2.11

Propagation of wavepacket across FP.

incident wavepacket:

- $E_{inc}(t) = E_{inc}^{0} \cdot \exp\left(-t^2/2\sigma^2\right) \exp(-i\omega_0 t)$
- $\tilde{E}_{inc}(\omega) = \sqrt{2\pi}\,\sigma\, E_{inc}^{0} \exp\left(-(\omega-\omega_0)^2\sigma^2/2\right)$

$\hookrightarrow$ carrier at ω_0, pulse length σ,

pulse width in ω: $\Delta\omega = 1/\sigma$

transmission amplitude:

$$t(\omega) = \frac{t^2 e^{i\omega L/c}}{1 - r^2 e^{2i\omega L/c}} \simeq$$

$$\simeq t^2 \cdot e^{i\omega L/c} \sum_n \frac{1}{1 - r^2 - 2i(\omega-\omega_n) L/c\, r^2}$$

with $\omega_n = \frac{\pi c}{L} \cdot n$

$$= e^{i\omega L/c} \cdot \sum_n \frac{i \frac{c}{2L} T}{\omega - \omega_n + i\frac{cT}{2L}}$$

transmitted spectrum

$$\tilde{E}_{tr}(\omega) = t(\omega)\cdot \tilde{E}_{inc}(\omega) =$$

$$= \sqrt{2\pi}\,\sigma\, E^{\circ}_{inc} \cdot \exp\left[-(\omega-\omega_0)^2\,\sigma^2/2\right]\, e^{i\omega L/c}$$

$$\cdot \sum_n \frac{i\, cT/2L}{\omega-\omega_n + i\, cT/2L}$$

three regimes

i) $1/\sigma \ll cT/L$, i.e. $\sigma \gg \tau_{decay} = \Gamma^{-1}$

very long pulse, almost monochromatic

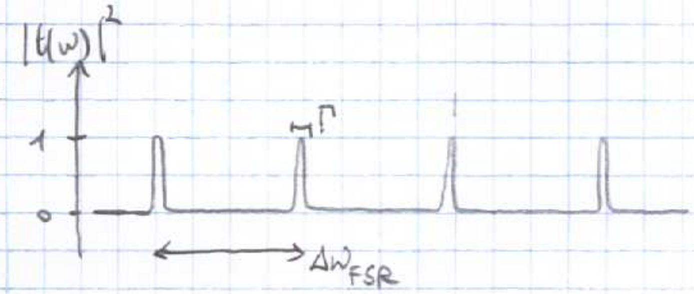
ii) $\frac{cT}{L} \ll 1/\sigma \ll \Delta\omega_{FSR}$ i.e. $\tau_{decay} \gg \sigma \gg L/c$

intermediate pulse:

- short w/r cavity ring-down.
- long w/r cavity round-trip time
  $\longrightarrow$ overlaps with at most 1 mode

iii) $\Delta\omega_{FSR} \ll 1/\sigma$ i.e. $\sigma \ll L/c \ll \tau_{decay}$

- short pulse, almost $\delta(t)$-shaped.
  $\longrightarrow$ overlaps with many cavity modes.



$$E_{tr}(t) = \int \frac{d\omega}{2\pi} \bar{E}_{tr}(\omega) =$$

$$= \int \frac{d\omega}{2\pi} \sqrt{2\pi}\, \sigma\, E_{inc}^{0}\, e^{-(\omega-\omega_0)^2 \sigma^2/2}\, e^{i\omega L/c} \cdot e^{-i\omega t}$$

$$\sum_n \frac{i\, cT/2L}{\omega - \omega_n + i\, cT/2L}$$

$$\int \frac{d\omega}{2\pi} \tilde{f}(\omega) \cdot \tilde{g}(\omega)\, e^{-i\omega t} = \int dt'\, f(t')\, g(t-t')$$

$$\tilde{f}(\omega) = \sqrt{2\pi}\, \sigma \cdot E_{inc}^{0}\, e^{-(\omega-\omega_0)^2 \sigma^2/2}$$

$$\hookrightarrow f(t) = E_{inc}^{0} \exp\left(-(t - L/c)^2 / 2\sigma^2\right) \exp(-i\omega_0 t)$$

$$\tilde{g}(\omega) = \sum_n \tilde{g}_n(\omega) = \sum_n \frac{i\, cT/2L}{\omega - \omega_n + i\, cT/2L}\, e^{i\omega L/c}$$

$$\hookrightarrow g_n(t) = -\frac{cT}{2L}\, \Theta\left(t - \frac{L}{c}\right) \exp\left[-i\omega_n (t - L/c)\right] \exp\left[-\Gamma (t - L/c)/2\right]$$

Response function of simple harmonic oscillator.

Case iii)

incident spectrum almost flat within each transm. peak.

$$\int \frac{d\omega}{2\pi} \tilde{f}(\omega) \tilde{g}(\omega)\, e^{-i\omega t} \simeq \sum_n \tilde{f}(\omega_n) \int \frac{d\omega}{2\pi} e^{-i\omega t}\, \tilde{g}_n(\omega) = \sum_n \tilde{f}(\omega_n) \cdot g_n(t)$$

$$= -\frac{cT}{2L} \sqrt{2\pi}\, \sigma\, E_{inc}^{0} \sum_n e^{-(\omega_n - \omega_0)^2 \sigma^2/2} \cdot \Theta(t - L/c)\, e^{-i\omega_n (t - L/c)}\, e^{-\Gamma (t - L/c)/2}.$$

* overall amplitude proportional to T :
  coupling of cavity modes with external incid. field

* many modes excited ; relative amplitude follows
  spectrum of incident wave packet

* beatings in transmitted field at $\omega_n - \omega_{n'}$

* ring-down with characteristic time $\tau_{decay} = \Gamma^{-1}$

* causality ensured by Θ function.

* delay L/c in crossing the cavity.

<u>Lemma</u> : Poisson's sum formula

$$\sum_n f(t - n\tau) = \frac{1}{\tau} \sum_m \tilde{f}\left(\frac{2\pi}{\tau} m\right) e^{-i\frac{2\pi m}{\tau} t}$$

thus $\displaystyle\sum_n \delta(t - n\tau) = \frac{1}{\tau} \sum_m e^{-i\frac{2\pi m}{\tau} t}$

So : $\displaystyle\sqrt{2\pi}\sigma \sum_n e^{-(\omega_n - \omega_0)^2 \sigma^2/2} e^{-i\omega_n t}$ with $\omega_n = 2\pi c/2L \cdot n$ is :

$$= \frac{2L}{c} \sum_n e^{-(t - n2L/c)^2/2\sigma^2}$$

$$E_{tr}(t) = \frac{cT}{2L} E^0_{inc} \quad \Theta(t - L/c) \quad e^{-\Gamma(t-L/c)/2} \cdot$$

$$\cdot \frac{2L}{c} \sum_n e^{-(t - n\,2L/c)^2/2\sigma^2} =$$

$$= T E^0_{inc} \; \Theta(t - L/c)\, e^{-\Gamma(t-L/c)/2} \sum_n e^{-(t-n\frac{2L}{c})^2/2\sigma^2}$$

→ series of gaussian wavepackets at distance $2L/c$

→ envelopped by exponential ring-down.

Physically: $\sigma \ll L/c$ means short wavepacket w/ 2 round trip

every time it hits each mirror, some of it is emitted

overall loss rate Γ

proportional to $T = t^2$: two mirror transmissions needed to enter and then exit

Case ii) $\tilde{f}(\omega_n) \neq 0$ only for single mode at $\bar{\omega}$

$$E_{tr}(t) = \frac{cT}{2L} \cdot \sqrt{2\pi}\,\sigma\, E^0_{inc}\, e^{-(\omega_0 - \bar{\omega})^2\sigma^2/2} \;\Theta(t - L/c) \cdot$$

$$\cdot\, e^{-i\bar{\omega}(t - L/c)} \; e^{-\Gamma(t-L/c)/2}$$

* frequency mismatch factor $e^{-(\omega_0-\bar{\omega})^2\sigma^2/2}$

* no beating, "monochromatic" version with exponential envelope

Case i)

only single resonance at $\bar{\omega}$ matters.

expand $\dfrac{i\Gamma/2}{\omega-\bar{\omega}+i\Gamma/2}$ around $\omega=\omega_0$.

More precisely:

$$\log \frac{i\Gamma/2}{\omega-\bar{\omega}+i\Gamma/2} \simeq \log \frac{i\Gamma/2}{\omega_0-\bar{\omega}+i\Gamma/2} + (\omega-\omega_0)\cdot(\alpha_R+i\alpha_I)+ \dots$$

$$E_{tr}(\omega) \simeq \sqrt{2\pi}\,\sigma\, E^0_{inc}\,\frac{i\Gamma/2}{\omega_0-\bar{\omega}+i\Gamma/2}\cdot e^{\alpha_R(\omega-\omega_0)}\, e^{i\alpha_I(\omega-\omega_0)}\cdot e^{-(\omega-\omega_0)^2\sigma^2/2}\, e^{i\omega L/c}$$

$$= \sqrt{2\pi}\,\sigma\, E^0_{inc}\,\frac{i\Gamma/2}{\omega_0-\bar{\omega}+i\Gamma/2}\,\exp\left[-\tfrac{1}{2}\left((\omega-\omega_0)^2\sigma^2-\alpha_R(\omega-\omega_0)\right)\right]\cdot$$

$$\cdot\exp\left[i\omega\left(\tfrac{L}{c}+\alpha_I\right)\right]\exp\left(-i\alpha_I\omega_0\right)$$

$$= \sqrt{2\pi}\,\sigma\, E^{0}_{inc} \frac{i\Gamma/2}{\omega_0 - \bar{\omega} + i\Gamma/2} \exp\left[-\frac{\sigma^2}{2}\left(\omega - \omega_0 - \frac{\alpha_R}{\sigma^2}\right)^2 + \frac{\alpha_R^2}{2\sigma^2}\right]$$

$$\cdot \exp\left[i\omega\left(\frac{L}{c} + \alpha_I\right)\right] \exp\left(-i\alpha_I\omega_0\right)$$

* shift of spectral c.o.m by $\frac{\alpha_R}{\sigma^2}$

* additional delay α_I.

$$E_R(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{E}_R(\omega) = E^{0}_{inc} \exp\left[-\frac{(t - L/c - \alpha_I)^2}{2\sigma^2}\right] \cdot$$

$$\cdot \exp\left(\alpha_R^2/2\sigma^2\right) \exp\left(-i\left(\omega_0 + \frac{\alpha_R}{\sigma^2}\right)\left(t - L/c - \alpha_I\right)\right)$$

$$\cdot \exp\left(-i\alpha_I\omega_0\right) \frac{i\Gamma/2}{\omega_0 - \bar{\omega} + i\Gamma/2}$$

$$= E^{0}_{inc} \exp\left[-(t - L/c - \alpha_I)^2/2\sigma^2\right] \exp\left(\alpha_R^2/2\sigma^2\right) \exp\left(-i\left(\omega_0 + \alpha_R/\sigma^2\right)\left(t - L/c\right)\right) \cdot$$

$$\cdot \frac{i\Gamma/2}{\omega_0 - \bar{\omega} + i\Gamma/2}$$

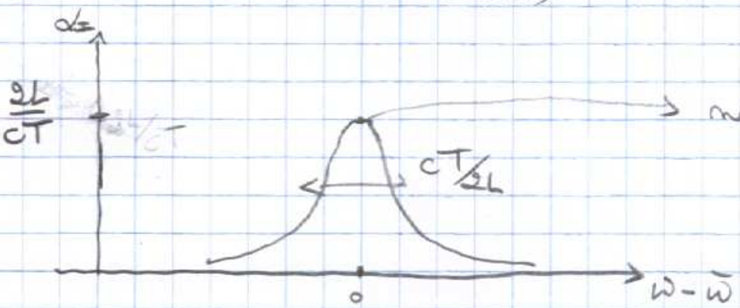
$$\alpha_R + i\,\alpha_I = \frac{\dfrac{d}{d\omega}\,\dfrac{\cancel{i\,cT/2L}}{\omega-\bar\omega+i\,cT/2L}}{\dfrac{\cancel{i\,cT/2L}}{\omega-\bar\omega+i\,cT/2L}} =$$

$$= \left(\omega-\bar\omega+i\,\frac{cT}{2L}\right)\cdot\left[-\frac{1}{\left(\omega-\bar\omega+i\,\frac{cT}{2L}\right)^2}\right] = -\frac{1}{\omega-\bar\omega+i\,\frac{cT}{2L}}$$

$$= -\frac{\omega-\bar\omega-i\,\frac{cT}{2L}}{(\omega-\bar\omega)^2+\left(\frac{cT}{2L}\right)^2}$$

$$\Longrightarrow \quad \alpha_R = \frac{\bar\omega-\omega}{(\omega-\bar\omega)^2+\left(\frac{cT}{2L}\right)^2} \longrightarrow \text{spectral shift}$$

$$\alpha_I = \frac{\frac{cT}{2L}}{(\omega-\bar\omega)^2+\left(\frac{cT}{2L}\right)^2} \longrightarrow \text{delay time.}$$



→ max delay time on resonance

$\tau_{delay} = \frac{cT}{2L} = \Gamma^{-1}$