Non-associative Gröbner Bases, Finitely-presented Lie Rings and the Engel Condition

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ABSTRACT

We give an algorithm for constructing a basis and a multiplication table of a finite-dimensional finitely-presented Lie ring. We apply this to construct the biggest t generator Lie rings that satisfy the n-Engel condition, for $(t, n) = (t, 2), (2, 3), (3, 3), (2, 4)$.

Categories and Subject Descriptors

G.4 [**Mathematical software**]: Algorithm design and analysis

General Terms

Algorithms

Keywords

Lie ring, Gröbner basis, Engel condition.

1. INTRODUCTION

A Lie ring L is a $\mathbb{Z}$-module equipped with a multiplication, $[\ ,\ ] : L \times L \to L$, $(x, y) \mapsto [x, y]$, that is anticommutative and satisfies the Jacobi identity. Lie rings appear naturally in several areas of group theory. Examples are the theory of nilpotent groups ([8]), the classification of p-groups ([11], [12]), and the restricted Burnside problem (see for example [9], [18]). Also [19] contains an account of some striking Lie ring techniques in group theory. On many occasions these Lie rings are given by a presentation by means of generators and relations (for a precise definition of this concept we refer to Section 5). Therefore it would be of great interest to have an algorithm for constructing a basis and multiplication table for a Lie ring given in this way. It is the objective of this paper to describe such an algorithm.

We say that a Lie ring is finite-dimensional if it is finitely generated as an abelian group. Of course it is only possible to construct a basis and multiplication table for Lie rings that are finite-dimensional. It is undecidable whether a given finitely presented Lie ring is finite dimensional, so we cannot require a general algorithm to terminate for any finitely-presented Lie ring. Our algorithm will terminate whenever the input defines a finite-dimensional Lie ring. Otherwise it will run forever.

Recently Gröbner bases in general non-associative algebras have been studied (see e.g., [3], [5], [13]). In this paper we use these to deal with finitely-presented Lie rings. However, because we are working over $\mathbb{Z}$ rather than over a field, the straightforward reduction algorithm does not work. Therefore, instead of a Gröbner basis we construct two sets of elements: a set of monic elements that is a Gröbner basis (and since its elements are monic we can use it in combination with the reduction algorithm), and a set that is merely linearly independent. This last set can be seen as taking care of the non-monic elements that occur. We call a pair of such sets (satisfying some additional conditions) a reduction pair. With a reduction pair of an ideal of $A_{\mathbb{Z}}(X)$ we can perform some of the tasks that are usually associated with Gröbner bases, like deciding ideal membership, and constructing a basis of the quotient. In the first part of the paper we describe an algorithm for constructing a reduction pair for a finitely-presented Lie ring that is finite dimensional.

There are algorithms known for constructing Gröbner bases of ideals of polynomial rings over $\mathbb{Z}$ (cf. [1]). For dealing with finitely-presented Lie rings a similar approach could be possible. This will be the subject of future research. However, it is likely that this will lead to a very similar algorithm.

There are a few algorithms known for constructing finitely-presented Lie algebras (e.g., [5], [6], [10]). These bear some similarity to the algorithms described here. The main difference lies in the treatment of the non-monic elements. Since the former algorithms work with Lie algebras over fields the problem of non-monic elements does not occur there. In [14] an algorithm is described to compute so-called nilpotent quotients of finitely-presented Lie rings. However, the approach via reduction pairs leads to a more general algorithm, that will work whenever the finitely-presented Lie ring is finite-dimensional.

In the second half of the paper we study Lie rings that satisfy the n-Engel identity, i.e., Lie rings L such that

$$[x, [x, \dots, [x, y] \dots]] = 0$$

for all $x, y \in L$ (n factors x). The study of these Lie rings goes back at least to [7]. It follows from a result of Zel'manov (see for example [18]) that a finitely-generated Lie ring that

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satisfies an n-Engel identity is nilpotent. By $E(t,n)$ we denote the "freest" t-generator Lie ring that satisfies the n-Engel identity. Now a natural question is what the structure of $E(t,n)$ is. For example, in [7], [16], [17] for various t,n upper bounds for the nilpotency class of $E(t,n)$ are given (with the difference that in these references the $E(t,n)$ are defined over fields). One problem when dealing with the n-Engel condition is that it is not a multilinear relation. In the second half of this paper (Theorem 14) we describe a set of relations with the following property: a Lie ring satisfies the n-Engel condition if and only if its basis elements satisfy the relations given in Theorem 14. In combination with the algorithms in the first half of the paper, this yields an algorithm to construct a basis and a multiplication table for $E(t,n)$. We illustrate this by constructing a basis of $E(t,2)$. Using an implementation of the algorithms in the computer algebra system GAP4 ([2]), we have constructed $E(2,3)$, $E(3,3)$ and $E(2,4)$. At the end of the paper we list the terms of the lower central series of these Lie rings.

The GAP4 implementations of the algorithms will be released as a GAP package in the near future.

This paper is arranged as follows. In Section 2 we describe the notation that we use, and review some of the theory of non-associative Gröbner bases. In Section 3 we introduce the notion of reduction pair, and study some of its properties. Section 4 has two lemmas that help to deal with anticommutativity and the Jacobi identity. Section 5 contains the main algorithms. Then in Sections 6, 7 we study Lie rings that satisfy the n-Engel condition.

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2. GRÖBNER BASES IN THE FREE ALGEBRA

Throughout X will be a finite set of symbols, also called letters. The free magma $M(X)$ on X is defined as follows. Firstly, $X \subset M(X)$, and secondly if $m,n \in M(X)$ then $(m,n) \in M(X)$. So $M(X)$ is the set of all bracketed words in the letters in X. The free magma is equipped with a binary operation: $m \cdot n = (m,n)$. The degree of elements of $M(X)$ is defined in the obvious way: $\deg(x) = 1$ for $x \in X$ and $\deg((m,n)) = \deg(m) + \deg(n)$.

We use a total order $<$ on $M(X)$ that is defined as follows. Firstly, the elements of X are ordered arbitrarily. Secondly, $\deg(m) < \deg(n)$ implies that $m < n$. Finally, if $m = (m', m'')$, $n = (n', n'')$ and $\deg(m) = \deg(n)$ then $m < n$ if and only if $m' < n'$ or $m' = n'$ and $m'' < n''$. We note that this ordering is multiplicative, i.e., $m < n$ implies $(p,m) < (p,n)$ and $(m,p) < (n,p)$ for all $p \in M(X)$. Furthermore, every subset of $M(X)$ has a minimal element.

The free algebra on X over $\mathbb{Z}$ is the $\mathbb{Z}$-span of $M(X)$. We denote it by $A_{\mathbb{Z}}(X)$. The binary operation of $M(X)$ is bilinearly extended to $A_{\mathbb{Z}}(X)$. The elements of $M(X)$ that occur in an $f \in A_{\mathbb{Z}}(X)$ are called the monomials of f. The leading monomial of f, denoted $\mathrm{LM}(f)$, is the biggest monomial of f. We say that f is monic if the coefficient of $\mathrm{LM}(f)$ is 1. The degree of f will be the degree of $\mathrm{LM}(f)$.

Now let $\sigma = (m_1, \dots, m_k)$ be a sequence of elements of $M(X)$ and $\delta = (d_1, \dots, d_k)$ a sequence of letters $d_i \in \{\mathrm{l}, \mathrm{r}\}$ (for "left" and "right"). Then we call the pair $\alpha = (\sigma, \delta)$ a *product prescription*. Corresponding to α there is a map $P_\alpha : M(X) \to M(X)$ defined inductively. If $k = 0$ then $P_\alpha(m) = m$ for all m. If $k > 0$ then set $\beta = ((m_2, \dots, m_k), (d_2, \dots, d_k))$, and $P_\alpha(m) = P_\beta((m_1, m))$ if $d_1 = \mathrm{l}$, and $P_\alpha(m) = P_\beta((m, m_1))$ if $d_1 = \mathrm{r}$. We extend P_α linearly to $A_{\mathbb{Z}}(X)$.

An $m \in M(X)$ is said to be a factor of $n \in M(X)$ if there is a product prescription α such that $P_\alpha(m) = n$. Let $G \subset A_{\mathbb{Z}}(X)$ be a set of monic elements. Let $f \in A_{\mathbb{Z}}(X)$ and suppose that there is a $g \in G$ such that $\mathrm{LM}(g)$ is a factor of a monomial m occurring in f. Let λ be the coefficient of m, and let α be a product prescription such that $P_\alpha(\mathrm{LM}(g)) = m$. Then we say that f reduces modulo G to $f - \lambda P_\alpha(g)$. From the properties of $<$ it follows that any sequence of reduction steps modulo G terminates with an element that cannot be reduced further. This element is called a normal form of f modulo G. We denote it f_G.

Here all ideals of $A_{\mathbb{Z}}(X)$ that we consider will be two sided.

Let $J \subset A_{\mathbb{Z}}(X)$ be an ideal. We call a $G \subset J$ a *Gröbner basis* of J if it consists of monic elements and for every $f \in J$ there is a $g \in G$ such that $\mathrm{LM}(g)$ is a factor of $\mathrm{LM}(f)$. If G is a Gröbner basis of J then all $f \in J$ reduce to zero modulo G. Furthermore, every $f \in A_{\mathbb{Z}}(X)$ has a unique normal form modulo G. Also, the monomials that do not reduce modulo G form a basis of $A_{\mathbb{Z}}(X)/J$. They are called the normal monomials modulo G.

Let $f, g \in A_{\mathbb{Z}}(X)$ be monic. If $\mathrm{LM}(f)$ is a factor of $\mathrm{LM}(g)$ then we let α be such that $P_\alpha(\mathrm{LM}(f)) = \mathrm{LM}(g)$ and set $S(f,g) = P_\alpha(f) - g$. Otherwise $S(f,g) = 0$. The following theorem is proved in [3].

Theorem 1. *Let $G \subset A_{\mathbb{Z}}(X)$ be a set of monic elements, that generates the ideal I. Then G is a Gröbner basis of I if and only if $S(f,g)$ reduces to zero modulo G for all $f, g \in G$.*

A set $G \subset A_{\mathbb{Z}}(X)$ is said to be self-reduced if there are no $g_1, g_2 \in G$, with $g_1 \neq g_2$ and such that $\mathrm{LM}(g_1)$ is a factor of $\mathrm{LM}(g_2)$.

Corollary 2. *Let $G \subset A_{\mathbb{Z}}(X)$. Suppose that all elements of G are monic, and that G is self-reduced. Then G is a Gröbner basis of the ideal it generates.*

3. REDUCTION PAIRS

It is clear that not every ideal in $A_{\mathbb{Z}}(X)$ has a Gröbner basis, since not every ideal is generated by monic elements. In this section we amend this by introducing the notion of reduction pair.

Let $G \subset A_{\mathbb{Z}}(X)$ be a Gröbner basis of the ideal it generates. Let $b_1, \dots, b_m \in A_{\mathbb{Z}}(X)$ be in normal form modulo G. Then we consider computing a basis of the space spanned by the b_i. First, let $m_1, \dots, m_r$ be the totality of monomials that occur in the b_i, with $m_1 > m_2 > \cdots > m_r$. Then we let an element b_i correspond to a vector of length r; the k-th coefficient being the coefficient of m_k in b_i. We let the vectors that we get be the rows of a matrix, and compute its Hermite normal form (cf. [15]). Then we transform the rows of this matrix back to elements of $A_{\mathbb{Z}}(X)$ and obtain a basis of the space spanned by the b_i. We call a basis computed in this way a *normal basis*.

Now let $G, B \subset A_{\mathbb{Z}}(X)$. Suppose that G is a Gröbner basis (of the ideal it generates), and B is a normal basis of a subspace of the space spanned by the normal monomials modulo G. Then we call $R = (G, B)$ a *reduction pair*.

Let $R = (G, B)$ be a reduction pair. Let $f \in A_{\mathbb{Z}}(X)$ be such that f_G lies in the $\mathbb{Z}$-span of B. Then we say that f reduces to zero modulo R. Let I be an ideal of $A_{\mathbb{Z}}(X)$, and $R = (G, B)$ a reduction pair such that $G, B \subset I$ and every $f \in I$ reduces to zero modulo R. Then we say that R is a reduction pair for I.

We call the reduction pair $R = (G, B)$ *closed* if for all product prescriptions α, and $b \in B$ we have that $P_\alpha(b)$ reduces to zero modulo R.

Lemma 3. *Let $R = (G, B)$ be a closed reduction pair. Let $I, J \subset A_{\mathbb{Z}}(X)$ be the ideals generated by $G \cup B$ and G respectively. Then the image of B in $A_{\mathbb{Z}}(X)/J$ is a basis of I/J. In particular, R is a reduction pair for I.*

Proof. Note that the normal monomials modulo G span $A_{\mathbb{Z}}(X)/J$. Hence B is linearly independent modulo J. Also the ideal I/J is generated by B. From the fact that R is closed it follows that $P_\alpha(b) \bmod J$ lies in the span of B for all $b \in B$ and product prescriptions α. Hence the span of B is an ideal in $A_{\mathbb{Z}}(X)/J$. It follows that this ideal is exactly I/J. For the last statement let $f \in I$. Let $\pi : A_{\mathbb{Z}}(X) \to A_{\mathbb{Z}}(X)/J$ be the projection map. Then π is bijective when restricted to the span of the normal monomials modulo G. Moreover, $\pi(B)$ is a basis of I/J. Hence $\pi(f_G)$ lies in the $\mathbb{Z}$-span of $\pi(B)$. This implies that f_G lies in the $\mathbb{Z}$-span of B. $\square$

Since in general it is rather difficult to check whether a given reduction pair $R = (G, B)$ is closed, we consider a weaker notion. Let $d > 0$. Suppose that R is such that $P_\alpha(b)$ reduces to zero modulo R for all product prescriptions α and $b \in B$ such that $\deg(P_\alpha(b)) \leq d$. Then we say that R is *d-closed*.

If we are given a reduction pair $R = (G, B)$, with G and B finite sets, then we can easily compute a d-closed reduction pair $R' = (G', B')$ such that $G' \cup B'$ generates the same ideal as $G \cup B$. Indeed, we compute all elements $(m \cdot b)_G$ and $(b \cdot m)_G$ where $b \in B$ and $m \in M(X)$ are such that $\deg(m) + \deg(b) \leq d$. By (possibly) enlarging B we can ensure that all such elements lie in the $\mathbb{Z}$-span of B. By continuing this process we eventually get a reduction pair $R' = (G', B')$ that is d-closed. We call R' the *d-closure* of R.

By $A_{\mathbb{Z}}(X)_d$ we denote the subspace of $A_{\mathbb{Z}}(X)$ spanned by all $m \in M(X)$ with $\deg(m) \leq d$.

Lemma 4. *Let $R = (G, B)$ be a d-closed reduction pair. Set $V_R = \{f \in A_{\mathbb{Z}}(X)_d \mid f \text{ reduces to zero modulo } R\}$. Then V_R is a subspace of $A_{\mathbb{Z}}(X)_d$ such that $P_\alpha(f) \in V_R$ for all $f \in V_R$ and product prescriptions α with $\deg(P_\alpha(f)) \leq d$*

Proof. An $f \in A_{\mathbb{Z}}(X)_d$ lies in V_R if and only if f can be written as

$$f = \lambda_1 P_{\alpha_1}(g_1) + \cdots + \lambda_r P_{\alpha_r}(g_r) + \mu_1 b_1 + \cdots + \mu_s b_s,$$

with $g_i \in G$, $b_j \in B$, $\lambda_i, \mu_j \in \mathbb{Z}$ and $\deg(b_j), \deg(P_{\alpha_i}(g_i)) \leq d$. This immediately shows that V_R is a linear subspace of $A_{\mathbb{Z}}(X)_d$. Since R is d-closed also the second statement follows. $\square$

4. ANTICOMMUTATIVITY AND JACOBI IDENTITIES

Let J be the ideal of $A_{\mathbb{Z}}(X)$ generated by all elements (m, m) and $(m, n) + (n, m)$ for $m, n \in M(X)$. Then J has a (infinite) Gröbner basis. Indeed, set $\mathcal{A}_2 = \{(x, x) \mid x \in X\} \cup \{(x, y) + (y, x) \mid x, y \in X, x < y\}$. For $k \geq 2$ let $\mathcal{A}_{k+1}$ consist of those elements (m, m), $(m, n) + (n, m)$ of degree $k + 1$ such that m, n are in normal form modulo $\cup_{i=2}^{k} \mathcal{A}_i$. Set $\mathcal{A} = \cup_{i \geq 2} \mathcal{A}_i$; then $\mathcal{A}$ is self-reduced, hence it is a Gröbner basis of the ideal it generates (Corollary 2). Now an $m \in M(X)$ reduces modulo $\mathcal{A}$ to $\pm m'$ (where $m' \in M(X)$), or to zero. This immediately implies that $\mathcal{A}$ generates J. We conclude that it is a Gröbner basis of J.

Lemma 5. *Let $G \subset A_{\mathbb{Z}}(X)$ consist of monic elements that are in normal form with respect to $\mathcal{A}$. Suppose that G is self reduced. Then $G \cup \mathcal{A}$ is a Gröbner basis (of the ideal it generates).*

Proof. Let $a_1, a_2 \in G \cup \mathcal{A}$. Then $S(a_1, a_2)$ can only be non trivial if $a_1 \in G$ and $a_2 \in \mathcal{A}$, with $\mathrm{LM}(a_1)$ a factor of $\mathrm{LM}(a_2)$. Suppose that this is the case, and let $S(a_1, a_2) = P_\alpha(a_1) - a_2$. According to Theorem 1 we have to show that $S(a_1, a_2)$ reduces to zero modulo $G \cup \mathcal{A}$.

First suppose that $a_2 = (m, n) + (n, m)$ with $m > n$, so that $\mathrm{LM}(a_2) = (m, n)$. Write $\alpha = ((m_1, \ldots, m_t), (d_1, \ldots, d_t))$. Here $t > 0$ so $m_t = m$, or $m_t = n$. Suppose that $m_t = n$ (and hence $d_t = \mathrm{r}$). Write $a_1 = p_1 + \sum_{i=2}^{k} \mu_i p_i$, where $p_i \in M(X)$ and $p_i > p_{i+1}$ for $1 \leq i < k$. Then $P_\alpha(a_1) = (m, n) + \sum_i \mu_i (P_\gamma(p_i), n)$, where $\gamma = ((m_1, \ldots, m_{t-1}), (d_1, \ldots, d_{t-1}))$ (note that $P_\gamma(p_1) = m$). Hence $P_\alpha(a_1) - a_2 = -(n, m) + \sum_i \mu_i (P_\gamma(p_i), n) = -(n, m + \sum_i \mu_i P_\gamma(p_i)) + f = -P_{\gamma'}(a_1) + f$, where f lies in the ideal generated by $\mathcal{A}$, and $\gamma' = ((m_1, \ldots, m_{t-1}, n), (d_1, \ldots, d_{t-1}, \mathrm{l}))$. So $S(a_1, a_2)$ reduces to f modulo a_1. Furthermore, f reduces to zero modulo $\mathcal{A}$. So in this case we are done. The case $m_t = m$ is proved analogously.

The case where $a_2 = (m, m)$ can be handled by similar arguments, and is left to the reader. $\square$

Note that it is easy to reduce elements of $A_{\mathbb{Z}}(X)$ modulo $\mathcal{A}$. For this we do not have to construct $\mathcal{A}$. In particular the fact that $\mathcal{A}$ is infinite is not a problem.

For $m, n, p \in M(X)$ we set

$$\mathrm{Jac}(m, n, p) = (m, (n, p)) + (p, (m, n)) + (n, (p, m)).$$

The proof of the next lemma is standard, see e.g., [4], Lemma 7.4.3.

Lemma 6. *Let $R = (G, B)$ be a d-closed reduction pair such that G contains $\mathcal{A}$. Suppose that $\mathrm{Jac}(x, m, n)$ reduces to zero modulo R for all $x \in X$ and all $m, n \in M(X)$ with $\deg(m) + \deg(n) + 1 \leq d$. Then $\mathrm{Jac}(m, n, p)$ reduces to zero modulo R for all $m, n, p \in M(X)$ with $\deg(m) + \deg(n) + \deg(p) \leq d$.*

5. THE ALGORITHM

A finitely-presented Lie ring is given by a set X of generators, and a set of relators $\mathcal{R} = \{h_1, \ldots, h_m\}$, where $h_i \in A_{\mathbb{Z}}(X)$. Let I be the ideal of $A_{\mathbb{Z}}(X)$ generated by the $h_i \in \mathcal{R}$ along with (m, m), $(m, n) + (n, m)$ and $\mathrm{Jac}(m, n, p)$ for $m, n, p \in M(X)$. Then the Lie ring with generators X and relations $\mathcal{R}$ is equal to the quotient $A_{\mathbb{Z}}(X)/I$.

It is the objective of this section to describe an algorithm for obtaining a reduction pair for I, in case $A_{\mathbb{Z}}(X)/I$ is finite-dimensional. Throughout, when we consider a reduction pair (G, B) we assume that $\mathcal{A} \subset G$, where $\mathcal{A}$ is as in the previous section. However, we do not construct $\mathcal{A}$, we just rewrite everything modulo $\mathcal{A}$, which is straightforward. This way the anticommutativity relations are being taken care of, and hence we only have to consider the h_i and the Jacobi identities.

We say that a reduction pair $R = (G, B)$ has property I_d if $G, B \subset I$, R is d-closed, and the generators of I of degree $\leq d$ reduce to 0 modulo R. Note that if $G, B = \emptyset$ then $R = (G, B)$ has I_0. Our strategy is to replace a reduction pair with I_d by one that has I_{d+1}. For this we have the following algorithm.

ALGORITHM 7. *Input: a reduction pair $R = (G, B)$ with property I_d.*
Output: a reduction pair $R' = (G', B')$ with I_{d+1}.

1. *Let $H \subset A_{\mathbb{Z}}(X)$ be the set of generators of I of degree $d + 1$.*
2. *Set $G' = G$ and $B' = B$.*
3. *For each $f \in H$ do the following:*
    - (a) *Set $\tilde{f} = f_{G'}$.*
    - (b) *If $\tilde{f}$ is not monic, or if there is a $g \in G'$ such that $\mathrm{LM}(\tilde{f})$ is a factor of a $\mathrm{LM}(g)$, then add $\tilde{f}$ to B'. Otherwise add it to G'.*
4. *Replace each $b \in B'$ by $b_{G'}$. Replace B' by a basis of the space spanned by B'. Replace (G', B') by its $(d + 1)$-closure, and return (G', B').*

PROPOSITION 8. *Let $R' = (G', B')$ be the output of Algorithm 7. Then R' is a reduction pair with property I_{d+1}. Moreover, if $f \in A_{\mathbb{Z}}(X)_d$ reduces to zero modulo R, then it reduces to zero modulo R'.*

PROOF. It is clear that G' consists of monic elements and that it is self-reduced. It is also clear that B' is a basis of a subspace of the space spanned by the normal monomials modulo G'. Hence R' is a reduction pair. It is also obvious that $G', B' \subset I$ and that R' is $(d + 1)$-closed. So the only thing that we have to show is that the generators of I of degree $\leq d + 1$ reduce to zero modulo R'.

Let $f \in A_{\mathbb{Z}}(X)_d$ reduce to zero modulo R. Then f_G lies in the $\mathbb{Z}$-span of B. We claim that this implies that $f_{G'}$ lies in the $\mathbb{Z}$-span of $b_{G'}$ for $b \in B$. Write $f_G = \mu_1 b_1 + \cdots + \mu_r b_r$, where $b_i \in B$, $\mu_i \in \mathbb{Z}$. Let f' be obtained from f_G by one reduction step modulo G'. So there is a $g \in G'$ such that $\mathrm{LM}(g)$ is a factor of a monomial m of f_G. Then m occurs in some of the b_i, with coefficient ν_i. Let α be such that $P_\alpha(\mathrm{LM}(g)) = m$ and set $b'_i = b_i - \nu_i P_\alpha(g)$. Let λ be the coefficient of m in f_G (so $\lambda = \sum_i \mu_i \nu_i$). Then $f' = f_G - \lambda P_\alpha(g)$, and $f' = \mu_1 b'_1 + \cdots \mu_r b'_r$. By repeating this argument we get the claim (note that since G' is a Gröbner basis, it makes no difference in which order we execute the reduction steps). Now since the $\mathbb{Z}$-span of B' includes the $\mathbb{Z}$-span of all $b_{G'}$ for $b \in B$, we have that f also reduces to zero modulo R'. In particular, all generators of I of degree $\leq d$ reduce to zero modulo R'.

Now let f be a generator of I of degree $d + 1$. When f is considered in Step 3 it is first replaced by $\tilde{f} = f - \sum_i \lambda_i P_{\alpha_i}(g_i)$, where $g_i \in G'$. Then either $\tilde{f}$ is added to G' or $\tilde{f}_{G'}$ lies in the $\mathbb{Z}$-span of B' in both cases f reduces to zero modulo R'. □

Now let $R = (G, B)$ be a reduction pair with I_d. Suppose that there is a monic $b \in B$. Then we perform the following operation. Let $g_1, \ldots, g_s$ be all elements of G such that $\mathrm{LM}(b)$ is a factor of $\mathrm{LM}(g_i)$. Set $\tilde{G} = (G \setminus \{g_1, \ldots, g_s\}) \cup \{b\}$, and $\tilde{B} = (B \setminus \{b\}) \cup \{g_1, \ldots, g_s\}$. We replace all elements of $\tilde{B}$ by their normal forms modulo $\tilde{G}$. Finally we let $R' = (G', B')$ be the d-closure of the reduction pair $(\tilde{G}, \tilde{B})$.

LEMMA 9. *R' is a reduction pair with property I_d.*

PROOF. We show that the generators of I of degree $\leq d$ reduce to zero modulo R'. The other requirements for I_d are quite clear. So let f be a generator of I of degree $\leq d$. Since f reduces to zero modulo R we can write f as

$$f = \lambda_1 P_{\alpha_1}(f_1) + \cdots + \lambda_r P_{\alpha_r}(f_r) + \mu_1 b_1 + \cdots + \mu_t b_t,$$

where $f_i \in G$, $b_i \in B$ and $\deg(b_i), \deg(P_{\alpha_i}(f_i)) \leq \deg(f)$. Let $V_{R'}$ be the space considered in Lemma 4. We show that every summand in the above expression lies in $V_{R'}$. Then by Lemma 4 the same is true for f, and we are done.

Consider a $P_{\alpha_i}(f_i)$. If f_i does not occur among the g_j then $P_{\alpha_i}(f_i)$ reduces to zero modulo G' (and hence lies in $V_{R'}$). If $f_i = g_j$ for some j then $g_j - \lambda P_\beta(b)$ (for some $\lambda \in \mathbb{Z}$ and β such that $\mathrm{LM}(P_\beta(b)) = \mathrm{LM}(g_j)$) lies in the $\mathbb{Z}$-span of B'. Since R' is d-closed, also $P_{\alpha_i}(g_j - \lambda P_\beta(b))_{G'}$ lies in the $\mathbb{Z}$-span of B'. Now $P_{\alpha_i}(f_i) = \lambda P_{\alpha_i} P_\beta(b) + P_{\alpha_i}(g_j - \lambda P_\beta(b))$. Therefore $P_{\alpha_i}(f_i)_{G'} = P_{\alpha_i}(g_j - \lambda P_\beta(b))_{G'}$ and hence lies in $V_{R'}$.

Now consider a b_i from the expression for f above. If it is equal to b then it reduces to zero modulo G' and hence lies in $V_{R'}$. Otherwise $b_i - \lambda P_\beta(b)$ (for certain λ and β) lies in the $\mathbb{Z}$-span of B'. And also in this case b_i reduces to zero modulo R'. □

Now we have the following algorithm.

ALGORITHM 10. *Input: the set of relators $\{h_1, \ldots, h_m\}$.*
Output: a reduction pair $R = (G, B)$ with I_d, and such that the $\mathbb{Z}$-span of B contains no monic elements.

1. *Using Algorithm 7 repeatedly, compute a reduction pair $R' = (G', B')$ with I_d.*
2. *Repeatedly move monic elements from B' to G' (see Lemma 9), until there are no monic elements left in B'.*
3. *Set $G = G'$, $B = B'$ and return $R = (G, B)$.*

PROPOSITION 11. *Algorithm 10 terminates, and produces the correct output.*

PROOF. Note that, when moving monic elements from B' to G' in Step 2, the set of normal monomials modulo G' of degree $\leq d$ decreases. So this step terminates, and hence the whole algorithm terminates. It follows that B has no monic elements. Since B is a normal basis (see Section 3) also the $\mathbb{Z}$-span of B contains no monic elements. □

Now suppose that $A_{\mathbb{Z}}(X)/I$ is finite-dimensional. For $d \geq 1$ let $R_d = (G_d, B_d)$ be the reduction pair computed with Algorithm 10. Let $V_d = V_{R_d}$ be the space of Lemma 4. Then $I = \cup_{d \geq 1} V_d$. Hence there are $d_0 \leq d_1$ such that for all $m \in M(X)$ of degree $d_0 + 1$ there is a monic $h \in V_{d_1}$ with $\mathrm{LM}(h) = m$. Since the $\mathbb{Z}$-span of B_{d_1} does not contain monic elements we conclude that all $m \in M(X)$ of degree $d_0 + 1$ reduce modulo G_{d_1} to linear combinations of monomials of degree $\leq d_0$. Now let e be such that $e \geq d_1$, $e \geq 2d_0 + 1$, and $\deg(h_i) \leq e$ for $1 \leq i \leq m$.

Proposition 12. *R_e is a reduction pair for I.*

Proof. First we claim that every $m \in M(X)$ reduces to a linear combination of monomials of degree $\leq d_0$ modulo G_e. This certainly holds for m of degree $d_0 + 1$. Modulo Jacobi identities of degree k every m of degree k can be rewritten as a linear combination of monomials (x, n), where $x \in X$ and $\deg(n) = k - 1$. So the claim holds by induction for m of degree $\leq e$. But if $\deg(m) > e$ then $m = (m', m'')$. By induction both m' and m'' reduce modulo G_e to elements of degree $\leq d_0$. But then m reduces to an element of degree $\leq 2d_0 < e$, and hence to an element of degree $\leq d_0$.

Set $W = \{f \in A_{\mathbb{Z}}(X) \mid f \text{ reduces to zero modulo } R_e\}$. Let $b \in B_e$, and $m \in M(X)$ a normal monomial modulo G_e. Then $\deg(b), \deg(m) \leq d_0$ so $\deg(mb) \leq e$. Hence mb reduces to zero modulo R_e (since R_e is e-closed). Similarly, bm reduces to zero modulo R_e. But then by the claim above the same holds when we replace m by any element of $M(X)$. In particular, R_e is closed. This implies that W is an ideal of $A_{\mathbb{Z}}(X)$.

Let $m, n \in M(X)$ and $x \in X$. Then modulo G_e, $\mathrm{Jac}(x, m, n)$ reduces to a linear combination of $\mathrm{Jac}(x, m', n')$ where $m', n' \in M(X)$ are of degree $\leq d_0$. So the degrees of the resulting elements are $\leq 2d_0 + 1 \leq e$. Since R_e has I_e, all these $\mathrm{Jac}(x, m', n')$ reduce to zero modulo R_e. Now by Lemma 6 all $\mathrm{Jac}(m, n, p)$ for $m, n, p \in M(X)$ reduce to zero modulo R_e. It follows that all generators of I lie in W. Hence $W = I$ and R_e is a reduction pair for I. $\square$

Now the main algorithm works as follows. With Algorithm 10 we compute reduction pairs R_d for $d = 1, 2, \ldots$. For each d we also compute the set of normal monomials of degree $\leq d$ modulo G_d. Since the quotient $A_{\mathbb{Z}}(X)/I$ is assumed to be finite-dimensional, at some point we find $d_0 < d_1$ such that there are no normal monomials of degree $d_0 + 1$ modulo G_{d_1}. Then we compute e as before, and R_e with Algorithm 10.

Now with R_e we can compute a basis of the quotient $A_{\mathbb{Z}}(X)/I$ (as abelian group). Let $U \subset A_{\mathbb{Z}}(X)$ be the space spanned by the normal monomials modulo G_e. Then reduction modulo G_e yields a surjective linear map $A_{\mathbb{Z}}(X) \to U$. Let $S \subset U$ be the $\mathbb{Z}$-span of B_e. Then by computing a Smith normal form (see [15]) we obtain a surjective linear map $U \to U/S$. Now we compose these maps and obtain a surjective linear map $\pi : A_{\mathbb{Z}}(X) \to U/S$, with kernel I. As abelian group $A_{\mathbb{Z}}(X)/I$ is isomorphic to U/S. Furthermore, we can use π to compute the Lie bracket on U/S.

Remark. It is also possible that the number of input relations, h_i, is infinite. (The next sections will have examples of that.) The algorithm can deal with this provided that we can only have a finite number of relations that involve a given finite set of monomials. Then the algorithm proceeds in exactly the same way. If the quotient algebra is finite-dimensional, then at some point a reduction pair (G, B) is computed such that there are no normal monomials modulo G of degree $> d_0$. Then the only h_i that are of interest are those that only involve the normal monomials of degree $\leq d_0$. And of those there is a finite number.

In particular this occurs when computing a nilpotent quotient. For this all monomials of degree exceeding some bound c are added to the relations. Then the quotient will be finite-dimensional, and nilpotent of class at most c.

In [14], Schneider has also developed an algorithm for computing nilpotent quotients of Lie rings. When the input relations are homogeneous (i.e., each h_i has monomials of the same degree), then it is possible to reformulate the algorithm described here in such a way that it becomes very similar to Schneider's algorithm. So for this case the two approaches yield similar algorithms.

6. THE N-ENGEL CONDITION

Let L be a Lie ring generated as abelian group by $\mathfrak{B} = \{x_1, \ldots, x_m\}$.

In the following we will use the right normed convention for iterated commutators. For example, $[xxxxy]$ will be the element $[x[x[x[xy]]]]$ of L.

Definition 13. *The Lie ring L satisfies the n-*Engel condition*, or L is n-Engel, if*

$$[\underbrace{x \ldots x}_{n} y] = 0,$$

for all $x, y \in L$. With $E(t, n)$ we denote the freest Lie ring with t generators which satisfies the n-Engel condition.

The n-Engel condition $[x \ldots xy] = 0$ is only linear in y. Hence in order to establish whether L is n-Engel it is not sufficient to check this condition for $x \in \mathfrak{B}$ only. Here we describe a set of conditions on the elements of $\mathfrak{B}$ only that are necessary and sufficient for L to be n-Engel.

Fix $(x_{i_1}, \ldots, x_{i_n}) \in \mathfrak{B}^n$. Suppose that among the i_l there are k_r indices equal to j_r, for $j_r \in \{i_1, \ldots, i_n\}$ (so $k_1 + \ldots + k_s = n$). Then we consider the sum of all elements $[x_{\sigma_1} \cdots x_{\sigma_n} y]$ where $(x_{\sigma_1}, \ldots, x_{\sigma_n})$ is a permutation of $(x_{i_1}, \ldots, x_{i_n})$. We denote this sum by

$$[(x_{j_1}^{(k_1)} \cdots x_{j_s}^{(k_s)})^* y].$$

Theorem 14. *The Lie ring L satisfies the n-Engel condition if and only if the following relations are satisfied*

$$\sum_{\substack{k_1, \ldots, k_s \geq 1 \\ k_1 + \ldots + k_s = n}} p_{j_1}^{k_1} \cdots p_{j_s}^{k_s} [(x_{j_1}^{(k_1)} \cdots x_{j_s}^{(k_s)})^* y] = 0, \qquad (1)$$

for all $y \in L$, $p_{j_r} = \pm 1$, $1 \leq j_1 \leq \ldots \leq j_s \leq m$, and $1 \leq s \leq n$.

Proof. Assume L to be n-Engel. By definition

$$[\underbrace{x \ldots x}_{n} y] = 0,$$

for all $x, y \in L$.

Note that every element of L can be written in the form

$$p_{i_1} x_{i_1} + \ldots + p_{i_r} x_{i_r},$$

where $p_{i_j} = \pm 1$ and $x_{i_j} \in \mathfrak{B}$. Hence

$$\underbrace{[(p_{j_1} x_{j_1} + \ldots + p_{j_s} x_{j_s}) \cdots (p_{j_1} x_{j_1} + \ldots + p_{j_s} x_{j_s})}_{n} y] = 0, \qquad (2)$$

for all $y \in L$, $x_{j_r} \in \mathfrak{B}$ with $1 \leq j_1 \leq \ldots \leq j_s \leq m$, and $s > 0$. It is sufficient to prove that from (2) we obtain (1). For this we use induction on s.

If $s = 1$ then (2) is

$$p_{j_1}^{n}[\underbrace{x_{j_1}\cdots x_{j_1}}_{n} y] = 0,$$

which is (1) for $s = 1$.

Assume $s > 1$. Then by expanding (2) we get

$$\begin{aligned}
0 \quad = \quad & \sum_{1\le j_1\le m} p_{j_1}^{n}[x_{j_1}\cdots x_{j_1} y] \\
+ \quad & \sum_{\substack{k_1,k_2\ge 1\\ k_1+k_2=n\\ 1\le j_1\le j_2\le m}} p_{j_1}^{k_1}p_{j_2}^{k_2}[(x_{j_1}^{(k_1)}x_{j_2}^{(k_2)})^{*}y] \\
\vdots \quad & \\
+ \quad & \sum_{\substack{k_1,\dots,k_{s-1}\ge 1\\ k_1+\dots+k_{s-1}=n\\ 1\le j_1\le\dots\le j_{s-1}\le m}} p_{j_1}^{k_1}\cdots p_{j_{s-1}}^{k_{s-1}}[(x_{j_1}^{(k_1)}\cdots x_{j_{s-1}}^{(k_{s-1})})^{*}y] \\
+ \quad & \sum_{\substack{k_1,\dots,k_s\ge 1\\ k_1+\dots+k_s=n}} p_{j_1}^{k_1}\cdots p_{j_s}^{k_s}[(x_{j_1}^{(k_1)}\cdots x_{j_s}^{(k_s)})^{*}y].
\end{aligned}$$

All but the last summand vanish by induction, therefore we get (1), concluding the proof of this implication.

Vice versa, suppose the relations (1) hold for all $s = 1, \dots, n$. We want to prove that L satisfies the n-Engel condition.

As before, the left part of (2) can be written as

$$\begin{aligned}
& \sum_{j_1} p_{j_1}^{n}[x_{j_1}\cdots x_{j_1} y] \\
+ \quad & \sum_{\substack{k_1,k_2\ge 1\\ k_1+k_2=n\\ 1\le j_1\le j_2\le m}} p_{j_1}^{k_1}p_{j_2}^{k_2}[(x_{j_1}^{(k_1)}x_{j_2}^{(k_2)})^{*}y] \\
\vdots \quad & \\
+ \quad & \sum_{\substack{k_1,\dots,k_{s-1}\ge 1\\ k_1+\dots+k_{s-1}=n\\ j_1\le\dots\le j_{s-1}}} p_{j_1}^{k_1}\cdots p_{j_{s-1}}^{k_{s-1}}[(x_{j_1}^{(k_1)}\cdots x_{j_{s-1}}^{(k_{s-1})})^{*}y] \\
+ \quad & \sum_{\substack{k_1,\dots,k_s\ge 1\\ k_1+\dots+k_s=n}} p_{j_1}^{k_1}\cdots p_{j_s}^{k_s}[(x_{j_1}^{(k_1)}\cdots x_{j_s}^{(k_s)})^{*}y]
\end{aligned}$$

but all the summands are zero by (1), therefore (2) holds. □

In the remainder of this section we explore the case $n = 2$. This also serves as an example for the algorithm in Section 5.

For 2-Engel (1) is equivalent to the relations

$$\begin{aligned}
[x_i[x_i y]] &= 0; \\
[x_i[x_j y]] + [x_j[x_i y]] &= 0,
\end{aligned}$$

for $y \in L$, $x_i, x_j \in \mathfrak{B}$ and $i < j$.

Let $X = \{y_1, \dots, y_t\}$ be a set of symbols which we order as $y_1 < \dots < y_t$. Let $I \subset A_{\mathbb{Z}}(X)$ be as in Section 5, where the set of h_i is infinite and consists of all

$$(m,(m,n)) \tag{3}$$

$$(n,(m,p)) + (m,(n,p)) \tag{4}$$

for $m, n, p \in M(X)$ and in (4) we have $m < n$.

Then $A_{\mathbb{Z}}(X)/I$ is equal to $E(t, 2)$.

We construct a reduction pair for I, using the algorithm of Section 5. We use "$\rightarrow$" for reduction, silently multiplying by -1 if convenient. Sometimes we add a subscript to the arrow to indicate the relation that we use.

Theorem 15. *The ideal I has a reduction pair $R = (G, B)$ where G consists of*

$$(y_i(y_i y_l)) \tag{5}$$

$$(y_j(y_i y_k)) + (y_i(y_j y_k)), \qquad (y_k(y_i y_j)) - (y_i(y_j y_k)) \tag{6}$$

for distinct $y_i, y_j, y_k, y_l \in X$ and $i < j < k$, along with all monomials of degree 4, 5 and 6 (that do not reduce modulo elements of G of lower degree) and B consists of

$$3(y_i(y_j y_k)) \tag{7}$$

for $i < j < k$.

Proof. First we analyse the relations of degree 3.

From (3) we get the elements (5). Let $i < j < k$. From (4) we get

$$(y_j(y_i y_k)) + (y_i(y_j y_k)) \tag{8}$$

$$(y_k(y_i y_j)) + (y_i(y_k y_j)) \quad \rightarrow \quad (y_k(y_i y_j)) - (y_i(y_j y_k)) \tag{9}$$

and

$$\begin{aligned}
(y_k(y_j y_i)) + (y_j(y_k y_i)) \quad &\rightarrow \quad (y_k(y_i y_j)) + (y_j(y_i y_k)) \\
&\underset{(9)}{\rightarrow} \quad (y_i(y_j y_k)) + (y_j(y_i y_k)) \\
&\underset{(8)}{\rightarrow} \quad 0
\end{aligned}$$

then every element of the form (4) of degree 3 reduces to 0 modulo (8), (9) (which are the same as (6)).

Also of degree 3 we have

$$\begin{aligned}
\mathrm{Jac}(y_i, y_j, y_k) \quad &= \quad (y_i(y_j y_k)) + (y_j(y_k y_i)) + (y_k(y_i y_j)) \\
&\rightarrow \quad (y_i(y_j y_k)) - (y_j(y_i y_k)) + (y_k(y_i y_j)) \\
&\rightarrow \quad 3(y_i(y_j y_k))
\end{aligned}$$

The monomials of degree ≤ 3 that are in normal form with respect to G are

$$\mathfrak{B} = \{y_1, \dots, y_t\} \cup \{(y_i y_j) \mid i < j\} \cup \{(y_i(y_j y_k)) \mid i < j < k\}.$$

The next step of the algorithm is to write down all relations of degree 4. This is rather a lot of work. Instead we show that every monomial of degree 4 that is a product of monomials in $\mathfrak{B}$ is contained in I. This shows that in this step of the algorithm all these monomials will be added to G.

By (3) we have, for $i < j < k$

$$\begin{aligned}
& (y_i(y_i(y_j y_k))) \\
& (y_j(y_j(y_i y_k))) \quad \underset{(6)}{\rightarrow} \quad (y_j(y_i(y_j y_k))) \\
& (y_k(y_k(y_i y_j))) \quad \underset{(6)}{\rightarrow} \quad (y_k(y_i(y_j y_k)))
\end{aligned}$$

The $\mathrm{Jac}(y_i, y_j, (y_i y_k))$, for $i < j$ and $i < k$, reduce to the monomials $((y_i y_j)(y_i y_k))$ modulo (6) and the relations above.

By (4) we have, for $i < j$ and $k < l$

$$(y_j(y_i(y_k y_l))) + (y_i(y_j(y_k y_l))). \tag{10}$$

Now, for $i < j < k < l$, the elements $\mathrm{Jac}(y_i, y_j, (y_k y_l))$ and $\mathrm{Jac}(y_k, y_l, (y_i y_j))$ reduce to $((y_i y_j)(y_i y_l)) - 2(y_i(y_j(y_k y_l)))$ and $((y_i y_j)(y_i y_l)) + 2(y_i(y_j(y_k y_l)))$ respectively, modulo (6) and (10). Taking the difference we see that $4(y_i(y_j(y_k y_l))) \in I$. But since $3(y_j(y_k y_l)) \in I$ it follows that $(y_i(y_j(y_k y_l))) \in I$.

Consequently, for $i < j < k < l$ we get

$$\begin{aligned}
&(y_j(y_i(y_k y_l))) \underset{(10)}{\to} (y_i(y_j(y_k y_l))) \to 0\\
&(y_k(y_i(y_j y_l))) \underset{(10)}{\to} (y_i(y_k(y_j y_l))) \underset{(6)}{\to} (y_i(y_j(y_k y_l))) \to 0\\
&(y_l(y_i(y_j y_k))) \underset{(10)}{\to} (y_i(y_l(y_j y_k))) \underset{(6)}{\to} (y_i(y_j(y_k y_l))) \to 0
\end{aligned}$$

therefore the elements $\mathrm{Jac}(y_i, y_j, (y_k y_l))$, $\mathrm{Jac}(y_i, y_k, (y_j y_l))$ and $\mathrm{Jac}(y_i, y_l, (y_j y_k))$ reduce to $((y_i y_j)(y_k y_l))$, $((y_i y_k)(y_j y_l))$ and $((y_i y_l)(y_j y_k))$ respectively.

We can stop because any monomial of degree 4 is contained in I therefore also any monomial of degree ≥ 4 is contained in I. This can be shown by induction on the degree. By the Jacobi identity a monomial of degree d is a linear combination (y_i, m) where $y_i \in X$ and m is a monomial of degree $d - 1$. Furthermore every 2-Engel relation is homogeneous, so if we analyse relations of degree > 4 we do not obtain relations of degree ≤ 4. Hence G contains all monomials of degree $4, 5, 6$ that do not reduce modulo elements of G of lower degree. Note that this means that all monomials of higher degree reduce to zero modulo G. $\square$

It follows that $E(t, 2)$ is spanned by

$$\mathfrak{B} = \{y_1, \ldots, y_t\} \cup \{(y_i y_j) \mid i < j\} \cup \{(y_i(y_j y_k)) \mid i < j < k\}.$$

Hence

$$\dim(E(t, 2)) = t + \binom{t}{2} + \binom{t}{3}.$$

In fact, there are t elements in $\{y_1, \ldots, y_t\}$ and $\binom{t}{2}$ elements in $\{(y_i y_j) \mid i < j\}$, and $\binom{t}{3}$ elements in $\{(y_i(y_j y_k)) \mid i < j < k\}$.

7. FURTHER N-ENGEL LIE RINGS

A finitely generated abelian group can uniquely be written as $(\mathbb{Z}/d_1\mathbb{Z})^{k_1} \oplus \cdots \oplus (\mathbb{Z}/d_r\mathbb{Z})^{k_r} \oplus \mathbb{Z}^m$, where d_i divides d_{i+1}. We denote this group by $d_1^{k_1} \cdots d_r^{k_r} 0^m$.

Let L be a Lie ring. Then we set $L^1 = L$ and for $k \geq 1$ we let L^{k+1} be the subring generated (as abelian group) by all $[x, y]$ for $x \in L$, $y \in L^k$.

We have implemented the algorithms described in this paper in the computer algebra system GAP4 ([2]). Using this implementation we have obtained the following results.

1. Let $L = E(2, 3)$. Then as abelian groups, $L^1 = 2^3 0^5$, $L^2 = 2^3 0^3$, $L^3 = 2^3 0^2$, $L^4 = 2^3$, $L^5 = 2^2$, $L^6 = 0$.

2. Let $L = E(3, 3)$. Then as abelian groups, $L^1 = 2^{40} 10^3 0^{17}$, $L^2 = 2^{40} 10^3 0^{14}$, $L^3 = 2^{40} 10^3 0^{11}$, $L^4 = 2^{40} 10^3 0^3$, $L^5 = 2^{33} 10^3$, $L^6 = 2^{18}$, $L^7 = 2^9$, $L^8 = 2^3$.

3. Let $L = E(2, 4)$ then $L = 5^{15} 10^8 0^{11}$, $L^2 = 5^{15} 10^8 0^9$, $L^3 = 5^{15} 10^8 0^8$, $L^4 = 5^{15} 10^8 0^6$, $L^5 = 5^{15} 10^8 0^3$, $L^6 = 5^{16} 10^7 0^1$, $L^7 = 5^{15} 10^5$, $L^8 = 5^{14} 10^2$, $L^9 = 5^{12}$, $L^{10} = 5^6$, $L^{11} = 5^3$, $L^{12} = 5^1$.

In particular, the nilpotency class of $E(2, 3)$ over a field is 3 if the characteristic is not 2, and 5 if the characteristic is 2. The nilpotency class of $E(3, 3)$ is 4 if the characteristic is different from $2, 5$, it is 5 in characteristic 5, and 8 in characteristic 2. The nilpotency class of $E(2, 4)$ is 12 over fields of characteristic 5, it is 8 over fields of characteristic 2, and it is 6 over other fields.

On a 2GHz processor constructing $E(2, 3)$, $E(3, 3)$, and $E(2, 4)$ cost respectively 0.1 seconds, 107 seconds, and about 12 hours. The main reason for this sharp increase in running times is that fact that the number of relations that describe the n-Engel property explodes. However, we feel that it should be possible to construct a few other of these n-Engel Lie rings. This will be a theme of future research.

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