Lecture 10 : Atoms and photons in a cavity.

Generation and detection of quantum entangled states

inspired from: Raimond, Brune, Haroche, Rev. Mod. Phys 73, 565 (2001)

QED in a cavity

free space : $A(r) = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \sqrt{\frac{2\pi\hbar c}{k}} \cdot \hat{\varepsilon} \left[\hat{a}_k e^{ikr} + \hat{a}_k^+ e^{-ikr} \right]$

in cavity with 3D confinement :

$$\hat{A}(r) = \sum_i \vec{a}_i(r)\, \hat{a}_i + \vec{a}_i^*(r)\, \hat{a}_i^+$$

with $a_i(r)$ = vector potential of i-th eigenmode of e.m. problem with suitable boundary conditions.

$$\hat{H} = \sum_i \hbar\omega_i\, \hat{a}_i^+ \hat{a}_i$$

with :

* $i = (i_x, i_y, i_z)$

* $\omega_i = c \sqrt{\frac{\pi^2}{L_x^2} i_x^2 + \frac{\pi^2}{L_y^2} i_y^2 + \frac{\pi^2}{L_z^2} i_z^2}$

* p. ex. TE mode : $\vec{a}_i(r) = a_i^0 \sin\left(\frac{\pi}{L_x} i_x x\right) \cos\left(\frac{\pi}{L_y} i_y y\right) \sin\left(\frac{\pi}{L_z} i_z z\right) \cdot \hat{y}$

  $\hookrightarrow$ satisfies boundary conditions at $x, y, z = 0, L_{x,y,z}$, requires $i_x, i_z > 0$, $i_y \geq 0$

* normalization factor a_i^0 such that energy of 1-photon state by $\frac{1}{8\pi} \int d^3r\, |E_i(r)|^2 + |B_i(r)|^2$ gives one quantum of energy $\hbar\omega_i$

→ in empty, metallic cavities $\quad a_n^0 = \sqrt{\dfrac{8\pi\hbar c^2}{\omega_n L_x L_y L_z}}$.

→ with periodic boundary conditions : $\quad a_i(r) = \sqrt{\dfrac{2\pi\hbar c}{k L_x L_y L_z}}\, e^{ik\cdot z}$

<u>In particular</u>: field of single photon scales down as $1/\sqrt{V}$

smaller cavity ⟶ higher field.

<u>Single atom in a cavity:</u>

$$H = \frac{1}{2m}\left(P - \frac{q}{c}A(R)\right)^2 + V(R) + \sum_i \hbar\omega_i\, \hat{a}_i^+ \hat{a}_i$$

dipole approx.

$$\simeq \frac{1}{2m}\left(P - \frac{q}{c}A(r_{at})\right)^2 + V(R) + \sum_i \hbar\omega_i\, \hat{a}_i^+ \hat{a}_i$$

atom is located at r_{at}

$$= \frac{1}{2m}P^2 + V(R) + \sum_i \hbar\omega_i\, \hat{a}_i^+ \hat{a}_i - \frac{q}{mc}P\, A(r_{at}) + \frac{q^2}{2mc^2}A(r_{at})^2$$

* at lowest order in q ⟶ neglect $\frac{q^2}{2mc^2}A(r_{at})^2$ term

* two-level approx. $\quad \hat{P} \simeq P_{eg}\left(|e\rangle\langle g| + |g\rangle\langle e|\right)$

$$\frac{1}{2m}P^2 + V(R) \simeq \hbar\omega_{eg}\, |e\rangle\langle e|$$

* single-mode approx. ⟶ only mode i is close to resonance

$$H \simeq \hbar\omega_{eg}|e\rangle\langle e| + \hbar\omega_i\, a_i^{+} a_i - \frac{q}{mc} p_{eg}\left[|e\rangle\langle g| + |g\rangle\langle e|\right] \times$$

$$\times \left[a_i(r_{at})\, \hat{a}_i + a_i(r_{at})^{*}\, \hat{a}_i^{+} \right]$$

$$\simeq \hbar\omega_{eg}|e\rangle\langle e| + \hbar\omega_i\, a_i^{+} a_i - \frac{q p_{eg}}{mc}\left[a_i(r_{at})\,|e\rangle\langle g|\, a_i + \right.$$

$$\left. + a_i(r_{at})^{*}\,|g\rangle\langle e|\, a_i^{+} \right]$$

↳ Rotating wave approx : processes where photon is emitted and atom goes up to $|e\rangle$ are energetically disfavoured and neglected

$$\simeq \hbar\omega_{eg}|e\rangle\langle e| + \hbar\omega_i\, a_i^{+} a_i - \overbrace{\frac{q p_{eg}}{mc} a_i^{0} \sin\left(\frac{\pi}{L_x} i_x x_{at}\right) \cos\left(\frac{\pi}{L_y} i_y y_{at}\right) \sin\left(\frac{\pi}{L_z} i_z z_{at}\right)}^{\hbar\Omega_R \,\to\, \text{Rabi frequency}}$$

$$\cdot \left[\hat{a}_i\,|e\rangle\langle g| + \hat{a}_i^{+}\,|g\rangle\langle e| \right] = \tilde{H}$$

Jaynes - Cummings model

Eigenstates of JC model :

* Total # of excitations is conserved in $\hat{H}$, $N_T = |e\rangle\langle e| + a_i^{+} a_i$
* within each N_T subspace : 2 states $|N_T, g\rangle$ and $|N_T - 1, e\rangle$
* eigenmodes are linear superpositions of 2 states

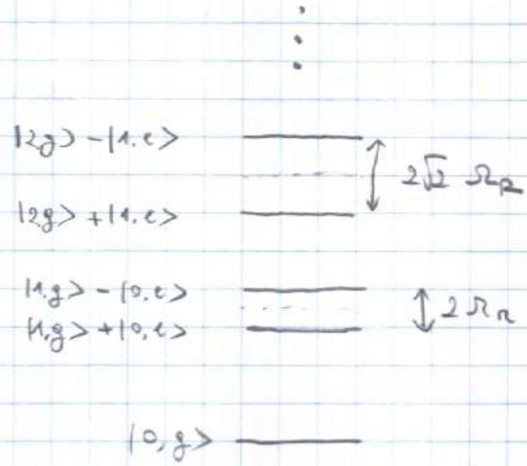
Resonant case $\omega_i = \omega_{eg}$:

$$H_{N_T} = -\hbar\Omega_R \sqrt{N_T} \left[|N_T - 1, e\rangle\langle N_T, g| + |N_T, g\rangle\langle N_T - 1, e| \right]$$

* eigenstates are $\frac{1}{\sqrt{2}}\left[|N_T, g\rangle \pm |N_T - 1, e\rangle \right]$

* eigenenergies are $\mp \hbar\Omega_R \sqrt{N_T}$

"Jaynes - Cummings ladder"



Starting from $|\psi(t=0)\rangle = |1,g\rangle$:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[\left(\frac{|1,g\rangle + |0,e\rangle}{\sqrt{2}} \right) e^{i\Omega_R t} + \left(\frac{|1,g\rangle - |0,e\rangle}{\sqrt{2}} \right) e^{-i\Omega_R t} \right]$$

$$= |1,g\rangle \cdot \cos\Omega_R t + |0,e\rangle \; i\sin\Omega_R t$$

→ Rabi flopping from $|1,g\rangle$ to $|0,e\rangle$ and reverse with period $T = \pi/\Omega_R$

→ reversible absorption - emission cycles.

→ different from free-space : irreversible scattering process.

Direct demonstration of field quantization:

create field in coherent state by incident laser pulse:

$$|\psi(t=0)\rangle = \sum_n e^{-|\alpha|^2/2} \cdot \frac{\alpha^n}{\sqrt{n!}} \, |n, g\rangle$$

at time t:

$$|\psi(t)\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \Big[\cos(\sqrt{n}\,\Omega_R t) \, |n, g\rangle +$$

$$+ i \sin(\sqrt{n}\,\Omega_R t) \, |n-1, e\rangle \Big]$$

$$P_e = \sum_n e^{-|\alpha|^2/2} \cdot \frac{|\alpha|^{2n}}{n!} \sin^2(\sqrt{n} \cdot \Omega_R t)$$

probability of detecting atom in e oscillates
at frequencies $2\sqrt{n}\,\Omega_R = \omega_n$

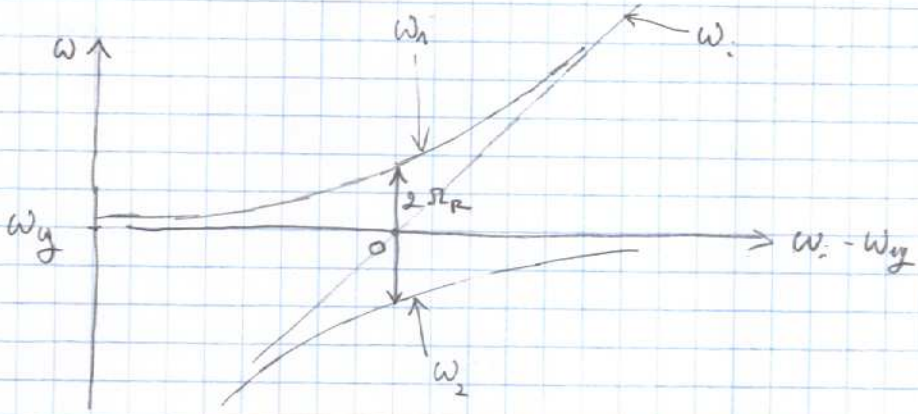
Experiment: measure P_e as a function t

↳ by Fourier transform ⟶ peaks at $2\sqrt{n}\,\Omega_R$

⟹ direct proof of field quantization!

* Photoelectric effect could be explained in terms of classical
e.m. field interacting with quantum matter

General case: level anticrossing



→ for $|\omega_l - \omega_{eg}| \gg \Omega_R$ eigenstates tend back to uncoupled $|1,g\rangle$ and $|0,e\rangle$

→ resonant region: anticrossing of amplitude Ω_R

→ width of resonant region $\approx \Omega_R$

An experimental realisation : circular Rydberg atoms in superconducting cavity.

circular Rydberg atoms :

* hydrogenoid system (e.g alkali atom) with external electron in $l = m = n \gg 1$ states.

* transitions between neighboring n states with circular polarized light.

- frequency of transition :

$$\Delta\nu = -\frac{1}{2\pi\hbar}\left[\frac{R_y}{n^2} - \frac{R_y}{(n-1)^2}\right] \simeq \frac{R_y}{2\pi\hbar}\cdot\frac{2}{n^3}$$

* for $n = 50$, $\Delta\nu \approx 54$ GHz in the microwave domain ($\lambda \approx 5$ mm)

* scales down as $1/n^3$ as ionisation threshold is approached.

- dipole moment :

* classical orbit $\frac{e^2}{r^2} = \frac{mv^2}{r}$

Coulomb attraction — centripetal force.

* quantization of l $\Longrightarrow$ $\hbar n = mvr$

$\Longrightarrow$ $r \simeq n^2 a_B$ $\left[a_B = \frac{\hbar^2}{me^2} = \text{Bohr radius}\right]$

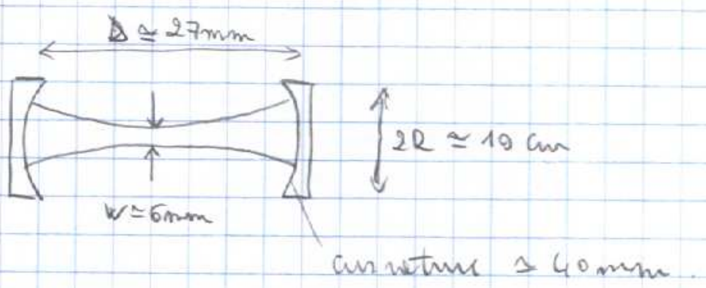
$\hookrightarrow$ for $n = 50$ $\longrightarrow$ $r \approx 2500\, a_B \simeq 125$ nm

* dipole moment (on the order of $d_{ey} \simeq q\cdot r$) is therefore very large

- lifetime : $n = l = m$ can only decay to $n-1 = l-1 = m-1$ emitting a σ^+ photon.

$$\Gamma_{sp} = \frac{4}{3}\frac{\omega_0^3}{\hbar c^3}\, d_{ey}^2 \sim \frac{1}{6\pi^3}\frac{mc^2}{\hbar}\alpha^5\cdot\frac{1}{n^5} \quad \text{very slow}$$

Superconducting cavity



TEM$_{900}$ mode : frequency $\nu \simeq \frac{c}{2D} \cdot 9 = 50$ GHz.

mode volume : field concentrated in waist region
effective $V_{eff} \simeq \pi w^2 D$ rather than $\pi R^2 D$
(one order magnitude decreased)

distance between modes (free spectral range $\Delta\nu \simeq \frac{c}{2D} \simeq 5$ GHz)

atom - field coupling Ω_R : atom at antinode to maximise ampl

inserting values $\longrightarrow$ $\Omega_R \simeq 2\pi \cdot 50$ kHz
* much smaller than F.S.R.

* still very fast as compared to cavity losses $\Gamma \simeq 10^3$ Hz.

* other losses $\longrightarrow$ atomic decay outside cavity $\Gamma_{at} \lesssim 30$ Hz

Detection : hard to directly count μ-wave photons.
selective ionisation allows to determine atomic state ;
atoms enter cavity individually, interaction time tunable.

Creation of an EPR pair :

$$|\psi\rangle = |\uparrow\rangle_a |\downarrow\rangle_b - |\downarrow\rangle_a |\uparrow\rangle_b$$

two qu. bits a, b are entangled

* measuring a in $\uparrow$ $\longrightarrow$ b will be found in $\downarrow$

* not simply classical hidden variable :
  - same property holds for spin measurement along any axis

Cavity - atom entanglement :

$$|\psi(t=0)\rangle = |e,0\rangle$$

$\downarrow$ ½ of Rabi flop , $\Omega_R t = \pi/4$

$$\frac{1}{\sqrt{2}}\left[|e,0\rangle - i|g,1\rangle\right] = |\psi_1\rangle$$

If expt performed with flying atoms : entanglement travels to large distance

* Detecting atom in $|g\rangle$ projects cavity into $|n=1\rangle$ Fock state
  $\hookrightarrow$ better preparation of $|n=1\rangle$ than with coherent state

* Apply a $\pi/2$ coherent laser pulse on atom :

$$|\psi_1\rangle \longrightarrow |\psi_2\rangle = \frac{1}{2}\left[\left(|e,0\rangle + |g,0\rangle\right) - i\left(|g,1\rangle - |e,1\rangle\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[|e\rangle \cdot \left(\frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)\right) + |g\rangle\left(\frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)\right)\right]$$

→ detecting atom in $|e\rangle$ projects cavity into superposition $\frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$ with finite field amplitude.

→ phase of superposition can be controlled using $\pi/2$ coherent pulses with different phases

* Read-out cavity state with second atom:

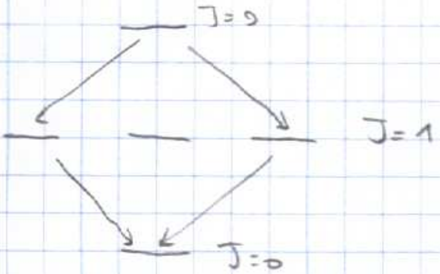
$$\begin{array}{l} \pi \text{ rotation} \\ (\Omega_R t = \pi/2) \end{array} \quad \begin{cases} |g_2, 1\rangle \longrightarrow |e_2, 0\rangle \\ |g_2, 0\rangle \longrightarrow |g_2, 0\rangle \end{cases}$$

$$\frac{1}{\sqrt{2}}\Big[|\underset{\text{1st atom}}{e}, 0, \underset{\text{2nd atom}}{g}\rangle - i|g, 1, g\rangle\Big] \longrightarrow \frac{1}{\sqrt{2}}\Big[|e, 0, g\rangle - i|g, 0, e\rangle\Big] =$$

$$= \frac{1}{\sqrt{2}}\Big[|e, g\rangle - i|g, e\rangle\Big] \otimes |0\rangle$$

* cavity disentangles from atoms
* entanglement is transferred to pair of atoms.

Analogous for 2-photon decay → generates pair of EPR-entangled photons



$$|\psi_+\rangle = \frac{1}{\sqrt{2}}\Big[|\sigma_+\sigma_-\rangle + |\sigma_-\sigma_+\rangle\Big] =$$

$$= \frac{1}{\sqrt{2}}\Big[|xx\rangle + |yy\rangle\Big]$$

Quantum phase gate

* 2π Rabi flop in cavity with 1 photon

$$|g,1\rangle \longrightarrow \underset{\uparrow}{-}|g,1\rangle \qquad (\Omega_R t = \pi)$$

typical -1 of 2π rotation of $S = 1/2$

* third atomic state $|i\rangle$

$$|\psi_{at}(t=0)\rangle = c_g|g\rangle + c_i|i\rangle$$

cavity in $|0\rangle$ → $c_g|g\rangle + c_i|i\rangle$

cavity in $|1\rangle$ → $-c_g|g\rangle + c_i|i\rangle$

* preserves state of cavity
* imprints a phase on atomic state that depends on cavity state.

↳ basic building block of QI processing

* between a pair of $\pi/2$ coherent laser pulses produces control-not gate.

$$|g,0\rangle \longrightarrow \frac{1}{\sqrt{2}}\left[|g,0\rangle + |i,0\rangle\right] \longrightarrow \frac{1}{\sqrt{2}}\left[|g,0\rangle + |i,0\rangle\right] \longrightarrow |g,0\rangle$$

$$|g,1\rangle \longrightarrow \frac{1}{\sqrt{2}}\left[|g,1\rangle + |i,1\rangle\right] \longrightarrow \frac{1}{\sqrt{2}}\left[-|g,1\rangle + |i,1\rangle\right] \longrightarrow |i,1\rangle$$

→ information on the cavity state is encoded into atom (provided one restricts to $n=0,1$ subspace)

→ cavity is left in the original state
↳ Non-Destructive measurement (QND)

→ it is meaningless to ask whether the final photon is the "same" as initial one or a copy.
(QM implies indistinguishability of particles)

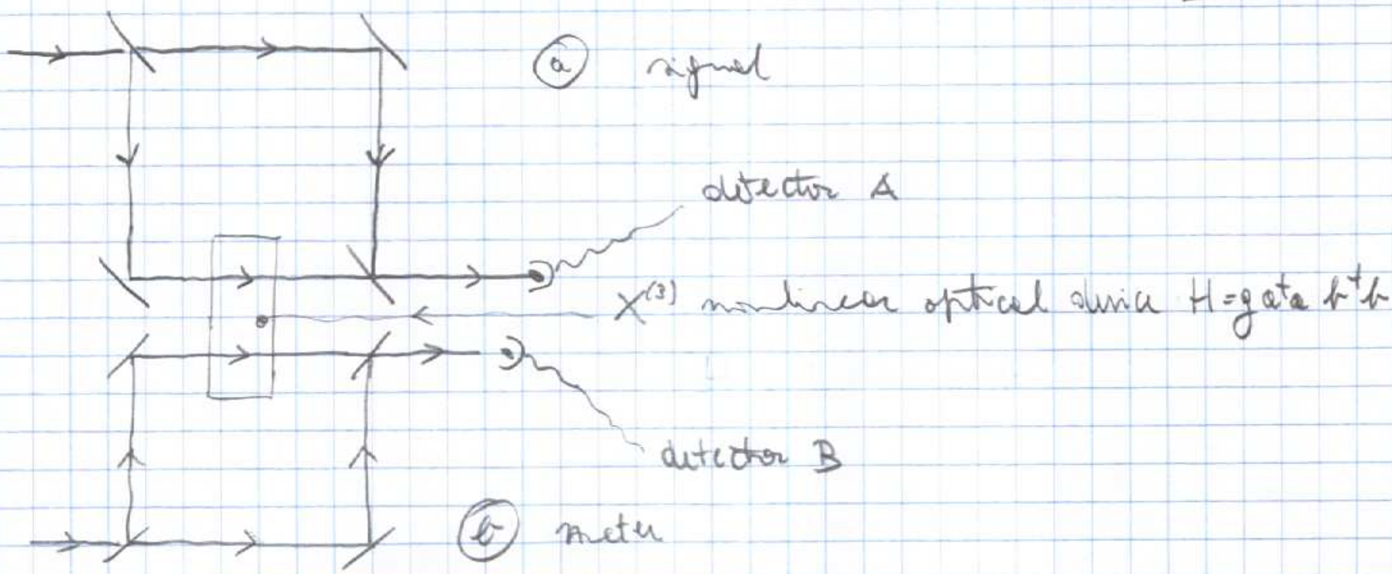
→ different from standard photodetection where the photon is absorbed and disappears forever

→ measurement can be repeated finding again the same value of photon number.

→ information on the field phase is however lost
$\mathrm{Tr}_{at}[\rho_{fin}] = P_0|0\rangle\langle 0| + P_1|1\rangle\langle 1|$ as in usual which-way detection scheme
↳ saves Heisenberg's uncertainty principle

A microscopic model of measurement back-action

10.12



signal : quantum system $\longrightarrow$ 1 photon

meter : classical system $\longrightarrow$ coherent state $\bar{n} \pm \sqrt{\bar{n}}$ photons.

Intensities in A and B $\longrightarrow$ sensitive to phase difference in the two Mach-Zehnders

$$N_B = \bar{n} \sin^2(\phi_B) \quad \text{with } \phi_B = \phi_B^0 + g\, a^\dagger a .$$

* shot-noise on N_B of the order $\sqrt{N_B} = \Delta N_B$
* max sensitivity if $\phi_B^0 \sim \pi/4$, $N_B = \bar{n}/2$
* determination of phase has $\bar{n}\,\Delta\phi_B \sim \Delta N_B = \sqrt{\bar{n}/2}$

$$\Longrightarrow \Delta\phi_B \sim 1/\sqrt{\bar{n}} .$$

* phase shift in B due to 1 photon in A is $\sim g$

$\Longrightarrow$ $\bar{n} \gtrsim 1/g^2$ to be able to detect it !

Fluctuations of phase of a due to fluctuations in b

$$\Delta\phi_B \sim g\sqrt{\bar{n}} \geq g\sqrt{1/g^2} \sim 1$$

$\Longrightarrow$ any measurement on B that is able to detect a single photon in A introduces a phase shift on A that spoils interference fringes at the exit of A M-Z interferometer.

Which-way information collected by B spoils wave properties of photon propagating in A