# The LASER (Light Amplification by Stimulated Emission of Radiation)

2 level atom in $|e\rangle$


spontaneous emission:

emitted photon
* random direction
* random phase
* frequency within natural linewidth of $e \to g$

what happens if a photon is already present?

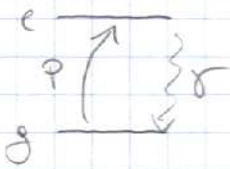
stimulated emission
* same direction, phase, frequency of incident photon
* works as an amplifier

Einstein rate equations: $\frac{d}{dt} P_e = - A - B \cdot n$

spontaneous emission

stimulated emission by $n$ existing photons.

Here, different perspective: O.B.E




$\gamma$ = spontaneous emission

$P$ = pumping rate. Simple model of more complicate physics involving many internal atomic levels.

$$\frac{d}{dt}\hat{\rho}_{at} = -i\left[H_0, \hat{\rho}_{at}\right] + \frac{\gamma}{2}\left[2\sigma^- \rho_{at}\sigma^+ - \sigma^+\sigma^-\rho_{at} - \rho_{at}\sigma^+\sigma^-\right]$$

$$+ \frac{P}{2}\left[2\sigma^+\rho_{at}\sigma^- - \sigma^-\sigma^+\rho_{at} - \rho_{at}\sigma^-\sigma^+\right]$$

$\hookrightarrow$ <u>pumping</u> term

<u>OBE</u>

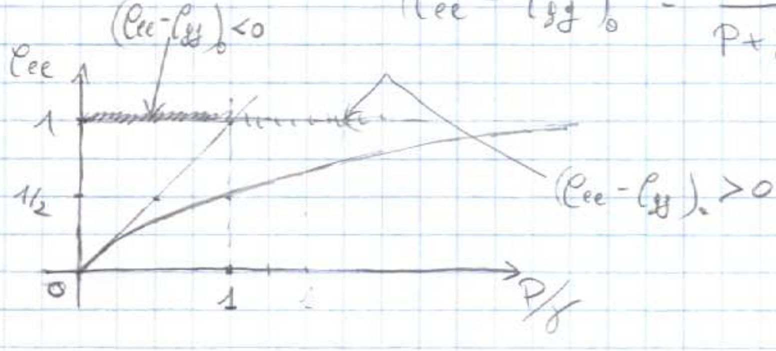
$$\begin{cases} \dot{\rho}_{ee} = -\gamma\rho_{ee} + P\rho_{gg} + i\left(\Omega_L(t)\rho_{ge} - \Omega^*(t)\rho_{eg}\right) \\ \dot{\rho}_{gg} = \gamma\rho_{ee} - P\rho_{gg} - i\left(\Omega_L(t)\rho_{ge} - \Omega^*(t)\rho_{eg}\right) \\ \dot{\rho}_{eg} = -\frac{\gamma+P}{2}\rho_{eg} + i\Omega_L(t)\left(\rho_{gg}-\rho_{ee}\right) - i\omega_{eg}\rho_{eg} \end{cases}$$

<u>steady state at $\Omega_L = 0$</u> :

$$\rho_{ee}^0 = \frac{P}{\gamma}\rho_{gg}^0 = \frac{P}{\gamma}\left(1-\rho_{ee}^0\right)$$

$$\Longrightarrow \rho_{ee}^0 = \frac{P}{P+\gamma}, \quad \rho_{gg}^0 = \frac{\gamma}{P+\gamma}$$

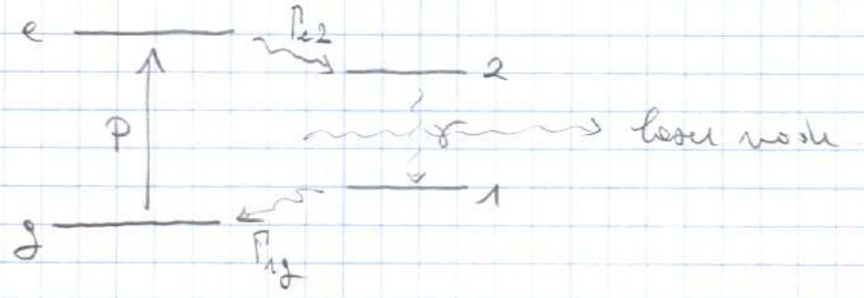
$$\left(\rho_{ee}-\rho_{gg}\right)_0 = \frac{P-\gamma}{P+\gamma}$$




<u>POPULATION INVERSION</u>

<u>population inversion requires $P > \gamma$</u>

Other level schemes to obtain population inversion at lower P:



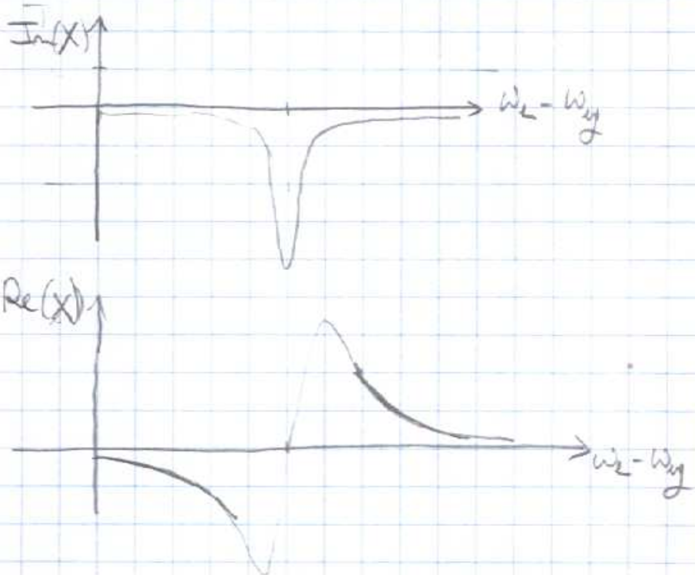

if $\Gamma_{e2} >> \Gamma_{1g}$ population accumulates in 2, state 1 quickly depleted

$\Longrightarrow$ easier inversion of population between 1 and 2.

For small $\Omega$ (weak saturation):

$$\langle d \rangle = \frac{d^2}{\omega_{eg} - \omega_L - i\,(\gamma+P)/2}\,(\rho_{gg} - \rho_{ee})_0\, E_L .$$

* if $(\rho_{gg} - \rho_{ee})_0 > 0$ : absorption
* if $(\rho_{gg} - \rho_{ee})_0 < 0$ : amplification



NOTE: tail of amplification line can be used for superluminal propagation at $v_g > c$ !!

* L. J. Wang, A. Kuzmich, A. Dogariu, Nature 406, 277 (00)

Saturation

$$\langle d \rangle = \frac{d^2}{\omega_{g}-\omega_L - i(\gamma+P)/2} \left(\rho_{gg}-\rho_{ee}\right) E_L$$

with $\rho_{gg}-\rho_{ee}$ to be evaluated self-consistently

$$\frac{d}{dt}\left(\rho_{ee}-\rho_{gg}\right) = -2\gamma\rho_{ee} + 2P\rho_{gg} + 2i\left(\Omega\rho_{ge} - \Omega^{*}\rho_{eg}\right)$$

$$= -2(\gamma+P)\left[\left(\rho_{ee}-\rho_{gg}\right) - \left(\rho_{ee}-\rho_{gg}\right)_0\right] +$$

$$-2|E_L|^2 d^2 \frac{(\gamma+P)}{(\omega_{g}-\omega_L)^2 + (\gamma+P)^2/4}\left(\rho_{ee}-\rho_{gg}\right)$$

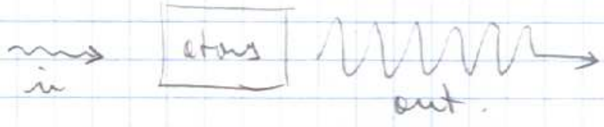
$$\left(\rho_{ee}-\rho_{gg}\right)\left[2\cancel{(\gamma+P)} + 2|E_L|^2 d^2 \frac{\cancel{(\gamma+P)}}{(\omega_{g}-\omega_L)^2 + (\gamma+P)^2/4}\right] = 2\cancel{(\gamma+P)}\left(\rho_{ee}-\rho_{gg}\right)_0$$

$$\left(\rho_{ee}-\rho_{gg}\right) = \frac{\left(\rho_{ee}-\rho_{gg}\right)_0}{1 + \frac{d^2|E_L|^2}{(\omega_{g}-\omega_L)^2 + (\gamma+P)^2/4}}$$

→ whatever the sign of $(\rho_{ee}-\rho_{gg})_0$, $E_L$ tends to equilibrate populations towards $\rho_{ee} = \rho_{gg}$

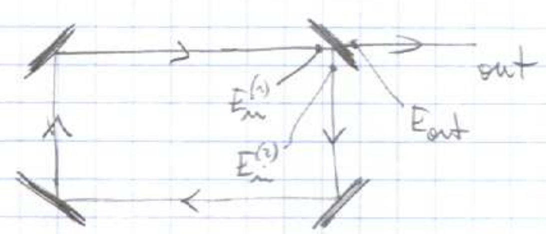
## Amplification:

* incident beam get amplified by stimulated emission.
* frequency, direction and phase preserved




## Laser oscillation

ring-cavity system (for simplicity of description):




* amplified field is recycled
* if amplification overcomes losses, self-oscillation is expected.

$$\left.\begin{aligned} E_{out} &= t \cdot E_{in}^{(1)} \\ E_{in}^{(2)} &= r\, E_{in}^{(1)} \end{aligned}\right\}$$ partially transmitting output mirror.

$E_{in}^{(1)} = \exp(i k \cdot L)\, E_{in}^{(2)}$, with $L$ = length of ring

$$k = \frac{\omega}{c}\sqrt{1 + 4\pi n \chi_{at}}$$

We assume:

- ring cavity tuned on atomic transition $\frac{\omega_{eg}}{c} L = 2\pi M$
- laser field oscillates at $\omega_L = \omega_{eg}$.
- reflection amplitude $r \in \mathbb{R}^+$

$$k = \frac{\omega_L}{c}\sqrt{1+4\pi n \chi_{el}} \simeq \frac{\omega_{eg}}{c} + \frac{2\pi n\,\omega_{eg}}{c}\,\frac{d^2}{\omega_{eg}-\omega_L - i\,(\gamma+P)/2}\,(\rho_{gg}-\rho_{ee})$$

$$\overset{\omega_{eg}=\omega_L}{\simeq} \frac{\omega_{eg}}{c} + \frac{4\pi n\,\omega_{eg}\,d^2}{(\gamma+P)c}\,(\rho_{gg}-\rho_{ee})\,i$$

with $\rho_{gg}-\rho_{ee}$ to be calculated self-consistently
(nonlinear saturation effect)

$$E_n^{(2)} = r\,E_n^{(1)} = r\,\exp(i\,k\,L)\,E_n^{(1)}$$

$$= \underbrace{r}_{\to\ 1 \geq r \geq 0} \cdot \underbrace{\cancel{\exp\left(i\frac{\omega_{eg}}{c}L\right)}}_{1} \cdot \exp\left(-\frac{4\pi n\,d^2}{(\gamma+P)}\,\frac{\omega_{eg}}{c}\,L\,(\rho_{gg}-\rho_{ee})\right)E_n^{(1)}$$

$$\Rightarrow\quad 1 = r\cdot\exp\left[-\frac{4\pi n\,d^2}{(\gamma+P)}\,\frac{\omega_{eg}}{c}\,L\,(\rho_{gg}-\rho_{ee})\right] \quad \text{or} \quad E_n^{(1)} = 0.$$

$$0 < -\log r = \frac{4\pi n\,d^2}{(\gamma+P)}\,\frac{\omega_{eg}}{c}\,L\,(\rho_{ee}-\rho_{gg}) =$$

$$= \frac{4\pi n\,d^2}{(\gamma+P)}\,\frac{\omega_{eg}}{c}\,L\,\frac{(\rho_{ee}-\rho_{gg})_0}{1+\dfrac{d^2\,|E_n^{(1)}|^2}{\underbrace{(\omega_{eg}-\omega_L)^2}_{0} + (\gamma+P)^2/4}}$$

→ approximate field as uniform around cavity or if $r \simeq 1$.

$$(-\log r)\left[1+\frac{4d^2}{(\gamma+P)^2}\,|E_n^{(1)}|^2\right] = \frac{4\pi n\,d^2}{\gamma+P}\,\frac{\omega_{eg}}{c}\,L\,(\rho_{ee}-\rho_{gg})_0$$

$$\boxed{|E_n^{(1)}| = \frac{(\gamma+P)^2}{4d^2}\cdot\left\{\frac{4\pi n\,d^2}{(\gamma+P)\,(-\log r)}\,\frac{\omega_{eg}}{c}\,L\,(\rho_{ee}-\rho_{gg})_0 - 1\right\}}$$

* if $\frac{4\pi n d^2}{\gamma+P} \frac{\omega_{eg}}{c} L \, (\rho_{ee}-\rho_{gg})_0 < (-\log r)$ :

  * only $E_{in}^{(2)} = 0$ solution exists

  * no emitted radiation $E_t = 0$
    (more precisely : no coherent radiation)

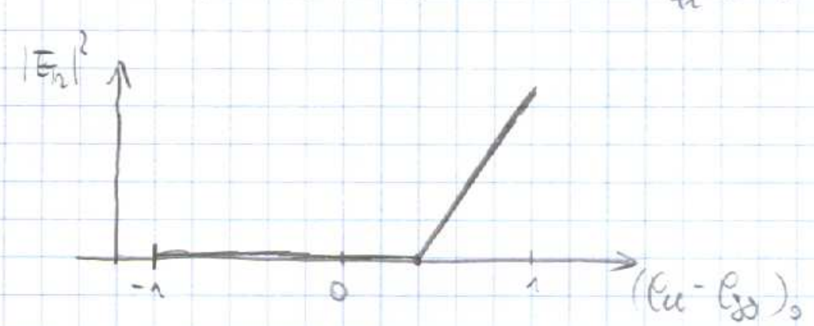
* if $\frac{4\pi n d^2}{\gamma+P} \frac{\omega_{eg}}{c} (\rho_{ee}-\rho_{gg})_0 > (-\log r)$ :

  * two solutions possible
    - $E_{in}^{(2)} = 0$ (turns out dynamically unstable)
    - $|E_{in}^{(2)}|^2 = \frac{(\gamma+P)^2}{4d^2} \left\{ \frac{4\pi n d^2}{(\gamma+P)(-\log r)} \frac{\omega_{eg}}{c} L \, (\rho_{ee}-\rho_{gg})_0 - 1 \right\}$

    (turns out dynamically stable $\rightarrow$ on).

* phase of $E_{in}^{(2)}$ remains undetermined :
  equations of motion are invariant under
  $U(1)$ symmetry $E_{in}^{(2)} \rightarrow E_{in}^{(2)} e^{i\phi}$

* coherent emission $E_{tr} = t \, E_{in}^{(2)}$



Is there an analogy with phase transitions in statistical mechanics?

see e.g.:

- V. DeGiorgio and Scully, Phys Rev A 2, 1170 (70)
- R. Graham and H. Haken, Z. Phys. 237, 31 (70)
- H. Haken, Rev Mod. Phys. 47, 67 (1975)
- A. Gatti and L. Lugiato, Phys. Rev. A 52, 1675 (95)
- I. Carusotto and C. Ciuti, Phys. Rev B 72, 125335 (05)

Critical point

* discontinuity in some (thermodynamic) quantity
* change in symmetry, e.g. spontaneous symmetry breaking
  → long range order
  → macroscopic diffusion time of order parameter

examples:

ferromagnetism breaks rotational symmetry SO(3)
Ising model breaks $Z_2$ (discrete!)
BEC breaks U(1) of matter field phase

$$\psi(r) \to \psi(r)\, e^{i\theta}$$

long-range order
$$\begin{cases} \lim_{|r-r'|} \langle \psi^\dagger(r)\, \psi(r') \rangle \to \text{finite} \\ \text{decay time } \tau_c \text{ of } \langle \psi^\dagger(t+\tau)\, \psi(t) \rangle \text{ goes to } \infty \text{ for large system limit} \end{cases}$$

Is lasing a phase transition?

* discontinuity of some observable?
  - YES at mean-field level of semiclassical equations (as done here)
  - NO in exact theory of single-mode cavity

* change in symmetry
  - NO. Concept of long-range order is not applicable as single-mode cavity is necessarily finite-size object
  - MAY-BE. Phase diffusion time scales as $\tau \sim N_{ph}$, $N_{ph}$ being number of photons in lasing mode (Schawlow-Townes linewidth).

Another crucial issue: laser is not equilibrium phenomenon
- energy pumped in system and then lost
- stationary state: dynamical balance rather than thermal equilibrium
- statistical distribution not Boltzmann $e^{-\beta\hat{H}}$

Open question

what happens in spatially-extended laser devices?

e.g. VCSELs (vertical cavity, surface emitting lasers)

- Long-range order concept well defined.
- full calculation including fluctuations still to be built.
- not clear what critical properties should be

Simplified model of laser field dynamics

Along the lines of 2.9 - 2.10:

$$i\frac{d}{dt}E = \left(\omega_0 - i\frac{\Gamma}{2}\right)E + \underbrace{\frac{ic}{2L}t_m}_{\eta} E_{inc}(t) + i\frac{P_0}{2}\underbrace{\frac{E}{1+\beta|E|^2}}$$

saturated amplification
(can be derived in the "good cavity" limit where atomic $\gamma_e \gg \Gamma$)

No incident field $E_{inc} = 0$

→ solution $E = 0$ always present.

dynamically stable only for $P_0 < \Gamma$.

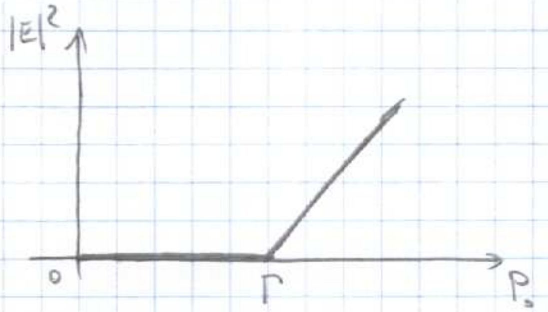
then becomes unstable to amplification:

$$E \sim E_0 \exp\left[(P_0 - \Gamma)\, t/2\right]$$

→ for $P_0 > \Gamma$, another solution appears such

that $|E|^2 = \frac{1}{\beta} \frac{P_0 - \Gamma}{\Gamma}$ ;

phase of $E$ undetermined, eq. of motion is symmetric under $E \to E e^{i\theta}$.

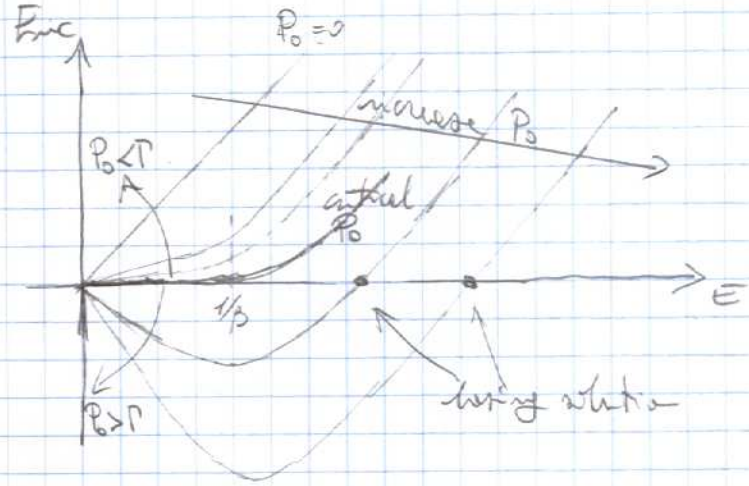



In the presence of incident field $E_{inc}$ :

assume $E_{inc}(t) = E^0_{inc}\, e^{-i\omega_0 t}$ with $E^0_{inc} \in \mathbb{R}^+$

solution $E(t) = E^0 e^{-i\omega_0 t}$ satisfying equation of state:

$$\left(\Gamma - \frac{P_0}{1+\beta|E|^2}\right) E^0 = \eta\, E^0_{inc}$$

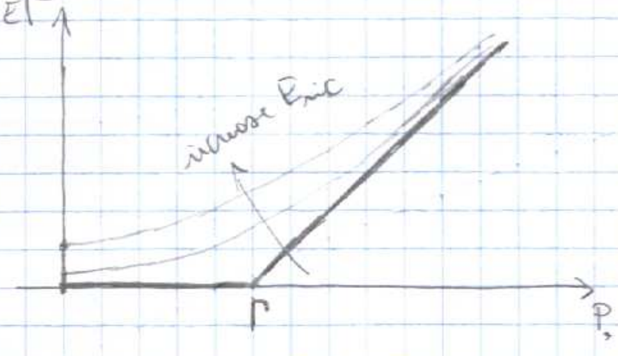



<u>effect of incident field</u>

* fix phase of laser emission
  - explicit symmetry breaking

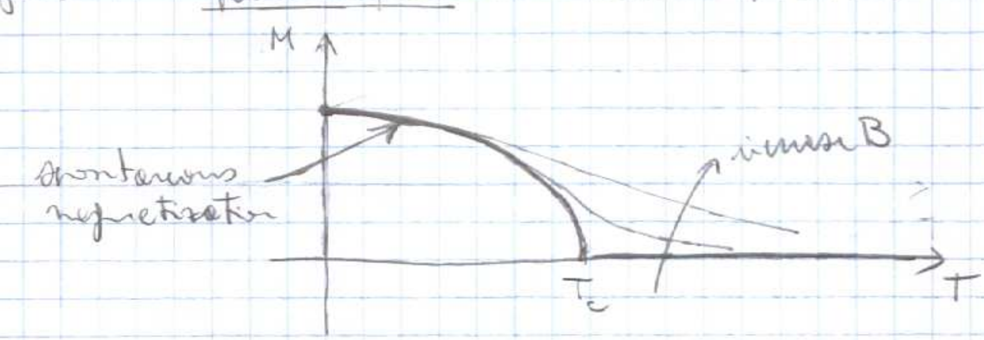
* smooth out critical point

* reinforce amplitude of emission




↳ <u>"externally injected laser"</u>

* Analogous to <u>ferromagnet</u> in external field $B$:




↳ direction of $\vec{M}$ pinned by direction of $\vec{B}$.