Lecture 8 : - States of the quantized e m field: coherent states vs single photons
- Spontaneous emission.

$$H = \sum_\alpha \frac{1}{2m_\alpha}\left(P_\alpha - \frac{e_\alpha}{c} A(R_\alpha)\right)^2 + \int \frac{d^3k}{(2\pi)^3} \sum_\varepsilon \hbar c k \; a^\dagger_{k,\varepsilon} a_{k\varepsilon}$$

with $$A(R_\alpha) = \int \frac{d^3k}{(2\pi)^3} \sum_\varepsilon \sqrt{\frac{2\pi\hbar c}{k}} \left[\hat{\varepsilon} e^{ik\cdot z} a_{k,\varepsilon} + \right.$$
$$\left. + \hat{\varepsilon}^* e^{-ik\cdot z} a^\dagger_{k,\varepsilon} \right]$$

where $\hat{\varepsilon}$ = unit vector of polarization state.

For instance, if $\underline{k} = k\cdot\hat{z}$:

* $\hat{\varepsilon} = \{\hat{x}, \hat{y}\}$ (linear basis)

[alternative choice $\hat{\varepsilon} = \{ \hat{e}_\pm = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})\}$ (circular basis)]

Single-photon states

$$|\psi\rangle = \int \frac{d^3k}{(2\pi)^3} \sum_\varepsilon \phi(k,\varepsilon)\cdot a^\dagger_{k,\varepsilon} |vac\rangle$$

$$\langle\psi|\psi\rangle = \int \frac{d^3k}{(2\pi)^3} \sum_\varepsilon |\phi(k,\varepsilon)|^2 = 1$$

[in fact $\langle vac| a_{k,\varepsilon} a^\dagger_{k',\varepsilon'} |vac\rangle = (2\pi)^3 \delta(k-k')\, \delta_{\varepsilon\varepsilon'}$]

$\phi(k,\varepsilon)$ is photon wavefunction in a first-quantization picture!

Probability of having photon in k,ε : $P(k,\varepsilon) = |\phi(k,\varepsilon)|^2$

Photon is vectorial, <u>transverse</u> particle:

vector wavefunction $\vec{\phi}(\underline{k}) = \sum_{\varepsilon} \hat{\varepsilon} \cdot \phi(k, \varepsilon)$ is a transverse field such that $\underline{k} \cdot \vec{\phi}(k) = 0$.

in real space: $\vec{\phi}(\underline{r}) = \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot r} \vec{\phi}(k)$ is such that $\nabla \cdot \vec{\phi} = 0$.

It is impossible to build a spatially localised, divergenceless function $\Rightarrow$ no "position operator" for photon.

Defining $\vec{\alpha}(r) = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \hat{\varepsilon} \cdot a(k, \varepsilon) e^{ikr}$, we have:

$$[\alpha_i(r), \alpha_j^\dagger(r')] = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \varepsilon_i \varepsilon_j e^{ik(r-r')} =$$

$$= \int \frac{d^3k}{(2\pi)^3} P_\perp(k) e^{ik(r-r')} = \delta_{ij}^\perp(r-r')$$

transverse-delta function.

<u>Explicit form</u> $\delta_{ij}^\perp(r) = \frac{2}{3}\delta_{ij}\,\delta(r) + \frac{\eta(r)}{4\pi r^3}\left(\frac{3 r_i r_j}{r^2} - \delta_{ij}\right)$

with $\eta(r) = \begin{cases} \to 0 & \text{for } r \to 0 \\ = 1 & \text{for large } r \end{cases}$ a kind of P.P.

$\hookrightarrow$ Cohen-Tannoudji, Dupont-Roc, Grynberg, chap A_I.

$\Rightarrow$ Localised photon state $|\psi_r\rangle = \alpha^\dagger(r)|vac\rangle$ <u>not really localised</u>

$\langle\psi_r| n(r') |\psi_r\rangle = \langle vac| \alpha(r)\, \alpha^\dagger(r')\, \alpha(r')\, \alpha^\dagger(r) |vac\rangle \neq 0$ also for $r \neq r'$!

Time-evolution of free e.m. field

$$|\psi(t)\rangle = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \phi(k,\varepsilon;t)\, a^{\dagger}_{k,\varepsilon} |vac\rangle$$

$$H|\psi(t)\rangle = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \phi(k,\varepsilon;t)\, \hbar c k\, a^{\dagger}_{k,\varepsilon} |vac\rangle$$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} i\hbar \frac{d}{dt}\phi(k,\varepsilon;t)\, a^{\dagger}_{k,\varepsilon} |vac\rangle$$

$$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle \implies i\hbar \frac{d}{dt}\phi(k,\varepsilon;t) = \hbar c k\, \phi(k,\varepsilon;t)$$

$\downarrow$

k-space "Schrödinger"-eq for photon

to be compared to:

$$i\hbar \frac{d}{dt}\phi(k,t) = +\frac{\hbar^2 k^2}{2m}\phi(k,t) \quad \text{for}$$

standard massive particle

In real space:

i) simplified case: $\phi(k,\varepsilon) = \delta_{\varepsilon,\hat{x}}\,(2\pi)^2\,\delta(k_x)\,\delta(k_y)\cdot\bar{\phi}(k_z)$,

$\bar{\phi}(k_z) \neq 0$ only for $k_z > 0$

$\longrightarrow$ { models one-dimensional field propagating in $+\hat{z}$ direct.
plane wave fronts

$$\vec{\alpha}(r) = \hat{x}\,\bar{\alpha}(r)$$

$$i\hbar \frac{\partial}{\partial t} \vec{\alpha}(r) = \hbar c \left(-i\partial_z\right) \vec{\alpha}(r)$$

$$\Longrightarrow \quad \frac{d}{dt} \vec{\alpha}(r) = -c\,\partial_z \vec{\alpha}$$ rigid translation in time

$$\vec{\alpha}(r,t) = \vec{\alpha}(\vec{r} - ct)$$

ii) general case $\longrightarrow$ Try to recover Maxwell's eqs.

$$\vec{B}(r) = \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{2\pi\hbar c}{k}}\, i\left(\mathbf{k} \times \vec{\alpha}(k)\right) e^{ik\cdot r} = \nabla \times \vec{\alpha}(r)$$

$$\frac{d}{dt} \vec{B}(r) = \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{2\pi\hbar c}{k}}\, i\left(k \times \dot{\vec{\alpha}}(k)\right) e^{ikr} = \nabla \times \dot{\vec{\alpha}}(r)$$

if we set $\vec{\mathcal{E}}(r) = -\frac{1}{c}\frac{\partial \vec{\alpha}}{\partial t}$

$$\Longrightarrow \quad \nabla \times \vec{\mathcal{E}} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

for free-field:

$$\vec{\mathcal{E}}(r) = \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{2\pi\hbar c}{k}} \left(-\frac{1}{c}\right)(-ick)\, \vec{\alpha}(k)\, e^{ikr} =$$

$$= \int \frac{d^3k}{(2\pi)^3}\, i \sqrt{2\pi\hbar c k}\; \vec{\alpha}(k)\, e^{ikr}$$

$$\frac{d}{dt} \vec{\mathcal{E}}(r) = \int \frac{d^3k}{(2\pi)^3}\, i\,(-ick) \sqrt{2\pi\hbar c k}\; \vec{\alpha}(k)\, e^{ikr} =$$

$$= \int \frac{d^3k}{(2\pi)^3} \left(-ck^2\right) \sqrt{\frac{2\pi\hbar c}{k}}\; \vec{\alpha}(k)\, e^{ikr} =$$

$$= c \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{2\pi\hbar c}{k}} \; \underline{k} \times (\underline{k} \times \vec{\alpha}(k)) \, e^{ikr} =$$

$$= c \;\; \nabla \times \mathcal{B}(r)$$

$$\Longrightarrow \quad \nabla \times \mathcal{B}(r) = \frac{1}{c} \frac{\partial \mathcal{E}}{\partial t}$$

Maxwell's eqs are the "analog" of Schrödinger eq. for the photon field

$$\vec{\alpha}(r,t) = \int d^3r' \; U(r-r', t-t') \vec{\alpha}(r', t'), \quad U = \text{Maxwell propagator}$$

* Free-field dynamics conserves total # of photons.

* In addition: <u>absorption</u> and <u>emission</u> processes

$\hookrightarrow$ not included in "Schrödinger" eq. for single-photon wavefunction.

<u>NOTE</u>: Schrödinger eq. for single photon can be generalized to several photons

$$|\psi\rangle = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \sum_{\varepsilon} \sum_{\varepsilon'} \phi(k,\varepsilon; k', \varepsilon') . \; a^{\dagger}(k,\varepsilon) \, a^{\dagger}(k', \varepsilon') \, |vac\rangle$$

$\hookrightarrow$ only symmetric part of $\phi(k,\varepsilon; k', \varepsilon')$ with respect to $(k,\varepsilon) \leftrightarrow (k', \varepsilon')$ matters

$$i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle \quad \Longrightarrow \quad i\hbar \frac{d}{dt} \phi = \hbar c (k + k') \phi$$

which is analogous to

$$i\hbar \frac{d}{dt}\psi(k_1,k_2) = \left(\frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m}\right)\psi(k_1,k_2)$$

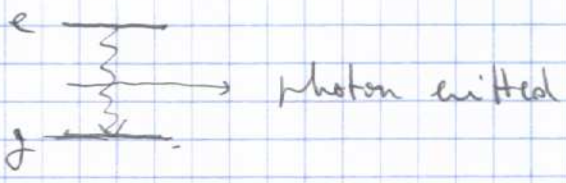
for 2 massive particles.

In <u>real space</u>: in the limiting 1D of (*)

$$i\hbar \frac{d}{dt}\alpha(r_1,r_2) = -ic\left(\partial_{r_1} + \partial_{r_2}\right)\alpha(r_1,r_2)$$

<u>In general</u>: $\alpha(r_1,r_2;t) = \int d^3r_1'\, d^3r_2'\; U(r_1-r_1',t-t')\, U(r_2-r_2',t-t') \cdot \alpha(r_1',r_2',t')$

$\boxed{\text{Emission from 2-level atom:}}$



$$H = \int \frac{d^3k}{(2\pi)^3}\sum_{\varepsilon} \hbar ck\; a^\dagger(k,\varepsilon)\, a(k,\varepsilon) + \frac{1}{2m}\left(P - \frac{q}{c}A(R)\right)^2 + V_{coul}(R)$$

Hydrogen atom $\rightarrow \begin{cases} \{R,P\} = \text{coordinate/momentum of electron} \\ V_{coul}(R) = \text{Coulomb potential of nucleus} \end{cases}$

↳ state $e = 2P$, state $g = 1S$ ⟶ <u>Lyman α</u> transition of H

$V_{coul} + \frac{P^2}{2m}$ gives $E_e - E_g = -\frac{Ry}{2^2} - \left(-\frac{Ry}{1^2}\right) = \frac{3}{4}Ry$

with $Ry \simeq \frac{1}{2}mc^2\cdot\alpha^2 = 13{,}6\,eV$

$$H = H_0 + H_{int} = \frac{P^2}{2m} + V_{coul}(R) + \int \frac{d^3k}{(2\pi)^3}\sum_{\varepsilon}\hbar ck\; a^\dagger(k,\varepsilon)\, a(k,\varepsilon) +$$

$$- \frac{q}{2mc}\left(P\cdot A(R) + A(R)\cdot P\right) + \frac{q^2}{2mc^2}A(R)^2$$

At lowest order in q: emission process occurs via $\left(P \cdot A(R) + A(R)\,P\right)$ term.

Emitted photon has energy $E_e - E_g = \frac{3}{8} mc^2\alpha^2 \simeq 10{,}2\,eV$.
and wavelength $\lambda = \frac{2\pi\hbar c}{E_e - E_g} \simeq \frac{2\pi\hbar c}{\frac{3}{8}mc^2\alpha^2} = \frac{16\pi}{3}\frac{\hbar}{mc}\frac{1}{\alpha^2} \simeq$
$\simeq 120\,nm$

$\Longrightarrow \quad \lambda \gg a_B$, with Bohr radius $a_B = \frac{\hbar}{mc\alpha} \simeq 0{,}52\,\mathring{A}$

NOTE: $\lambda_c = \frac{\hbar}{mc}$ is "Compton wavelength"
→ wavelength of photon of energy mc^2
→ minimal distance over which particle can be localized before QFT

Long-wavelength approx. $A(R) \simeq A(0)$
$$H_{int} = -\frac{q}{mc} A(0) \cdot P$$

Fermi golden rule: $\Gamma = \frac{2\pi}{\hbar} \int df \; |\langle f | H_{int} | i \rangle|^2 \; \delta(E_f - E_i)$

in our case $\int df : \int \frac{d^3k}{(2\pi)^3} \cdot \sum_{\varepsilon}$ of possible states of emitted photon.

$$\begin{cases} |k,\varepsilon\rangle = a^{+}(k,\varepsilon)\,|vac\rangle\,|g\rangle \\ |i\rangle = |e\rangle \end{cases}$$

$$\langle k,\varepsilon | H_{int} | i \rangle = -\frac{q}{mc} \langle k,\varepsilon | A(0) \cdot P | i \rangle =$$

$$= -\frac{q}{mc} \langle vac | a(k,\varepsilon)\, A(0) | vac \rangle \times$$

$$\times \langle 1S | P | 2P \rangle =$$

$$= -\frac{q}{mc} \langle vac | a(k,\varepsilon) \int \frac{d^3k'}{(2\pi)^3} \sum_{\varepsilon'} \sqrt{\frac{2\pi\hbar c}{k}}\; \hat{\varepsilon}' \left(\cancel{a(k',\varepsilon')} + a^{+}(k',\varepsilon') \right) |vac\rangle$$
$$\times \langle 1S | P | 2P \rangle$$

$$= -\frac{q}{mc}\left(\hat{\varepsilon}\cdot \langle 1S|\vec{P}|2P\rangle\right)\sqrt{\frac{2\pi\hbar c}{k}}$$

From Hydrogen atom theory:

$$\langle 1S|\vec{P}|2P\rangle = \hat{x}\cdot\left[-i\frac{\hbar}{a_0}\frac{32}{81\sqrt{2}}\right]$$ for initial 2P state polarized $\parallel \hat{x}$

$$= \hat{x}\cdot P_{eg}$$

$$\Gamma = \frac{2\pi}{\hbar}\int\frac{d^3k}{(2\pi)^3}\cdot\sum_{\varepsilon}\left|-\frac{q}{mc}\sqrt{\frac{2\pi\hbar c}{k}}\,(\hat{\varepsilon}\cdot\hat{x})\,P_{eg}\right|^2\cdot\delta(E_e-E_g-\hbar ck)$$

$$= \frac{2\pi}{\hbar}\frac{1}{(2\pi)^3}\frac{q^2}{m^2c^2}\int_0^\infty dk\,k^2\,\frac{2\pi\hbar c}{k}\int d^2\Omega\;\hat{x}\cdot P_\perp(\Omega)\hat{x}\cdot|P_{eg}|^2\cdot\delta(E_e-E_g-\hbar ck)$$

where $P_\perp(\Omega) = \sum_\varepsilon \hat{\varepsilon}\cdot\hat{\varepsilon}$ = projector orthogonal to direction of Ω

$$= \frac{1}{2\pi}\frac{q^2}{m^2c}|P_{eg}|^2\int_0^\infty dk\cdot k\;\delta(E_e-E_g-\hbar ck)\cdot$$

$$\cdot\int d^2\Omega\cdot\left[\hat{x}^2-(\hat{x}\cdot\hat{n})^2\right] =$$

$$= \frac{1}{2\pi}\frac{q^2}{m^2c}|P_{eg}|^2\frac{1}{\hbar^2c^2}(E_e-E_g)\cdot\left[4\pi-\int_{-1}^{1}d\cos\vartheta\,d\phi\;\cos^2\vartheta\right] =$$

$$= \frac{1}{2\pi}\frac{q^2}{m^2c^3\hbar^2}(E_e-E_g)\left[4\pi-2\pi\cdot\frac{2}{3}\right]|P_{eg}|^2 =$$

$$= \frac{4q^2}{3m^2c^3\hbar^2}(E_e-E_g)\,|P_{eg}|^2$$

For a generic atom:

$$\langle e | P | g \rangle = \langle e | m \frac{1}{i\hbar} [R, H_0] | g \rangle =$$

$$= \frac{m}{i\hbar} (E_g - E_e) \langle e | R | g \rangle = + i \frac{m}{\hbar} (E_e - E_g) R_{eg}$$

$$\Gamma = \frac{4q^2}{3 m^2 c^3 \hbar^2} (E_e - E_g) \cdot m^2 \left(\frac{E_e - E_g}{\hbar}\right)^2 |R_{eg}|^2 =$$

$$= |q \cdot R_{eg}|^2 \frac{4}{3\hbar} \frac{\omega_{eg}^3}{c^3}$$

As $\hbar\omega_{eg} \simeq mc^2 \alpha^2$

$R_{eg} \simeq \frac{1}{\alpha} \frac{\hbar}{mc} \simeq$ Bohr radius.

$\alpha = \frac{q^2}{\hbar c} = 1/137$
fine structure constant
↓
small parameter of QED.

$$\Longrightarrow \Gamma \simeq q^2 \cdot \left(\frac{1}{\alpha} \frac{\hbar}{mc}\right)^2 \frac{1}{\hbar} \left(\frac{mc^2\alpha^2}{\hbar}\right)^3 \frac{1}{c^3} \simeq$$

$$\simeq \alpha^4 q^2 \frac{1}{\hbar^2} c\, m = \frac{mc^2\alpha^2}{\hbar} \alpha^2 \frac{q^2}{\hbar c}$$

i.e. $\frac{\Gamma}{\omega_{eg}} \simeq \alpha^3 \ll 1$ which validates perturbative approach

Quantitatively: $\hbar\Gamma(H: 2P \to 1S) = 4,12 \cdot 10^{-7}\, eV \longrightarrow \begin{cases} \Gamma = 6.26 \cdot 10^8\, s^{-1} \\ \Delta\nu = 9.9 \cdot 10^7\, Hz \end{cases}$ to be compared with $\nu = 2,44 \cdot 10^{15}\, Hz$

see Exercise 1 of Cohen-Tannoudji, Dupont-Roc, Grynberg, "Photons and atoms ..."

NOTE: same calculation straightforwardly extended to harmonic oscillator:

$|g\rangle \longrightarrow |n=0\rangle$, $|e\rangle \longrightarrow |n=1\rangle$. Decay rate = radiation force!

Single excited atom $\rightarrow$ generates single photon.

$\rightsquigarrow$ $\hbar\omega = E_e - E_g$

↳ SPONTANEOUS EMISSION process

Can a single photon create interference fringes?

* emitted photon has wavefunction $\alpha(k,t)$;
  real space wavepacket $\vec{\alpha}(r)$.

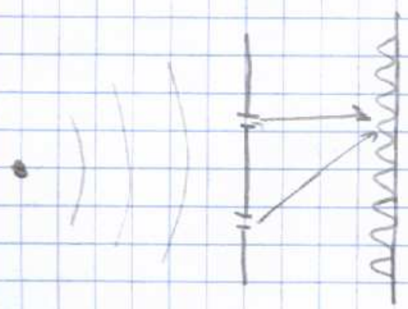
* Typically: $\vec{\alpha}(r)$ = $j=1$ multipolar wave with a radial profile:

$$\bar{\alpha}(r,t) \simeq \alpha_0\, \Theta(ct-r)\, \exp\left[2c(t-\tfrac{r}{c})/\gamma\right] \cdot \exp(ikr)$$

$\rightarrow$ atom excited into $|e\rangle$ at $t=0$.

$\rightarrow$ wavevector $k = \frac{1}{c\hbar}(E_e - E_g)$

* if spherical wave incides onto double slit



↳ fringes appear in $\vec{\alpha}(r)$ on screen

$\rightarrow$ $|\alpha(r)|^2$ gives probability distribution of detecting photon at r

exp : single-photon detection "clicks" (i.e. black spots on photographic film) are distributed according to $|\vec{\alpha}(r)|^2$

the longer the exp lasts, more photons are detected and clearer is the fringe pattern

see e.g. Grangier, Roger, Aspect, Europhys. Lett. 1, 173 ('86)

Single photon state $$|\psi\rangle = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \alpha(k,\varepsilon)\, a^\dagger(k,\varepsilon)\,|vac\rangle .$$

1) what is the physical meaning of the overall phase $\alpha(k,\varepsilon) \longrightarrow e^{i\phi}\,\alpha(k,\varepsilon)$?

2) what is the phase of the emitted E field ?

As usual in quantum mechanics, the global phase of the state vector is not physically relevant

The average value of the E field on a $N=1$ photon state is rigorously zero as $\hat{E}$ is the sum of a and $a^\dagger$.

How can one describe the classical e.m. emission of a radio circuit ?

Field radiated by a classical source

Lagrangian term coupling field to external current $j(r,t)$

$$\mathcal{L}_{ext} = \frac{1}{c}\int d^3r \;\; j(r,t)\; A(r)$$

which provides $H_{ext} = -\frac{1}{c}\int d^3r \;\; j(r,t) \;\; \hat{A}(r)$

$$H = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \hbar c k \;\; a^{\dagger}(k,\varepsilon)\, a(k,\varepsilon) - \frac{1}{c}\int d^3r \;\; j(r,t) \cdot$$

$$\cdot \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \sqrt{\frac{2\pi\hbar c}{k}} \;\; \hat{\varepsilon} \cdot \left(e^{ikr}\; a(k,\varepsilon) + e^{-ikr}\; a^{\dagger}(k,\varepsilon)\right)$$

$$= H_0 - \frac{1}{c}\int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \left\{ \left[\int d^3r \;\; (\hat{\varepsilon}\cdot j(r,t))\, e^{ikr}\right] a(k,\varepsilon) \; + \right.$$

$$\left. + \left[\int d^3r \;\; (\hat{\varepsilon}\cdot j(r,t))\, e^{-ikr}\right] a^{\dagger}(k,\varepsilon) \right\}$$

define: $\quad \bar{j}_{k,\varepsilon}(t) = \sqrt{\frac{2\pi\hbar}{ck}} \int d^3r \;\; (\hat{\varepsilon}\cdot j(r,t))\; e^{ikr}$

$$H = \int \frac{d^3k}{(2\pi)^3} \sum_{\varepsilon} \left[\hbar c k \;\; \hat{a}^{\dagger}(k,\varepsilon)\, \hat{a}(k,\varepsilon) \; - \; \bar{j}_{k,\varepsilon}(t)\; \hat{a}^{\dagger}(k,\varepsilon) \; + \right.$$

$$\left. + \; \bar{j}^{*}_{k,\varepsilon}(t)\; \hat{a}(k,\varepsilon) \right]$$

* different k modes are decoupled
* external current provides source terms proportional to $\hat{a}$, $\hat{a}^{\dagger}$.

* analogous to external force onto harmonic oscillator

Evolution equations in Heisenberg picture

* $\hat{O}_H(t) = \exp(iHt/\hbar)\,\hat{O}_S\,\exp(-iHt/\hbar)$

$\uparrow$ Schrödinger picture operator

* expectation value $\langle \hat{O}(t)\rangle = \langle \psi(t)|\,\hat{O}_S\,|\psi(t)\rangle =$

$= \langle \psi|\,\hat{O}_H(t)\,|\psi\rangle$

* $i\hbar \frac{d}{dt}\hat{O}_H(t) = \left[\hat{O}_H(t), \hat{H}\right]$

Our case:

$$i\hbar \frac{d}{dt}\hat{a}_H(k,\varepsilon;t) = \hbar c k\,\hat{a}_H(k,\varepsilon;t) - \bar{j}_{k,\varepsilon}(t)$$

$\hookrightarrow \mathbb{C}$ - number

$$\hat{a}_H(k,\varepsilon;t) = \hat{a}_H(k,\varepsilon;t=0)\,e^{-ickt} - \int_0^t dt'\,\bar{j}_{k,\varepsilon}(t')\,e^{-ick(t-t')}$$

Expectation value:

$$\alpha(k,\varepsilon;t) = \langle vac|\,a_H(k,\varepsilon;t)\,|vac\rangle =$$

$$= \cancel{\langle vac|\,a_H(k,\varepsilon;t=0)\,e^{-ickt}\,|vac\rangle} - \int dt'\,\bar{j}_{k,\varepsilon}(t')\,e^{-ick(t-t')} \times$$

initially: vacuum state $\quad 0$

$$\times \underbrace{\langle vac|vac\rangle}_{1}$$

$$= -\int dt'\,\bar{j}_{k,\varepsilon}(t')\,e^{-ick(t-t')}$$

Expectation value of physical quantities:

$$A(r,t) = \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{2\pi\hbar c}{k}}\, \hat{\varepsilon} \left(e^{ikr} \alpha(k,\varepsilon;t) + e^{-ikr} \alpha^*(k\,\varepsilon;t) \right)$$

$$E(r,t) = -\frac{1}{c}\frac{\partial A}{\partial t}$$

$$B(r,t) = \nabla \times A$$

all of them are different from 0.

good model of classical e.m. emission from antenna driven by classical electric current.

<u>Quantum state</u>:

$$a_H(k,\varepsilon;t)\,|vac\rangle = \cancel{a_H(k,\varepsilon;t=0)\, e^{-ickt}\,|vac\rangle}^{\,0} + $$
$$- \int_0^t dt'\, \bar{j}_{k,\varepsilon}(t')\, e^{-ick(t-t')}\, |vac\rangle$$

$\Longrightarrow$ $|vac\rangle$ is eigenstate of $a_H(k,\varepsilon;t)$

$$U(t,0)\, a_H(k,\varepsilon;t) \underbrace{U^\dagger(t,0)\, U(t,0)}_{\mathbb{1}}\, |vac\rangle = a_S(k,\varepsilon)\cdot U(t,0)\,|vac\rangle$$
$$= a_S(k,\varepsilon)\, \underbrace{|\psi_S(t)\rangle}$$

$\hookrightarrow$ in Schröd picture

$$\parallel$$

$$U(t,0)\;\; a_H(k,\varepsilon;t)\,|vac\rangle$$

$$\parallel$$

$$U(t,0)\left[-\int_0^t dt'\, \bar{j}_{k,\varepsilon}(t')\, e^{-ick(t-t')}\right]|vac\rangle = \left[\;\dots\;\right]|\psi_S(t)\rangle$$

$\Rightarrow$ $|\Psi_S(t)\rangle$ is eigenvector of $\hat{a}_S$
with eigenvalue:

$$- \int_0^{t} dt' \, f_{n\varepsilon}(t') \, e^{-i\omega (t-t')}$$

COHERENT (or SEMICLASSICAL) state
of harmonic oscillator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Explicit form of $|\alpha\rangle$ in Fock basis $|n\rangle$:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \cdot |n\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n \hat{a}^{\dagger n}}{n!} |n\rangle$$

$$= e^{-|\alpha|^2/2} \exp\left(\alpha \hat{a}^\dagger\right) |vac\rangle$$

$\rightarrow$ probability distribution of having n photons

$$P(n) = \exp(-|\alpha|^2) \cdot \frac{|\alpha|^{2n}}{n!} = \frac{\bar{n}^n}{n!} \exp(-\bar{n})$$

[average #: $\bar{n} = |\alpha|^2$]

$$\langle a^\dagger a\rangle = |\alpha|^2$$

$$\langle a^\dagger a \, a^\dagger a\rangle = \langle a^\dagger a^\dagger a a\rangle + \langle a^\dagger a\rangle = |\alpha|^4 + |\alpha|^2$$

$$\Rightarrow \langle n^2\rangle - \langle n\rangle^2 = |\alpha|^4 + |\alpha|^2 - (|\alpha|^2)^2 = |\alpha|^2$$

$$\frac{\langle \Delta n^2\rangle}{\langle n\rangle^2} = \frac{\langle n^2\rangle - \langle n\rangle^2}{\langle n\rangle^2} = \frac{1}{|\alpha|^2} \rightarrow 0$$ in macroscopic limit $|\alpha|^2 \rightarrow \infty$.

Motion eqns for fields:

$$\hat{H} = \hat{H}_0 - \frac{1}{c}\int d^3r \; j(r,t) \; \hat{A}(r)$$

↳ only modification is in:

$$i\hbar \frac{d}{dt}\hat{E}_i = i\hbar c \; \nabla\times\hat{B}\Big|_i - \frac{1}{c}\int d^3r' \, j_j(r',t) \; [\hat{E}_i(r), \hat{A}_j(r')]$$

$$= i\hbar c \; \nabla\times\hat{B}\Big|_i - \frac{1}{c}\int d^3r' \, j_j(r',t) \; 4\pi i c\hbar \, \delta^{\perp}_{ij}(r-r')$$

$$\Rightarrow \quad \nabla\times\hat{B} = \frac{1}{c}\frac{\partial\hat{E}}{\partial t} + \underbrace{\frac{4\pi}{c} j_{\perp}(r,t)}_{\text{source term}}$$

To summarize:

* external classical source provides source term in evolution eqs for field operators

* quantum state of field is a coherent state such that $\hat{a}(k,\varepsilon)|t\rangle = \alpha(k,\varepsilon)|t\rangle$ for all modes (k,ε)

* this result is general as long as:
  - field is non-interacting, i.e. H contains only quadratic terms in field operators
  - source term couples linearly to field

⇒ field state fully determined by 1-body expectation values.

NOTE: physical phase of field is related to relative phase of given-N components

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$|e^{i\phi}\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \cdot \underbrace{e^{in\phi}}_{\text{relative phase of different } n\text{'s}} |n\rangle$$

→ keep in mind the distinction to the global (physically irrelevant) phase of $|\psi\rangle$!!