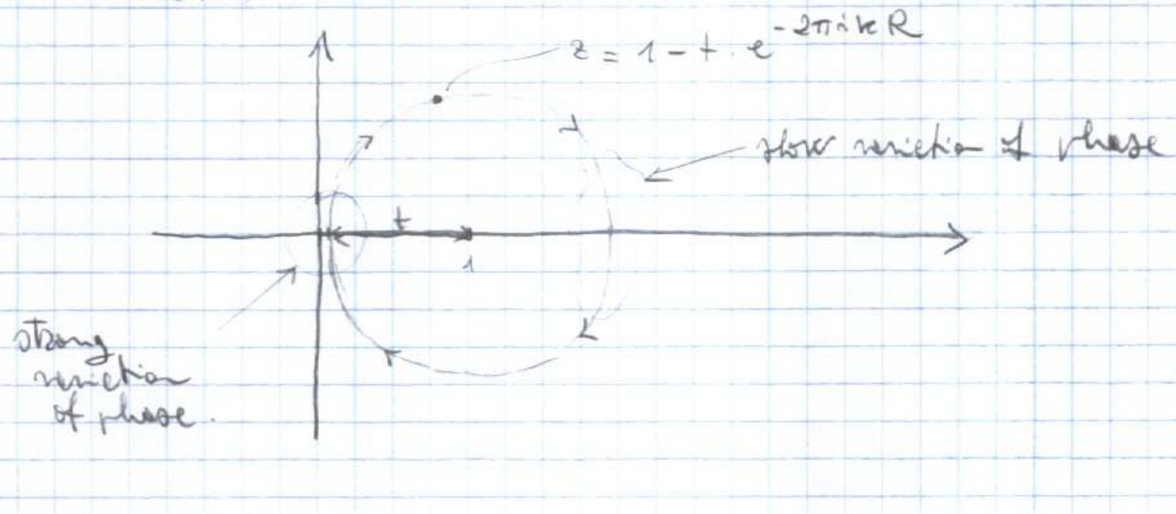
$$= - E_1^i \; e^{2\pi i k R} \left[\frac{1 - t\, e^{-2\pi i k R}}{1 - t\, e^{2\pi i k R}} \right]$$

as we have assumed $t \in \mathbb{R}$

$$\Rightarrow \quad |E_1^o|^2 = |E_1^i|^2 \quad \text{for all frequencies !}$$

but transmission phase varies

$$\mathrm{Arg}\left(\frac{E_1^o}{E_1^i}\right) = \pi + 2\pi k R + 2 \cdot \mathrm{Arg}\left(1 - t\, e^{-2\pi i k R}\right)$$



$$\frac{d}{dk} \mathrm{Arg}\left(\frac{E_1^o}{E_1^i}\right) = \frac{4\pi R}{1-t} \simeq \frac{8\pi R}{r^2}$$

Physical interpretation

$$\tau = \frac{d}{d\omega}\phi(\omega) \simeq \frac{1}{c}\frac{d}{dn}\phi(n) = \frac{8\pi R}{c}\cdot\frac{1}{r^2} =$$

$$= 4 \times T_{round} \times N_{round}$$

→ # round trips before exiting
(each round trip, probability r^2 of exiting.)

Stationary-phase argument.

$$E_1^o(t) = \int d\omega\, E_1^i(\omega)\cdot t(\omega)\, e^{-i\omega t} \simeq$$

$$\simeq \int d\omega\, E_1^i(\omega)\cdot|t(\omega)|\cdot e^{i\phi(\omega)}\, e^{-i\omega t}$$

$$\simeq \int d\omega\, E_1^i(\omega)\, |t(\omega)|\, e^{-i(\omega-\omega_0)(t-d\phi/d\omega)}\, e^{-i\omega_0 t}$$

assuming $E_1^i(\omega)$ smooth and real

$$\Longrightarrow E_1^o(t) \text{ peaked at } t-\frac{d\phi}{d\omega}=0$$

$$\text{i.e. } t = \frac{d\phi}{d\omega}$$

on the other hand:

$$E_1^i(t) \simeq \int d\omega\, E_1^i(\omega)\, e^{-i\omega t} \quad \text{peaked at } t=0.$$

$$\Rightarrow \text{DELAY TIME} \quad \tau = \frac{d\phi}{d\omega}$$

In the presence of dissipation:

$$t_{tot} = -\left[\frac{e^{2\pi i \hat{n} R} - t}{1 - t\, e^{2\pi i \hat{n} R}}\right]$$

where $\hat{n}$ = complex wavevector in ring

Single-round-trip loss $\exp\left[-2\pi R k_i\right]$

weak loss condition $2\pi R k_i \ll 1$

Non-radiative decay rate $\Gamma_{nr}/2 = \frac{c}{\cancel{2\pi R}} \cancel{2\pi R} k_i = c k_i$

i.e. $\exp\left[-2\pi R k_i\right] = \exp\left[-2\pi R \Gamma_{nr}/2c\right]$

At resonance $\exp\left[2\pi R k_r\right] = 1$

$$t_{tot} = \frac{t - e^{-2\pi R \Gamma_{nr}/2c}}{1 - t\, e^{-2\pi R \Gamma_{nr}/2c}} \simeq \frac{t - 1 + 2\pi R \Gamma_{nr}/2c}{1 - t + t\, 2\pi R \Gamma_{nr}/2c}$$

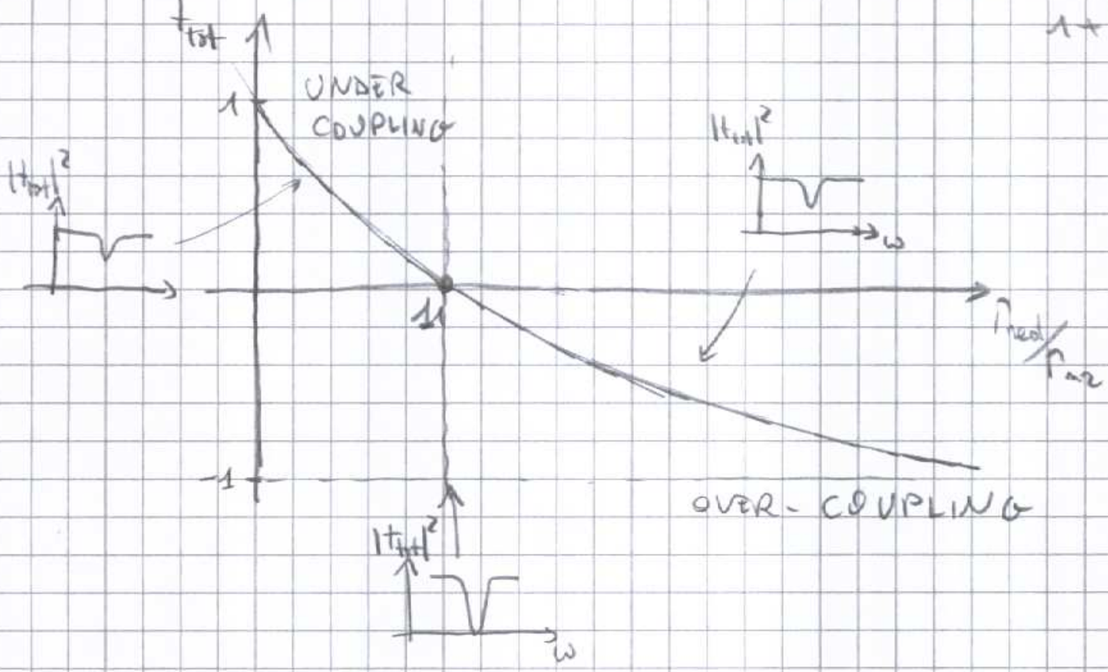
Radiative width:

denominator $\frac{1}{1 - t e^{2\pi i k R}} \simeq \frac{1}{1 - t - 2\pi i (k - k_0) R t} =$

$$= \frac{1}{1 - t - 2\pi i R t\, \delta\omega/c} = \frac{1}{i \frac{2\pi R t}{c}} \frac{1}{-\delta\omega - i \frac{c}{2\pi R t}(1-t)}$$

$$\Longrightarrow \frac{\Gamma_{rad}}{2} = \frac{c}{2\pi R t}(1-t) \simeq \frac{c}{2\pi R}(1-t)$$

$$t_{tot} = \frac{-2\pi \frac{R}{c} \frac{\Gamma_{rad}}{2} + 2\pi \frac{R}{c} \frac{\Gamma_{mr}}{2}}{\frac{2\pi R}{c}\frac{\Gamma_{rad}}{2} + \frac{2\pi R}{c}\frac{\Gamma_{mr}}{2}} = \frac{\Gamma_{mr} - \Gamma_{rad}}{\Gamma_{rad} + \Gamma_{mr}} = \frac{1 - \Gamma_{rad}/\Gamma_{mr}}{1 + \Gamma_{rad}/\Gamma_{mr}}$$



Complete injection of energy into ring cavity

→ peak depth maximum at OPTIMAL COUPLING $\Gamma_{mr} = \Gamma_{rad}$

then decreases on both sides.

Addendum to solution of Ex. 1

Energy conservation $\Rightarrow \quad \| S|v\rangle \|^2 = \| |v\rangle \|^2 \quad \forall |v\rangle$

$$\langle v | S^\dagger S | v \rangle = \langle v | v \rangle \qquad \forall v.$$

$$\Rightarrow \quad S^\dagger S = \mathbb{1}$$

Symmetry $1 \leftrightarrow 2$:

$$T^{-1} S T = S \qquad \text{with} \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$T^{-1} S T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} S_{22} & S_{21} \\ S_{12} & S_{11} \end{pmatrix}$$

$$T^{-1} S T = S \quad \Rightarrow \quad \underbrace{S_{11} = S_{22}}_{r}, \quad \underbrace{S_{12} = S_{21}}_{t}.$$

Unitarity:

$$S^\dagger S = \begin{pmatrix} r^* & t^* \\ t^* & r^* \end{pmatrix} \begin{pmatrix} r & t \\ t & r \end{pmatrix} = \mathbb{1}$$

$$\Rightarrow \begin{cases} |r|^2 + |t|^2 = 1 \\ r^* t + r t^* = 0 \end{cases}$$

Write $t = |t| e^{i\vartheta} \quad \Rightarrow \quad r^* |t| e^{i\vartheta} + r |t| e^{-i\vartheta} = 0.$

$$\Rightarrow r = i |r| e^{i\vartheta} \quad \text{with} \quad |r|^2 = 1 - |t|^2.$$

Transmittivity of device:

$$E_{i1} = E_{o1} \, e^{i 2\pi R\omega/c}$$

$$E_{o1} = i|r| e^{i\vartheta} E_{i1} + |t| e^{i\vartheta} E_{i2} = e^{-i2\pi R\omega/c} E_{i1}$$

$$E_{i1} = \frac{|t| e^{i\vartheta} E_{i2}}{e^{-i2\pi R\omega/c} - i|r| e^{i\vartheta}}$$

$$E_{o2} = i|r| e^{i\vartheta} E_{i2} + \frac{|t|^2 e^{2i\vartheta}}{e^{-i2\pi R\omega/c} - i|r| e^{i\vartheta}} E_{i2}$$

$$= \left[i|r| e^{i\vartheta} + \frac{|t|^2 e^{2i\vartheta} e^{i2\pi R\omega/c}}{1 - i|r| e^{i\vartheta} e^{i2\pi R\omega/c}} \right] E_{i2}$$

$$\simeq \left[i|r| e^{i\vartheta} + |t|^2 e^{2i\vartheta} e^{i2\pi R\omega/c} + |t|^2 i|r| e^{3i\vartheta} e^{i4\pi R\omega/c} + \dots \right] \times$$

$$\times E_{i2}$$

$$= \left[\frac{i|r| e^{i\vartheta} + |r|^2 e^{2i\vartheta} e^{i2\pi R\omega/c} + |t|^2 e^{2i\vartheta} e^{i2\pi R\omega/c}}{1 - i|r| e^{i\vartheta} e^{i2\pi R\omega/c}} \right] E_{i2} =$$

$$= \underbrace{e^{2i\vartheta} e^{i2\pi R\omega/c} \left[\frac{1 + i|r| e^{-i\vartheta} e^{-i2\pi R\omega/c}}{1 - i|r| e^{i\vartheta} e^{i2\pi R\omega/c}} \right]}_{\tau(\omega)} E_{i2}$$

$$|r(\omega)| = 1 \qquad \text{es} \qquad r(\omega) = e^{i\phi}\left(\frac{1+A}{1+A^*}\right)$$

$$r(\omega) = e^{i\phi(\omega)}$$

$$\phi(\omega) = 2\vartheta + 2\pi R\omega/c + \mathrm{Arg}\left[\frac{1+i|r|\dots}{1-i|r|\dots}\right]$$

$$= 2\vartheta + 2\pi R\omega/c + 2\,\mathrm{Arg}\left[1+i|r|e^{-i\vartheta}e^{-i2\pi R\omega/c}\right]$$

$$= 2\vartheta + 2\pi R\omega/c + 2\,\mathrm{Arg}\left[1+|r|\,e^{i\overbrace{\left[\frac{\pi}{2}-\vartheta-2\pi R\omega/c\right]}^{\lambda}}\right]$$

$$\frac{d\phi}{d\omega} = 2\pi\frac{R}{c} + 2\frac{d}{d\omega}\,\mathrm{Arg}\left[1+|r|\cos\lambda(\omega) + i|r|\sin\lambda(\omega)\right]$$

$$= 2\pi\frac{R}{c} + 2\frac{d}{d\omega}\,\mathrm{atan}\left[\frac{|r|\sin\lambda(\omega)}{1+|r|\cos\lambda(\omega)}\right] =$$

$$= 2\pi\frac{R}{c} + 2\,\frac{|r|\left[|r|+\cos\lambda\right]}{1+|r|^2+2|r|\cos\lambda}\,\frac{d\lambda}{d\omega} =$$

$$= 2\pi\frac{R}{c} + 2\,\frac{|r|\left[|r|+\cos\lambda\right]}{1+|r|^2+2|r|\cos\lambda}\left(-2\pi\frac{R}{c}\right) =$$

$$= 2\pi\frac{R}{c}\cdot\left[1-\frac{2|r|^2+2|r|\cos\lambda}{1+|r|^2+2|r|\cos\lambda}\right] = 2\pi\frac{R}{c}\left[\frac{1+|r|^2+\cancel{2|r|\cos\lambda}-2|r|^2\,\cancel{-2|r|\cos\lambda}}{1+|r|^2+2|r|\cos\lambda}\right]$$

$$= 2\pi\frac{R}{c}\,\frac{1-|r|^2}{1+|r|^2+2|r|\cos\lambda}$$

Max $\frac{d\phi}{d\omega}$ for $\cos\lambda = -1$

$$\Rightarrow \left.\frac{d\phi}{d\omega}\right|_{max} = 2\pi \frac{R}{c} \frac{1-|r|^2}{1+|r|^2-2|r|} =$$

$$= 2\pi \frac{R}{c} \left[\frac{1+|r|}{1-|r|}\right] \simeq \frac{4\pi R}{c[1-|r|]}$$

Note: for $|r| = 1 \Rightarrow \frac{d\phi}{d\omega} = 0 \longrightarrow$ ring decoupled.

Lifetime of ring mode: $\Gamma = \frac{1}{2\pi R/c}(1-|r|^2) \simeq \frac{1-|r|}{\pi R/c}$

$$\left(\left.\tau_{del}\right|_{max} = 4\,\Gamma^{-1} \right)$$

NOTE: delay is max for $\cos\lambda = -1$, i.e.

$$\frac{\pi}{2} - \theta - \frac{2\pi R\omega}{c} = (2M+1)\pi$$

$$\frac{2\pi R\omega}{c} = \frac{\pi}{2} + (2M+1)\pi = 2M\pi - \underbrace{\frac{\pi}{2} - \theta}_{\text{reflection phase.}}$$

With dissipation:

$$e^{2\pi i R\omega/c} \longrightarrow e^{2\pi i R\omega/c} \; e^{-2\pi R \Gamma_{nr}/2c}$$

Γ_{nr} = non-radiative decay rate $= 2c \cdot k_i$

$$\tau(\omega) = e^{i\vartheta} \cdot \frac{e^{i2\pi R\omega/c}\, e^{-2\pi R\Gamma_{nr}/2c}\, e^{i\vartheta} + i|r|}{1 - i|r|\, e^{i\vartheta}\, e^{2\pi i R\omega/c}\, e^{-2\pi R\Gamma_{nr}/2c}}$$

$\longrightarrow \in \mathbb{R}^+$ on resonance.

$$\tau|_{res} = i e^{i\vartheta} \cdot \left. \frac{|r| - i e^{i2\pi R\omega/c}\, e^{i\vartheta}\, e^{-2\pi R\Gamma_{nr}/2c}}{1 - i e^{i2\pi R\omega/c}\, e^{i\vartheta}\, e^{-2\pi R\Gamma_{nr}/2c}\, |r|} \right|_{res} \longrightarrow$$

$$\longrightarrow \frac{|r| - e^{-2\pi R\Gamma_{nr}/2c}}{1 - |r|\, e^{-2\pi R\Gamma_{nr}/2c}} \simeq$$

$$\simeq \frac{|r| - 1 + \frac{2\pi R\Gamma_{nr}}{2c}}{1 - |r| + \frac{2\pi R\Gamma_{nr}}{2c}|r|} =$$

$$= \frac{-\frac{\pi R}{c}\Gamma_{rad} + \frac{\pi R}{c}\Gamma_{nr}}{\frac{\pi R}{c}\Gamma_{rad} + \Gamma_{nr}\frac{\pi R}{c}} = \frac{\Gamma_{nr} - \Gamma_{rad}}{\Gamma_{rad} + \Gamma_{nr}}$$