Constructing canonical bases of quantized enveloping algebras

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Abstract

An algorithm for computing the elements of a given weight of the canonical basis of a quantized enveloping algebra is described. Subsequently a similar algorithm is presented for computing the canonical basis of a finite-dimensional module.

1 Introduction

Since the invention of canonical bases of quantized enveloping algebras, one of the main problems regarding them has been to establish what they look like. Explicit formulas are only known in a few cases corresponding to root systems of low rank, namely A_1 (trivial), A_2 ([14]), A_3 ([20]), and B_2 ([21]). Furthermore, there is evidence suggesting that for higher ranks the formulas become so complicated that an explicit description is virtually impossible (see [3]). Therefore it is natural to start with less ambition, and try to find part of the canonical basis, say the part consisting of all elements of a given weight. That is the problem we deal with in this paper. We will describe an algorithm for computing the elements of a given weight of the canonical basis of the negative part of the quantized enveloping algebra of a finite-dimensional semisimple Lie algebra. We will also give an algorithm for computing the canonical basis of a finite-dimensional module.

To the best of my knowledge two algorithms for computing canonical bases of modules are known. In [11] an algorithm is described for computing the canonical basis of a $U_q(\mathfrak{sl}_n)$-module, and [19] contains an algorithm for computing the canonical basis of a fundamental module, when the root system is of type $A-D$. Our approach differs from the ones taken in [11], [19] by the fact that we work with PBW-type bases. This leads to algorithms that are more generally applicable: they work for any finite-dimensional module of the quantized enveloping algebra of any semisimple Lie algebra. The problem of computing elements of the canonical basis of the negative part of a quantized enveloping algebra has, to the best of my knowledge, not been considered before.

This paper is organized as follows. In Section 2 we recall some basic facts on canonical bases. In Section 3 we describe an algorithm for computing the action of the Kashiwara operators. In Section 4 we describe the notion of adapted string, following [13]. Then in Section 5 we describe an algorithm for computing the elements of the canonical basis of a given weight ν. In Section 6 we give a similar algorithm for computing the canonical basis of a finite-dimensional $U_q(\mathfrak{g})$-module. Finally, in Section 7 we apply the algorithm of Section 5 to the study of tight monomials of small weight.

The algorithms described here have been implemented in the computer algebra system GAP4 ([6]), as part of a package called QuaGroup, [8]. The main computations for Examples 3, 4 and for Table 1 were done using that package.

2 The canonical basis

First we recall some notation. Our main reference is [9]. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra with root system Φ. Let $\Delta = \{\alpha_1, \dots, \alpha_l\}$ be a simple system of Φ. Then $U_q(\mathfrak{g})$ is the corresponding quantized enveloping algebra, with generators F_α, $K_\alpha^{\pm 1}$, E_α, for $\alpha \in \Delta$, subject to the relations [9], 4.3. Furthermore, U^- will be the subalgebra of $U_q(\mathfrak{g})$ generated by the F_α. Let $\nu = \sum_{i=1}^{l} k_i\alpha_i$ be a linear combination of the simple roots, with non-negative integral coefficients. Then $U^-_{-\nu}$ will be the span of all elements $F_{\alpha_{m_1}} \cdots F_{\alpha_{m_t}}$ such that F_{α_i} appears exactly k_i times. Elements of $U^-_{-\nu}$ are said to be homogeneous of weight ν.

By $W(\Phi)$ we denote the Weyl group of Φ. It is generated by the simple reflections $s_i = s_{\alpha_i}$ for $1 \leq i \leq l$. Usually we will denote a reduced expression for the longest element in $W(\Phi)$ by w_0.

We work in the subalgebra U^- of $U_q(\mathfrak{g})$. Let $w_0 = s_{i_1} \cdots s_{i_t}$ be a reduced expression for the longest element in the Weyl group. For $1 \leq k \leq l$ let $T_k = T_{\alpha_k} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ be the automorphism described in [9], 8.13. For $1 \leq k \leq t$ set $F_k = T_{i_1} \cdots T_{i_{k-1}}(F_{\alpha_{i_k}})$. Then F_k is an element of weight $\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. We also denote F_k by F_{β_k}. As usual we set $F_k^{(m)} = F_k^m/[m]!_{\alpha_{i_k}}$ (where $[m]!_{\alpha_{i_k}}$ is the Gaussian factorial, defined in [9], 0.1, 4.2). Then the monomials

$$F_1^{(n_1)} \cdots F_t^{(n_t)} \tag{1}$$

form a basis of U^-. This basis is called a PBW-type basis, and we call a monomial of the form (1) a PBW-monomial (relative to the chosen reduced expression for the longest element of the Weyl group). We have algorithms for writing the product of any two PBW-monomials as a linear combination of PBW-monomials ([7]).

Let x be a monomial of the form (1). Then to stress the dependency of x on the choice of reduced expression for the longest element of the Weyl group, we say that x is a w_0-monomial. We refer to the exponents $n_1, \ldots, n_t$ as the first, second, ..., t-th exponent of x.

Now we let $^-$ be the unique automorphism of U^- (viewed as $\mathbb{Q}$-algebra) satisfying $\overline{q} = q^{-1}$ and $\overline{F}_{\alpha_i} = F_{\alpha_i}$. Elements that are invariant under $^-$ are said to be bar-invariant. The bar-invariant elements include all monomials of the form $F_{\alpha_{i_1}}^{(n_1)} \cdots F_{\alpha_{i_r}}^{(n_r)}$ (but not all PBW-monomials).

By results of Lusztig ([16] Theorem 42.1.10, [18], Proposition 8.2, see also [1]), there is a unique basis $\mathbf{B}$ of U^- with the following properties. Firstly, all elements of $\mathbf{B}$ are bar-invariant. Secondly, for any choice of reduced expression w_0 for the longest element in the Weyl group, and any element $X \in \mathbf{B}$ we have that $X = x + \sum \zeta_i x_i$, where x, x_i are different w_0-monomials, and $\zeta_i \in q\mathbb{Z}[q]$. The basis $\mathbf{B}$ is called the canonical basis. If we work with a reduced expression w_0 for the longest element in $W(\Phi)$, and write $X \in \mathbf{B}$ as above, then we say that x is the principal w_0-monomial of X (or just principal monomial of X, if it is clear which reduced expression we mean).

We let $\mathcal{L}(\infty)$ be the $\mathbb{Z}[q]$-lattice spanned by $\mathbf{B}$. Fix a reduced expression w_0 for the longest element in $W(\Phi)$. Then we have that $\mathcal{L}(\infty)$ is also spanned by the set of all w_0-monomials (cf. [16], Chapter 42; it will also follow from Proposition 2 in Section 5). We let $\pi : \mathcal{L}(\infty) \to \mathcal{L}(\infty)/q\mathcal{L}(\infty)$ be the projection map, and we let $\mathcal{B}(\infty)$ be the set of all $\pi(x)$, where x runs through all w_0-monomials. Then $\mathcal{B}(\infty)$ does not depend on the choice of reduced expression for the longest element in $W(\Phi)$. (Indeed, let $\widetilde{w}_0$ be a second reduced expression for the longest element in $W(\Phi)$. Let x be a w_0-monomial, and let X be the element of $\mathbf{B}$ with principal w_0-monomial x. Let x' be its principal $\widetilde{w}_0$-monomial, then $x = x' \bmod q\mathcal{L}(\infty)$.)

Let x be a w_0-monomial, then we write $b_x = \pi(x) \in \mathcal{B}(\infty)$. Also $G(b_x)$ will denote the element of $\mathbf{B}$ which has principal monomial x, i.e., such that $\pi(G(b_x)) = b_x$.

Remark. In this section we have worked with a $\mathbb{Z}[q]$-lattice $\mathcal{L}(\infty)$ in U^-. However, in some other places (e.g., [1], [16], [18]) a $\mathbb{Z}[q^{-1}]$-lattice in U^+ is used. In these references also a different PBW-type basis is used (compare the description of T_α in [9], 8.14 with the description of $T'_{i,-1}$ in [16], 37.1.3). These two approaches are equivalent. In order to see that we view $U_q(\mathfrak{g})$ as $\mathbb{Q}$-algebra, and we let ϕ be the automorphism of $U_q(\mathfrak{g})$ defined by $\phi(F_\alpha) = E_\alpha$, $\phi(K_\alpha) = K_\alpha$, $\phi(E_\alpha) = F_\alpha$, and $\phi(q) = q^{-1}$. (This is the composition of the automorphisms ω of [9], Lemma 4.6, and ψ of [9], Proposition 11.9.) Then $\phi(U^+) = U^-$ and $\phi(T'_{i,-1}(u)) = T_\alpha(\phi(u))$ for all $u \in U_q(\mathfrak{g})$. Therefore ϕ maps a PBW-type basis of U^+ (defined using the $T'_{i,-1}$) to a PBW-type basis of U^- (defined using the T_α), and interchanges q and q^{-1}. Also, ϕ maps bar-invariant elements of U^+ to bar-invariant elements of U^-, so that ϕ maps the canonical basis of U^+ to the canonical basis of U^-.

3 Kashiwara operators

Let $\alpha \in \Delta$. The Kashiwara operators $\widetilde{F}_\alpha, \widetilde{E}_\alpha : U^- \to U^-$ are defined as follows. Let $w_0 = s_{i_1} \cdots s_{i_t}$ be a reduced expression for the longest element of the Weyl group, such that $\alpha_{i_1} = \alpha$. Let u be a w_0-monomial with exponents $n_1, \dots, n_t$. Then $\widetilde{F}_\alpha(u) = F_1^{(n_1+1)} \cdots F_t^{(n_t)}$, and $\widetilde{E}_\alpha(u) = F_1^{(n_1-1)} \cdots F_t^{(n_t)}$, if $n_1 > 0$, and $\widetilde{E}_\alpha(u) = 0$ otherwise. (Note that $F_1 = F_\alpha$.) The action of $\widetilde{F}_\alpha$, $\widetilde{E}_\alpha$ is extended to the whole of U^- by linearity. It can be shown that this definition does not depend on the choice of reduced expression of the longest element in the Weyl group (cf. [9], 10.1).

We have that $\widetilde{F}_\alpha, \widetilde{E}_\alpha$ map PBW-monomials to PBW-monomials, relative to a reduced expression for the longest element in $W(\Phi)$ starting with s_α. However, $\mathcal{B}(\infty)$ does not depend on that choice, and therefore $\widetilde{F}_\alpha$, $\widetilde{E}_\alpha$ can be viewed as maps $\widetilde{F}_\alpha : \mathcal{B}(\infty) \to \mathcal{B}(\infty)$ and $\widetilde{E}_\alpha : \mathcal{B}(\infty) \to \mathcal{B}(\infty) \cup \{0\}$. This means that if x is a w_0-monomial, then $\widetilde{F}_\alpha(x) = x' \bmod q\mathcal{L}(\infty)$, where x' is a certain w_0-monomial. We consider the problem of obtaining x' from x.

First we note that if w_0 happens to start with s_α, then x' is constructed from x by increasing the first exponent of x by 1. Now suppose that w_0 does not start with s_α. Let $\widetilde{w}_0$ be a different reduced expression for the longest element of the Weyl group. Then there is a $\widetilde{w}_0$-monomial $\tilde{x}$ such that $x = \tilde{x} \bmod q\mathcal{L}(\infty)$. Following Lusztig's notation (see [15], [16]), we write $\tilde{x} = R_{w_0}^{\tilde{w}_0}(x)$. If we can find $\tilde{x}$ from x, then the problem of calculating $\widetilde{F}_\alpha(x)$ is solved. Indeed, let $\widetilde{w}_0$ be a reduced expression for the longest element of the Weyl group, starting with s_α. We find $\tilde{x} = R_{w_0}^{\tilde{w}_0}(x)$, and increase its first exponent by 1. Denote the resulting monomial by $\tilde{x}'$. Finally we construct $x' = R_{\tilde{w}_0}^{w_0}(\tilde{x}')$. Then $\widetilde{F}_\alpha(x) = x' \bmod q\mathcal{L}(\infty)$.

We may assume that $\widetilde{w}_0$ can be obtained from w_0 by applying one braid relation. Suppose that this relation amounts to replacing $s_\alpha s_\beta \cdots$ by $s_\beta s_\alpha \cdots$, where both words are of length d. Then $d = 2, 3, 4$ or 6. Suppose that the first word occurs in w_0 on positions $p, p+1, \ldots, p+d-1$. Write $x = F_1^{(m_1)} \cdots F_t^{(m_t)}$ and $\tilde{x} = F_1^{(m'_1)} \cdots F_t^{(m'_t)}$ (where the F_i in $\tilde{x}$ are defined relative to $\widetilde{w}_0$). We obtain the m'_i from the m_i in the following way.

1. If $d = 2$, then $m'_p = m_{p+1}$ and $m'_{p+1} = m_p$.

2. If $d = 3$, then set $\mu = \min(m_p, m_{p+2})$, and $m'_p = m_{p+1} + m_{p+2} - \mu$, $m'_{p+1} = \mu$, $m'_{p+2} = m_p + m_{p+1} - \mu$.

3. If $d = 4$ then suppose that the move consists of replacing $s_\alpha s_\beta s_\alpha s_\beta$ by $s_\beta s_\alpha s_\beta s_\alpha$. Set $a = m_p$, $b = m_{p+1}$, $c = m_{p+2}$, $d = m_{p+3}$.

   (a) If α is short, then set $n_1 = \max(b, \max(b,d) + c - a)$, $n_2 = \max(a,c) + 2b$, $n_3 = \min(c + d, a + \min(b,d))$, $n_4 = \min(a,c)$. Set $\mu = \max(2n_3, n_2 + n_4)$ and $m'_p = n_1$, $m'_{p+1} = \mu - n_2$, $m'_{p+2} = n_2 + n_3 - \mu$, $m'_{p+3} = n_4 - 2n_3 + \mu$.

   (b) If α is long, then set $p_1 = \max(b, \max(b,d) + 2c - 2a)$, $p_2 = \max(a,c) + b$, $p_3 = \min(2c + d, \min(b,d) + 2a)$, $p_4 = \min(a,c)$. Set $\mu = \max(p_3, p_2 + p_4)$, and $m'_p = p_1$, $m'_{p+1} = \mu - p_2$, $m'_{p+2} = p_3 + 2p_2 - 2\mu$, $m'_{p+3} = p_4 - p_3 + \mu$.

4. If $d = 6$, then we consider the root system of type D_4, along with its diagram automorphism ϕ of order 3. Let α_2 be the simple root fixed by

ϕ, and $\alpha_1, \alpha_3, \alpha_4$ the other three. Set $v = s_1 s_3 s_4$. We use the following two reduced expressions for the longest element in the Weyl group: $v_0 = v s_2 v s_2 v s_2$ and $\widetilde{v}_0 = s_2 v s_2 v s_2 v$. Let $\widetilde{U}^-$ be the algebra generated by the F_{α_i} for $1 \leq i \leq 4$, of the corresponding quantized enveloping algebra. In $\widetilde{U}^-$ we use the PBW-bases relative to v_0 and $\widetilde{v}_0$.

For simplicity assume that the root system of $U_q(\mathfrak{g})$ is of type G_2. Suppose that the braid relation amounts to replacing $w_0 = s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta$ by $\widetilde{w}_0 = s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha$, where α is long. Corresponding to a w_0-monomial x with exponents $m_1, \dots, m_6$ we construct the v_0-monomial $y = \psi_1(x)$ with exponents $m_1, m_1, m_1, m_2, m_3, m_3, m_3, m_4, m_5, m_5, m_5, m_6$. Furthermore, to a $\widetilde{w}_0$-monomial $\tilde{x}$ with exponents $m_1, \dots, m_6$ corresponds the $\widetilde{v}_0$-monomial $\tilde{y} = \psi_2(\tilde{x})$ with exponents $m_1, m_2, m_2, m_2, m_3, m_4, m_4, m_4, m_5, m_6, m_6, m_6$. Now starting with a w_0-monomial x we construct (using 1., and 2.) the $\widetilde{v}_0$-monomial $\tilde{y} = R_{v_0}^{\tilde{v}_0}(\psi_1(x))$. Then we have $R_{w_0}^{\tilde{w}_0}(x) = \psi_2^{-1}(\tilde{y})$.

Finally, if α is short, then we have $R_{w_0}^{\tilde{w}_0}(x) = \psi_1^{-1}(R_{\tilde{v}_0}^{v_0}(\psi_2(x)))$.

First of all, 1., and 2. are proved in [16], 3. can be proved using [15], 12.5, and 4. follows in the same way (see also [3], [16] Theorem 14.4.9). At the end of Section 5 we sketch a different proof of 2., 3.

We note that by the same methods we can calculate the action of $\widetilde{E}_\alpha$.

Example 1 Consider the root system of type A_3, with simple roots α, β, γ (where α, γ correspond to the outer vertices of the Dynkin diagram of type A_3, and β to the middle vertex). Then $w_0 = s_\alpha s_\beta s_\gamma s_\alpha s_\beta s_\alpha$ is a reduced expression for the longest element in the Weyl group. Let x be the w_0-monomial with exponents $(1,2,3,4,5,6)$. We calculate the action of $\widetilde{F}_\gamma$ on x. First of all, $\widetilde{w}_0 = s_\gamma s_\alpha s_\beta s_\gamma s_\alpha s_\beta$ is a reduced expression for the longest element in the Weyl group, starting with s_γ. By applying braid relations, w_0 is transformed to $\widetilde{w}_0$ in the following way:

$$w_0 \to s_\alpha s_\beta s_\gamma s_\beta s_\alpha s_\beta \to s_\alpha s_\gamma s_\beta s_\gamma s_\alpha s_\beta \to s_\gamma s_\alpha s_\beta s_\gamma s_\alpha s_\beta = \widetilde{w}_0.$$

Using 1. and 2. above, we see that x transforms to a monomial with exponents $(1,2,3,7,4,5)$, $(1,8,2,3,4,5)$, and $(8,1,2,3,4,5)$. Here the last sequence of exponents defines the $\widetilde{w}_0$-monomial $\tilde{x}$. We now increase its first exponent to 9, obtaining the $\widetilde{w}_0$-monomial $\tilde{x}'$. Transforming this back we obtain the w_0-monomial x' with exponents $(1,2,3,4,5,7)$, which is equal to $\widetilde{F}_\gamma(x)$.

4 Adapted strings

First we recall some facts on Littelmann's path model. For more details and proofs we refer to [12].

Let P denote the weight lattice, and let $P_{\mathbb{R}}$ be the vector space over $\mathbb{R}$ spanned by P (i.e., $P_{\mathbb{R}} = P \otimes_{\mathbb{Z}} \mathbb{R}$). Let Π be the set of all piecewise linear paths $\xi : [0,1] \to P_{\mathbb{R}}$, such that $\xi(0) = 0$. For $\alpha \in \Delta$ Littelmann defined operators $f_\alpha, e_\alpha : \Pi \to \Pi \cup \{0\}$. Let λ be a dominant weight and let ξ_λ be the path joining λ and the origin by a straight line. Let Π_λ be the set of all nonzero $f_{\alpha_{i_1}} \cdots f_{\alpha_{i_m}}(\xi_\lambda)$ for $m \geq 0$. Then $\xi(1) \in P$ for all $\xi \in \Pi_\lambda$. Let $\mu \in P$ be a weight, and let $V(\lambda)$ be the highest-weight module over $U_q(\mathfrak{g})$ of highest weight λ. A theorem of Littelmann states that the number of paths in $\xi \in \Pi_\lambda$ such that $\xi(1) = \mu$ is equal to the dimension of the weight space of weight μ in $V(\lambda)$ ([12], Theorem 9.1).

Let $\nu = \sum_{i=1}^{l} k_i \alpha_i$ be a linear combination of simple roots, with non-negative integral coefficients. Set $\lambda = \sum_{i=1}^{l} k_i \lambda_i$ (where the λ_i are the fundamental weights). Then the dimension of the weight space of weight $\lambda - \nu$ in $V(\lambda)$ is equal to the dimension of $U^-_{-\nu}$. In particular, the dimension of $U^-_{-\nu}$ is equal to the number of paths $\xi \in \Pi_\lambda$ such that $\xi(1) = \lambda - \nu$.

Let $w_0 = s_{i_1} \cdots s_{i_t}$ be a fixed reduced expression of the longest element in the Weyl group. Let ν, λ be as in the preceding paragraph, and let $\xi \in \Pi_\lambda$ be such that $\xi(1) = \lambda - \nu$. We define a sequence of integers $\eta_\xi = (n_1, \ldots, n_t)$ and a sequence of paths ξ_k in the following way. First we set $\xi_0 = \xi$. Suppose that the elements $\xi_0, \ldots, \xi_{k-1}$ and $n_1, \ldots, n_{k-1}$ are defined. Then let n_k be maximal such that $e^{n_k}_{\alpha_{i_k}}(\xi_{k-1}) \neq 0$, and set $\xi_k = e^{n_k}_{\alpha_{i_k}}(\xi_{k-1})$. Following [13] we call η_ξ the adapted string corresponding to ξ (relative to the fixed reduced expression of the longest element of the Weyl group). Let S_ν be the set of adapted strings corresponding to all $\xi \in \Pi_\lambda$ such that $\xi(1) = \lambda - \nu$.

Let $\eta = (n_1, \ldots, n_t) \in S_\nu$ and set

$$M_\eta = F^{(n_1)}_{\alpha_{i_1}} \cdots F^{(n_t)}_{\alpha_{i_t}},$$

and

$$b_\eta = \widetilde{F}^{n_1}_{\alpha_{i_1}} \cdots \widetilde{F}^{n_t}_{\alpha_{i_t}}(1) \in \mathcal{B}(\infty).$$

Let $<_{\text{lex}}$ be the lexicographical ordering on integer sequences of length t (i.e., $(m_1, \ldots, m_t) <_{\text{lex}} (n_1, \ldots, n_t)$ if there is a k such that $m_i = n_i$ for $i < k$, and

$m_k < n_k$). Then [13] Proposition 10.4 states

$$M_\eta = G(b_\eta) - \sum_{\substack{\eta' >_{\text{lex}} \eta \\ \eta' \in S_\nu}} c_{\eta,\eta'} G(b_{\eta'}), \tag{2}$$

where $c_{\eta,\eta'} \in \mathbb{Z}[q, q^{-1}]$.

Let $\eta = (n_1, \dots, n_t)$ be an adapted string, corresponding to the reduced expression $w_0 = s_{i_1} \cdots s_{i_t}$ of the longest element in $W(\Phi)$. Then we also write $f^\eta(\xi_\lambda)$ instead of $f_{\alpha_{i_1}}^{n_1} \cdots f_{\alpha_{i_t}}^{n_t}(\xi_\lambda)$.

5 Constructing canonical basis elements

Here we describe an algorithm for computing the elements of the canonical basis of a given weight ν. The main idea is similar to the one used in [11]: we "approximate" $G(b_\eta)$ with a bar-invariant element, i.e., M_η. Then we add multiples of the $G(b_{\eta'})$, that are already constructed, while making sure that the element remains bar-invariant. We are finished once the element is a $\mathbb{Z}[q]$-linear combination of PBW-monomials, where exactly one coefficient is 1, and the rest lies in $q\mathbb{Z}[q]$.

By $<_{\text{lex}}$ we denote the lexicographical ordering on the PBW-monomials of U^- (i.e., $F_1^{(m_1)} \cdots F_t^{(m_t)} <_{\text{lex}} F_1^{(n_1)} \cdots F_t^{(n_t)}$ if and only if $(m_1, \dots, m_t) <_{\text{lex}} (n_1, \dots, n_t)$).

Let x be a PBW-monomial. From Section 2 we recall that b_x denotes the element of $\mathcal{B}(\infty)$ such that $G(b_x)$ has principal monomial x. Also by $\varepsilon_\alpha(x)$ we denote the maximal integer n such that $\widetilde{E}_\alpha^n(b_x) \neq 0$. Note that if x is a w_0-monomial, where w_0 starts with s_α, then $\varepsilon_\alpha(x)$ is equal to the first exponent of x.

Proposition 2 *Let $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_r}}$ be a reduced word in the Weyl group of Φ. Let w_0 be any reduced expression for the longest element in the Weyl group starting with w. Let*

$$x = F_{\alpha_{i_1}}^{(n_1)} T_{\alpha_{i_1}}(F_{\alpha_{i_2}})^{(n_2)} \cdots (T_{\alpha_{i_1}} \cdots T_{\alpha_{i_{r-1}}})(F_{\alpha_{i_r}})^{(n_r)}$$

be a PBW-monomial in U^-. Then $G(b_x)$ is equal to x plus a $q\mathbb{Z}[q]$-linear combination of w_0-monomials y such that $y >_{\text{lex}} x$.

In the proof we use two direct sum decomposition of U^- relative to a simple root α:

$$U^- = U^- \cap T_\alpha(U^-) \oplus F_\alpha U^-, \tag{3}$$

$$U^- = U^- \cap T_\alpha^{-1}(U^-) \oplus U^- F_\alpha, \tag{4}$$

(cf. [9], 8.25, [18]). We have the corresponding projection maps $\pi_\alpha^+ : U^- \to U^- \cap T_\alpha(U^-)$ and $\pi_\alpha^- : U^- \to U^- \cap T_\alpha^{-1}(U^-)$. These maps can be described as follows. Let $w_0 = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_t}}$ be a reduced expression for the longest element in the Weyl group, where $\alpha_{i_1} = \alpha$. If (1) is a w_0-monomial, then $F_1 = F_\alpha$ and $F_\alpha U^-$ is the linear span of all w_0-monomials with first exponent ≥ 1. Also $U^- \cap T_\alpha(U^-)$ is the linear span of all w_0-monomials with first exponent equal to 0. Now let $u \in U^-$ and write u as a linear combination of w_0-monomials. Then $u = u_1 + u_2$, where u_1 consists of w_0-monomials with first exponent 0, and u_2 is a linear combination of w_0-monomials with first exponent ≥ 1. Hence $\pi_\alpha^+(u) = u_1$.

Set $v = s_{\alpha_{i_2}} \cdots s_{\alpha_{i_t}}$, and let β be a simple root such that $v(\beta) > 0$. We set $\widetilde{w}_0 = v s_\beta$; then $\widetilde{w}_0$ is also a reduced expression for the longest element of the Weyl group. We have $v(\beta) > 0$, but $s_\alpha v(\beta) < 0$, so that $v(\beta) = \alpha$. Hence $T_v(F_\beta) = F_\alpha$ (cf. [9], Proposition 8.20). So if we have a $\widetilde{w}_0$-monomial of the form (1), then $F_t = F_\alpha$; hence $U^- F_\alpha$ is the linear span of all $\widetilde{w}_0$-monomials with t-th exponent ≥ 1. Furthermore, $U^- \cap T_\alpha^{-1}(U^-)$ is the linear span of all $\widetilde{w}_0$-monomials with t-th exponent equal to 0. This means that we can decompose $u \in U^-$ according to the decomposition (4) by writing $u = u_1 + u_2$, where u_1 is a linear combination of $\widetilde{w}_0$-monomials with t-th exponent 0, and u_2 consists of $\widetilde{w}_0$-monomials with t-th exponent ≥ 1. Then $\pi_\alpha^-(u) = u_1$.

We have that $B_\alpha^+ = \pi_\alpha^+(\mathbf{B} \setminus \mathbf{B} \cap F_\alpha U^-)$ is a basis of $U^- \cap T_\alpha(U^-)$, and $B_\alpha^- = \pi_\alpha^-(\mathbf{B} \setminus \mathbf{B} \cap U^- F_\alpha)$ is a basis of $U^- \cap T_\alpha^{-1}(U^-)$ (cf. [18]). Now [18], Theorem 1.2 states that

$$T_\alpha(B_\alpha^-) = B_\alpha^+. \tag{5}$$

Proof. (Of Proposition 2). We use induction on r. Note that the result is trivial for $r = 1$ as in that case $x = F_{\alpha_{i_1}}^{(n_1)}$ and $G(b_x) = x$. Set $\alpha = \alpha_{i_1}$ and

$$x' = T_{\alpha_{i_1}}(F_{\alpha_{i_2}})^{(n_2)} \cdots (T_{\alpha_{i_1}} \cdots T_{\alpha_{i_{r-1}}})(F_{\alpha_{i_r}})^{(n_r)},$$

$$x'' = F_{\alpha_{i_2}}^{(n_2)} T_{\alpha_{i_2}} (F_{\alpha_{i_3}})^{(n_3)} \cdots (T_{\alpha_{i_2}} \cdots T_{\alpha_{i_{r-1}}})(F_{\alpha_{i_r}})^{(n_r)}.$$

(So that $x' = T_\alpha(x'')$.) We define $\widetilde{w}_0$ as above. Then x'' is a $\widetilde{w}_0$-monomial and by induction $G(b_{x''})$ is equal to x'' plus a $q\mathbb{Z}[q]$-linear combination of $\widetilde{w}_0$-monomials that are lexicographically bigger than x''. By the description of π_α^- we see that the same holds for $\pi_\alpha^-(G(b_{x''}))$. Now, by (5) we have that $T_\alpha(\pi_\alpha^-(G(b_{x''}))) = \pi_\alpha^+(G(b_y))$ for some $G(b_y) \in \mathbf{B} \setminus \mathbf{B} \cap F_\alpha U^-$. But $T_\alpha(\pi_\alpha^-(G(b_{x''})))$ is equal to $T_\alpha(x'') = x'$ plus a $q\mathbb{Z}[q]$-linear combination of w_0-monomials (lexicographically bigger than x'), and therefore $y = x'$. It follows that $\pi_\alpha^+(G(b_{x'}))$ is equal to x' plus a $q\mathbb{Z}[q]$-linear combination of w_0-monomials that are lexicographically bigger than x'. From the description above of the map π_α^+ we now see that $G(b_{x'})$ is equal to $\pi_\alpha^+(G(b_{x'}))$ plus a linear combination of w_0-monomials with non-zero first exponent, and these are lexicographically bigger than x'. Now by [9] 11.12(1), we have that $G(b_x) = F_\alpha^{(n_1)} G(b_{x'}) + R$ where R is a linear combination of elements $G(b_z)$, with $\varepsilon_\alpha(z) > n_1$. By [9], 11.3(2), 11.12(3) we have that $G(b_u) \in F_\alpha^{(\varepsilon_\alpha(u))} U^-$ for all PBW-monomials u. In particular, all w_0-monomials occurring in R have first exponent $> n_1$, and therefore they are bigger than x in the lexicographical ordering. □

Proposition (2) yields the following algorithm for constructing elements of the canonical basis. From (2) we get

$$G(b_\eta) = M_\eta + \sum_{\eta' >_{\mathrm{lex}} \eta} c_{\eta,\eta'} G(b_{\eta'}). \tag{6}$$

The M_η, $G(b_\eta)$ are all bar-invariant, and the latter form a basis of $U_{-\nu}^-$, hence the $c_{\eta,\eta'}$ are bar-invariant as well.

Let $\eta \in S_\nu$, and suppose that we have already constructed the elements $G(b_{\eta'})$ for $\eta' >_{\mathrm{lex}} \eta$. In order to construct $G(b_\eta)$ we need to know the coefficients $c_{\eta,\eta'}$ in (6). For $b_1, b_2 \in \mathcal{B}(\infty)$ we write $b_1 <_{\mathrm{lex}} b_2$ if the principal monomial of $G(b_1)$ is smaller with respect to $<_{\mathrm{lex}}$ than the principal monomial of $G(b_2)$. Order the elements occurring in the sum on the right hand side of (6) as $b_{\eta_1} <_{\mathrm{lex}} b_{\eta_2} <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} b_{\eta_r}$. We define a sequence of elements $G_k \in U^-$. First set $G_0 = M_\eta$. Suppose that $G_0, \ldots, G_{k-1}$ are defined. Let ζ_k be the coefficient of the principal monomial of $G(b_{\eta_k})$ in G_{k-1}, and let ζ_k' be the unique bar-invariant element of $\mathbb{Z}[q, q^{-1}]$ such that $\zeta_k + \zeta_k' \in q\mathbb{Z}[q]$. Set $G_k = G_{k-1} + \zeta_k' G(b_{\eta_k})$. By induction on k, and Proposition 2 we have that $c_{\eta,\eta_k} = \zeta_k'$. Hence $G_r = G(b_\eta)$.

Example 3 We consider the root system of type B_2, with simple roots α, β, where α is long. We use the reduced expression $s_\alpha s_\beta s_\alpha s_\beta$ of the longest element in the Weyl group. The generators of the corresponding PBW-type basis of U^- are F_α, $F_{\alpha+\beta}$, $F_{\alpha+2\beta}$, F_β. Let $\nu = 3\alpha + 2\beta$; we compute the elements of the canonical basis of weight ν.

The set S_ν consists of the adapted strings $\eta_1 = (3,2,0,0)$, $\eta_2 = (2,2,1,0)$, $\eta_3 = (2,1,1,1)$, $\eta_4 = (1,2,2,0)$ (in lexicographical order). First of all $M_{\eta_1} = F_\alpha^{(3)} F_\beta^{(2)} = G(b_{\eta_1})$. Now we consider η_2. Using the algorithms to compute products of PBW-monomials in U^- ([7]), which are implemented in [8], we get

$$M_{\eta_2} = F_\alpha^{(2)} F_\beta^{(2)} F_\alpha = F_\alpha^{(2)} F_{\alpha+2\beta} + qF_\alpha^{(2)} F_{\alpha+\beta} F_\beta + (1+q^4+q^8) F_\alpha^{(3)} F_\beta^{(2)}.$$

Here the coefficient of $F_\alpha^{(3)} F_\beta^{(2)}$ is not contained in $q\mathbb{Z}[q]$. We repair this situation, and we get that

$$G(b_{\eta_2}) = M_{\eta_2} - G(b_{\eta_1}) = F_\alpha^{(2)} F_{\alpha+2\beta} + qF_\alpha^{(2)} F_{\alpha+\beta} F_\beta + (q^4+q^8) F_\alpha^{(3)} F_\beta^{(2)}.$$

Thirdly, $M_{\eta_3} = F_\alpha^{(2)} F_{\alpha+\beta} F_\beta + (q^{-3}+q^{-1}+q+q^3+q^5+q^7) F_\alpha^{(3)} F_\beta^{(2)}$. Here we get

$$G(b_{\eta_3}) = M_{\eta_3} - (q^{-3}+q^{-1}+q+q^3) G(b_{\eta_1}) = F_\alpha^{(2)} F_{\alpha+\beta} F_\beta + (q^5+q^7) F_\alpha^{(3)} F_\beta^{(2)}.$$

Finally, $M_{\eta_4} = F_\alpha F_{\alpha+\beta}^{(2)} + (1+q^4) F_\alpha^{(2)} F_{\alpha+2\beta} + (q+q^5) F_\alpha^{(2)} F_{\alpha+\beta} F_\beta + (q^4 + q^8 + q^{12}) F_\alpha^{(3)} F_\beta^{(2)}$. Here the coefficient of $F_\alpha^{(2)} F_{\alpha+2\beta}$ does not lie in $q\mathbb{Z}[q]$. So we have to subtract the element of the canonical basis with that principal monomial, i.e., $G(b_{\eta_2})$. We get

$$G(b_{\eta_4}) = M_{\eta_4} - G(b_{\eta_2}) = F_\alpha F_{\alpha+\beta}^{(2)} + q^4 F_\alpha^{(2)} F_{\alpha+2\beta} + q^5 F_\alpha^{(2)} F_{\alpha+\beta} F_\beta + q^{12} F_\alpha^{(3)} F_\beta^{(2)}.$$

As a first application of the algorithm for constructing elements of the canonical basis we give an algorithm for constructing highest-weight modules. Let λ be a dominant weight. Let v_λ be a highest-weight vector of the highest weight module $V(\lambda)$. Then according to [9], Theorem 11.10 (d), the set $\{G(b) \cdot v_\lambda \mid b \in \mathcal{B}(\infty)\} \setminus \{0\}$ is a basis of $V(\lambda)$. Using the path method it is straightforward to decide which $b \in \mathcal{B}(\infty)$ satisfy $G(b) \cdot v_\lambda = 0$. Let $b = b_\eta$ for some adapted string η. Then $G(b) \cdot v_\lambda = 0$ if and only if $f^\eta \xi_\lambda = 0$ (this follows from Lemma 5 below, along with [9], Theorem 11.10(d)). Furthermore, we only have to check $b \in \mathcal{B}(\infty)$ of weight ν such that the multiplicity of $\lambda - \nu$ in $V(\lambda)$ is non-zero. By a standard algorithm we can calculate the set of all those ν (using the path method for example). Now the nonzero $G(b) \cdot v_\lambda$ form a basis of the highest-weight module, and we use the $G(b)$ such that $G(b) \cdot v_\lambda = 0$ to rewrite all other vectors to linear combinations of basis elements. We remark that this algorithm is rather inefficient because the dimension of $U^-_{-\nu}$ grows quickly as the level of ν increases. A more efficient algorithm for constructing highest-weight modules is indicated in [7]. However, using the algorithm described above, it is possible to investigate single weight spaces of a highest-weight module, without constructing the module first.

Example 4 We use the same notation as in Example 3. Let $\lambda = \lambda_1$ be the first fundamental weight. Then $V(\lambda)$ has a weight space of weight $-\lambda_1 = \lambda - 2\alpha - 2\beta$. The elements of the canonical basis of weight $2\alpha + 2\beta$ are

$$\begin{aligned}
G(b_1) &= F_\alpha^{(2)} F_\beta^{(2)} \\
G(b_2) &= F_\alpha F_{\alpha+\beta} F_\beta + (q^3 + q^5) F_\alpha^{(2)} F_\beta^{(2)} \\
G(b_3) &= F_\alpha F_{\alpha+2\beta} + q F_\alpha F_{\alpha+\beta} F_\beta + (q^2 + q^6) F_\alpha^{(2)} F_\beta^{(2)} \\
G(b_4) &= F_{\alpha+\beta}^{(2)} + q^2 F_\alpha F_{\alpha+2\beta} + q^3 F_\alpha F_{\alpha+\beta} F_\beta + q^8 F_\alpha^{(2)} F_\beta^{(2)}.
\end{aligned}$$

They correspond to the strings $\eta_1 = (2,2,0,0)$, $\eta_2 = (1,1,1,1)$, $\eta_3 = (1,2,1,0)$ and $\eta_4 = (0,2,2,0)$ respectively. Now only $f^{\eta_3} \xi_\lambda \neq 0$. So $G(b_i) \cdot v_\lambda = 0$ for $i = 1,2,4$. Let x_i denote the principal monomial of $G(b_i)$. We see that $x_i \cdot v_\lambda = 0$ for $i = 1,2$, and $x_4 \cdot v_\lambda = -q^2 x_3 \cdot v_\lambda$.

We end this section with a sketch of a proof of case 3. of the formulas for the exponents m'_i in Section 2. We have to study the case where the root system is of type B_2. We let α, β be the simple roots, where β is long. First suppose that we use the reduced expression $s_\alpha s_\beta s_\alpha s_\beta$. Then by [13], Corollary 2, the set $C_1^{s,r}$ of adapted strings of weight $s\alpha + r\beta$ consists of all $\eta_{l,m} = (s-m, r-l, m, l)$, such that $0 \leq m \leq s$, $0 \leq l \leq r$ and $2(r-l) \geq m \geq 2l$. Here we have $\eta_{l,m} >_{\text{lex}} \eta_{l',m'}$ if $m < m'$ or $m = m'$ and $l < l'$.

Now

$$F_\alpha^{(s-m)}F_\beta^{(r-l)}F_\alpha^{(m)}F_\beta^{(l)} = \sum_{\substack{i,j\geq 0\\ i+j\leq r-l\\ 2i+j\leq m}} q^{(m-2i-j)(2r-2l-2i-j)+2(r-l-i-j)i}$$
$$\begin{bmatrix} s-2i-j\\ s-m\end{bmatrix}_\alpha \begin{bmatrix} r-i-j\\ l\end{bmatrix}_\beta F_\alpha^{(s-2i-j)}F_{2\alpha+\beta}^{(i)}F_{\alpha+\beta}^{(j)}F_\beta^{(r-i-j)}.$$

By studying the coefficients in this expression, and following the algorithm for computing elements of the canonical basis it can be shown that the principal monomial of $G(b_{\eta_{l,m}})$ is $F_\alpha^{(s-m)}F_{2\alpha+\beta}^{(l)}F_{\alpha+\beta}^{(m-2l)}F_\beta^{(r-m+l)}$ if $m\leq r$, and $F_\alpha^{(s-m)}F_{2\alpha+\beta}^{(m+l-r)}F_{\alpha+\beta}^{(2r-2l-m)}F_\beta^{(l)}$ if $m\geq r$.

Now suppose that we use the reduced expression $s_\beta s_\alpha s_\beta s_\alpha$. In this case the set $C_2^{s,r}$ of adapted strings of weight $s\alpha+r\beta$ consists of all $\zeta_{l,m}=(r-m, s-l, m, l)$ such that $0\leq l\leq s$, $0\leq m\leq r$, $s-l\geq m\geq l$ (cf. [13], Corollary 2). We have that $\zeta_{l,m}>_{\rm lex}\zeta_{l',m'}$ if $m<m'$ or $m=m'$ and $l<l'$. In this case the principal monomial of $G(b_{\zeta_{l,m}})$ is $F_\beta^{(r-m)}F_{\alpha+\beta}^{(2m-s+l)}F_{2\alpha+\beta}^{(s-l+m)}F_\alpha^{(l)}$ if $s\leq 2m$, and $F_\beta^{(r-m)}F_{\alpha+\beta}^{(l)}F_{2\alpha+\beta}^{(m-l)}F_\alpha^{(s+l-2m)}$ if $s\geq 2m$.

Suppose that the braid relation consists of replacing $s_\alpha s_\beta s_\alpha s_\beta$ by $s_\beta s_\alpha s_\beta s_\alpha$. We start with a PBW-monomial $x=F_\alpha^{(a)}F_{2\alpha+\beta}^{(b)}F_{\alpha+\beta}^{(c)}F_\beta^{(d)}$. We form the adapted string η such that $G(b_\eta)$ has principal monomial x. By the description of the principal monomials above we have that $\eta=(a, c+\max(b,d), 2b+c, \min(b,d))$. Now we use the bijection $\phi: C_1^{s,r}\to C_2^{s,r}$, such that $f^\theta=f^{\phi(\theta)}$ for all $\theta\in C_1^{s,r}$. According to [13], Proposition 2.4 it is given by $\phi(\eta)=(n_1,n_2,n_3,n_4)$, where $n_1=\max(b,\max(b,d)+c-a)$, $n_2=\max(a,c)+2b$, $n_3=\min(c+d, a+\min(b,d))$, $n_4=\min(a,c)$. Now $\phi(\eta)$ corresponds to the PBW-monomial $F_\beta^{(n_1)}F_{\alpha+\beta}^{(2n_3-n_2)}F_{2\alpha+\beta}^{(n_2-n_3)}F_\alpha^{(n_4)}$ if $n_2+n_4\leq 2n_3$, and to $F_\beta^{(n_1)}F_{\alpha+\beta}^{(n_4)}F_{2\alpha+\beta}^{(n_3-n_4)}F_\alpha^{(n_2+2n_4-2n_3)}$ if $n_2+n_4\geq 2n_3$. This implies the formulas in the case 3(a); the case 3(b) is similar.

Also the formula in case 2. can be proved this way.

6 Canonical bases of modules

For a dominant weight λ, let $V(\lambda)$ be the finite-dimensional highest-weight module over $U_q(\mathfrak{g})$ with highest weight λ. Let $v_\lambda\in V(\lambda)$ be a fixed highest weight vector. Set $\mathbf{B}(\lambda)=\{G(b)\cdot v_\lambda \mid G(b)\in\mathbf{B}\}\setminus\{0\}$. Then $\mathbf{B}(\lambda)$ is a basis of $V(\lambda)$ (cf. [9], Theorem 11.10), called the canonical basis of $V(\lambda)$. We can compute $\mathbf{B}(\lambda)$ by computing elements of $\mathbf{B}$. However, this method is rather inefficient as for many $G(b) \in \mathbf{B}$ we have that $G(b) \cdot v_\lambda = 0$. In this section we describe an algorithm for computing $\mathbf{B}(\lambda)$ without computing elements of $\mathbf{B}$ first.

Let $\varphi_\lambda : U^- \to V(\lambda)$ be the map defined by $\varphi_\lambda(u) = u \cdot v_\lambda$. Set $\mathcal{L}(\lambda) = \varphi_\lambda(\mathcal{L}(\infty))$, then $\mathcal{L}(\lambda)$ is a $\mathbb{Z}[q]$-lattice in $V(\lambda)$ spanned by all nonzero $G(b) \cdot v_\lambda$ for $G(b) \in \mathbf{B}$.

By φ_λ we also denote the induced map $\varphi_\lambda : \mathcal{L}(\infty)/q\mathcal{L}(\infty) \to \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$, and we set $\mathcal{B}(\lambda) = \varphi_\lambda(\mathcal{B}(\infty)) \backslash \{0\}$. Then $\mathcal{B}(\lambda)$ consists of all $x \cdot v_\lambda \bmod q\mathcal{L}(\lambda)$, where x runs through all PBW-monomials such that $G(b_x) \cdot v_\lambda \neq 0$. Therefore, $|\mathcal{B}(\lambda)| = \dim V(\lambda)$.

For $\alpha \in \Delta$ we use the Kashiwara operator $\widetilde{F}_\alpha : V(\lambda) \to V(\lambda)$ as defined in [9], 9.2. For $u \in \mathcal{L}(\infty)$ we have $\widetilde{F}_\alpha(u \cdot v_\lambda) = \widetilde{F}_\alpha(u) \cdot v_\lambda \bmod q\mathcal{L}(\lambda)$ (where the second $\widetilde{F}_\alpha$ is the Kashiwara operator on U^-), cf. [9], Proposition 10.9. Therefore $\widetilde{F}_\alpha$ is also a map from $\mathcal{B}(\lambda)$ into $\mathcal{B}(\lambda) \cup \{0\}$.

Let $\eta = (n_1, \dots, n_t)$ be an adapted string, relative to the reduced expression $w_0 = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_t}}$. Then we write $\widetilde{F}^\eta$ for $\widetilde{F}_{\alpha_{i_1}}^{n_1} \cdots \widetilde{F}_{\alpha_{i_t}}^{n_t}$ (where the $\widetilde{F}_{\alpha_k}$ are the Kashiwara operators on U^- or the Kashiwara operators on $V(\lambda)$). From Section 4 we recall that ξ_λ denotes the path joining 0 with λ by a straight line.

Lemma 5 *Let η be an adapted string, and set $b = \widetilde{F}^\eta(1) \in \mathcal{B}(\infty)$. Then $\varphi_\lambda(b) = 0$ if and only if $f^\eta(\xi_\lambda) = 0$.*

Proof. Set $b_\lambda = \varphi_\lambda(1) \in \mathcal{B}(\lambda)$. By [10], Theorem 4.1 we have that $f^\eta \xi_\lambda = 0$ if and only if $\widetilde{F}^\eta b_\lambda = 0$. By [9], Proposition 10.9 this is equivalent to $\varphi_\lambda(\widetilde{F}^\eta(1)) = 0$. □

For an adapted string η we denote by x_η the PBW-monomial with the property $\widetilde{F}^\eta(1) = x_\eta \bmod q\mathcal{L}(\infty)$. Note that we can compute x_η by using the algorithm for computing the action of $\widetilde{F}_\alpha$, described in Section 3. So Lemma 5 gives a straightforward algorithm for computing the elements of $\mathcal{B}(\lambda)$ of a given weight $\lambda - \nu$. We loop over all $\eta \in S_\nu$ and for every η such that $f^\eta \xi_\lambda \neq 0$ we compute $x_\eta \cdot v_\lambda \bmod q\mathcal{L}(\lambda)$.

By $^{-}$ we denote the involution of $V(\lambda)$ defined by $\overline{u \cdot v_\lambda} = \overline{u} \cdot v_\lambda$, for $u \in U^-$ (this is well defined by [9], Proposition 11.9(b)).

Lemma 6 *Let $b \in \mathcal{B}(\lambda)$. Then there is a unique element $v(b) \in \mathcal{L}(\lambda)$ such that $v(b) = b \bmod q\mathcal{L}(\lambda)$ and $\overline{v(b)} = v(b)$. Let $b' \in \mathcal{B}(\infty)$ be such that $\varphi_\lambda(b') = b$; then $v(b) = \varphi_\lambda(G(b'))$.*

Proof. It is clear that $\varphi_\lambda(G(b'))$ has the listed properties. Suppose that the element $v \in \mathcal{L}(\lambda)$ also has these properties. Then we write v as a linear combination of elements $\varphi_\lambda(G(b''))$. Because v is bar-invariant, the coefficients in this expression must be bar-invariant as well. Because the $\varphi_\lambda(G(b''))$ form a basis of $\mathcal{L}(\lambda)$ over $\mathbb{Z}[q]$, the coefficients must lie in $\mathbb{Z}[q]$. This means that the coefficients are elements of $\mathbb{Z}$. Since $v = b \bmod q\mathcal{L}(\lambda)$ we have that the only $\varphi_\lambda(G(b''))$ that has a non-zero coefficient is $\varphi_\lambda(G(b'))$. □

Let ν be a weight such that $\lambda - \nu$ is a weight of $V(\lambda)$. Let $\xi_1, \ldots, \xi_r$ be the paths in Π_λ ending in $\lambda - \nu$. Let $\widetilde{S}_\nu = \{\eta_1, \ldots, \eta_r\}$ be the corresponding adapted strings (relative to some fixed reduced expression for the longest element in the Weyl group). Note that by [13], Lemma 1.3, $\widetilde{S}_\nu$ is the set of all $\eta \in S_\nu$ (defined as in Section 4) such that $f^\eta \xi_\lambda \neq 0$.

By taking images under φ_λ, we get by (6), for $\eta \in \widetilde{S}_\nu$,

$$v(b_\eta) = M_\eta \cdot v_\lambda + \sum_{\eta' >_{\mathrm{lex}} \eta} c_{\eta,\eta'} v(b_{\eta'}). \tag{7}$$

For $1 \leq i \leq r$ set $u_i = x_{\eta_i} \cdot v_\lambda$, and write $u_i <_{\mathrm{lex}} u_j$ if $x_{\eta_i} <_{\mathrm{lex}} x_{\eta_j}$. Then Proposition 2 implies that

$$v(b_{\eta_i}) = u_i + \sum_{u_{i_k} <_{\mathrm{lex}} u_i} \zeta_{i,i_k} u_{i_k} \tag{8}$$

where all $\zeta_{i,i_k} \in q\mathbb{Z}[q]$. We call u_i the principal vector of $v(b_{\eta_i})$.

By Lemma 6 we have that $v(b_{\eta_i})$ is the unique bar-invariant element of $\mathcal{L}(\lambda)$ of the form of the right hand side of (8). (Note that all $u_i \in \mathcal{L}(\lambda)$, so that any vector of the form (8) belongs to $\mathcal{L}(\lambda)$.) Hence by (7), and (8) we have an algorithm for computing the $v(b_{\eta_i})$ that is highly analogous to the algorithm for computing elements of the canonical basis of U^-. The u_i play the role of the PBW-monomials, and we use principal vectors instead of principal monomials. Furthermore, the role of S_ν is taken by $\widetilde{S}_\nu$, and M_η is replaced by $M_\eta \cdot v_\lambda$. The details of the algorithm are exactly the same; we leave them to the reader.

Note that here we do not need to use the algorithm for multiplying PBW-monomials in U^- (this in contrast to the algorithm given in Section 5). We only need to be able to compute the action of any given PBW-monomial on elements of the module $V(\lambda)$.

Remark. By the results in [5], the algorithm for computing canonical bases of modules can, in principle, be used to compute Kazhdan-Lusztig polynomials in the A_n-case. However, since there are specialized methods available for that, it seems unlikely that such an algorithm will beat the existing ones (see [4]).

7 Tight monomials of small weight

Following [17] we call a monomial $m = F_{\alpha_{i_1}}^{(n_1)} \cdots F_{\alpha_{i_r}}^{(n_r)}$ tight if $m \in \mathbf{B}$. The canonical bases of the quantized enveloping algebras of types A_1, A_2 consist entirely of tight monomials. But this is not the case for most other types. In this section we study (experimentally) the number of elements of $\mathbf{B}$ of small weight that are tight monomials, for a few examples of root systems. (We note that different monomials can define the same element of U^-, so that the number of tight monomials is usually higher than the number of elements of $\mathbf{B}$ that are tight monomials.)

We use a rather crude algorithm for computing all tight monomials of a given weight ν. First we write down all monomials in the generators F_{α_i} of weight ν. Then we calculate the set $\mathbf{B}_\nu$ of elements of the canonical basis of weight ν. Finally, we check which monomials are contained in $\mathbf{B}_\nu$, by writing each monomial on the PBW-type basis used to represent the elements of $\mathbf{B}_\nu$.

We consider the root systems of type D_4, E_6, F_4, and G_2. In each case, for a few small weights, we computed the number of elements of $\mathbf{B}$ that are tight monomials. The computations were done using the GAP4 package QuaGroup, on a Linux system with a 600MHz Pentium III processor and 32MB of working memory for GAP. The results are listed in Table 1. In this table we represent a weight by giving its coefficients when written as a linear combination of simple roots. For this we have used the same ordering of simple roots as in [2].

We see that the time needed to compute $\mathbf{B}_\nu$ can increase rapidly when the height of ν increases. This is seen most dramatically in the case of G_2. The most time consuming part of the algorithm is formed by the "straightening" algorithm, that writes the monomials M_η as linear combinations of PBW-monomials. The average cost of this algorithm is less for the higher rank

type	weight ν	dim U_ν^-	time (s)	tight
D_4	$(2,2,2,2)$	37	2.6	37
D_4	$(1,2,3,4)$	27	5.5	23
D_4	$(3,2,3,3)$	37	6.4	27
D_4	$(3,3,3,3)$	128	78.6	120
E_6	$(1,1,2,2,1,1)$	110	8.1	110
E_6	$(1,1,2,3,1,1)$	146	21.4	116
E_6	$(1,1,2,3,2,1)$	354	112.0	318
F_4	$(1,3,2,1)$	28	1.5	18
F_4	$(2,2,2,2)$	50	3.9	50
F_4	$(1,2,3,4)$	84	32.5	64
F_4	$(4,3,2,1)$	56	8.7	48
G_2	$(3,3)$	8	0.7	6
G_2	$(4,4)$	13	12.4	8
G_2	$(5,5)$	20	1017.2	10

Table 1: The number of tight monomials of small weight. The third column lists the dimension of U_ν^-, and the fourth column the time taken (in seconds) to calculate $\mathbf{B}_\nu$. The last column contains the number of element of $\mathbf{B}_\nu$ that are tight monomials.

cases.

In all cases, except G_2, by far the most elements of $\mathbf{B}_\nu$ are tight monomials. The question is whether this remains true for weights of higher level. In the case of G_2 the percentage of tight monomials drops rather sharply as the level of ν increases.

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