Constructing homomorphisms between Verma modules

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Abstract

We describe a practical method for constructing a nontrivial homomorphism between two Verma modules.

A theorem of Verma, Bernstein-Gel'fand-Gel'fand gives a straightforward criterion for the existence of a nontrivial homomorphism between Verma modules. Moreover, the theorem states that such homomorphisms are always injective. In this paper we consider the problem of explicitly constructing such a homomorphism if it exists.

In [6] a construction of homomorphisms between Verma modules is described. However, it is not easy to see how to carry out this construction in practice. The method described here can be seen as a variant of the construction in [6], the difference being that we are able to obtain the homomorphism explicitly.

This paper is organised as follows. In Section 1 we review the theoretical concepts and notation that we use, and describe the problem we deal with. In Section 2 we derive a few commutation formulas in the field of fractions of $U(\mathfrak{n}^-)$. Then in Section 3 the construction of a homomorphism between Verma modules is described. In Section 4 we compare this with the construction from [6].

1 Preliminaries

Let $\mathfrak{g}$ be a semisimple Lie algebra, with root system Φ, relative to a Cartan subalgebra $\mathfrak{h}$. We let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be a fixed set of simple roots. Let $\Phi^+ = \{\alpha_1, \ldots, \alpha_s\}$ be the set of positive roots (note that here the simple roots are listed first). Then there are root vectors $y_i = x_{-\alpha_i}$, $x_i = x_{\alpha_i}$, and basis vectors $h_i \in \mathfrak{h}$, such that the set $\{x_1, \ldots, x_s, y_1, \ldots, y_s, h_1, \ldots, h_l\}$ forms a Chevalley basis of $\mathfrak{g}$ (cf. [4]). We have that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^-$, $\mathfrak{n}^+$ are the subalgebras spanned by the y_i, x_i respectively.

In the sequel, if $\beta = \alpha_i \in \Phi^+$, then we also write y_β in place of y_i.

We let P denote the integral weight lattice spanned by the fundamental weights $\lambda_1 \ldots, \lambda_l$. Also $\mathbb{Q}P = \mathbb{Q}\lambda_1 + \cdots + \mathbb{Q}\lambda_l$. For $\lambda, \mu \in \mathbb{Q}P$ we write $\mu \leq \lambda$ if $\mu = \lambda - \sum_{i=1}^{l} k_i \alpha_i$, where $k_i \in \mathbb{Z}_{\geq 0}$. Then $\leq$ is a partial order on $\mathbb{Q}P$.

For $\alpha \in \Phi$ we have the reflection $s_\alpha : \mathbb{Q}P \to \mathbb{Q}P$, given by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$.

Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. We consider $U(\mathfrak{g})$ as a $\mathfrak{g}$-module by left multiplication. Let $\lambda = \sum a_i \lambda_i \in \mathbb{Q}P$, and let $J(\lambda)$ be the $\mathfrak{g}$-submodule of $U(\mathfrak{g})$

generated by $h_i - a_i + 1$ for $1 \leq i \leq l$ and x_i for $1 \leq i \leq s$. Then $M(\lambda) = U(\mathfrak{g})/J(\lambda)$ is a $\mathfrak{g}$-module. It is called a Verma module. As $U(\mathfrak{g}) = U(\mathfrak{n}^-) \oplus J(\lambda)$ we see that $U(\mathfrak{n}^-) \cong M(\lambda)$ (as $U(\mathfrak{n}^-)$-modules). Let v_λ denote the image of 1 under this isomorphism. Then $h_i \cdot v_\lambda = (a_i - 1)v_\lambda$, and $x_i \cdot v_\lambda = 0$. Furthermore, all other elements of $M(\lambda)$ can be written as $u \cdot v_\lambda$, where $u \in U(\mathfrak{n}^-)$.

Let $\nu = \sum_{i=1}^{l} k_i \alpha_i$, where $k_i \in \mathbb{Z}_{\geq 0}$. Then we let $U(\mathfrak{n}^-)_\nu$ be the span of all $y_{i_1} \cdots y_{i_r}$ such that $\alpha_{i_1} + \cdots + \alpha_{i_r} = \nu$.

For a proof of the following theorem we refer to [1], [2].

Theorem 1 (Verma, Bernstein-Gel'fand-Gel'fand) *Let $\lambda, \mu \in \mathbb{Q}P$, and set*

$$R = \mathrm{Hom}_{U(\mathfrak{g})}(M(\mu), M(\lambda)).$$

Then

1. $\dim R \leq 1$,
2. *non-trivial elements of R are injective,*
3. $\dim R = 1$ *if and only if there are positive roots $\alpha_{i_1}, \ldots, \alpha_{i_k}$ such that*

$$\mu \leq s_{\alpha_{i_1}}(\mu) \leq s_{\alpha_{i_2}} s_{\alpha_{i_1}}(\mu) \leq \cdots \leq s_{\alpha_{i_k}} \cdots s_{\alpha_{i_1}}(\mu) = \lambda.$$

The problem we consider is to construct a non-trivial element in R if $\dim R = 1$. This boils down to finding an element in R if $\mu = s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ and $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{>0}$. Suppose that we are in this situation, and set $h = \langle \lambda, \alpha^\vee \rangle$. An element $Y \cdot v_\lambda \in M(\lambda)$, where $Y \in U(\mathfrak{n}^-)$ is said to be *singular* if $x_\alpha \cdot (Y \cdot v_\lambda) = 0$ for $\alpha \in \Phi^+$. Let $\psi \in R$ be a non-trivial $U(\mathfrak{g})$-homomorphism. Then $\psi(v_\mu) = Y \cdot v_\lambda$ for some $Y \in U(\mathfrak{n}^-)$ with $Y \cdot v_\lambda$ singular. We have $h_i y = y h_i - \langle \nu, \alpha_i^\vee \rangle y$ for all $y \in U(\mathfrak{n}^-)_\nu$. Hence $h_i \cdot (y \cdot v_\lambda) = (\langle \lambda - \nu, \alpha_i^\vee \rangle - 1) y v_\lambda$. So, as $h_i \cdot (Y v_\lambda) = (\langle \mu, \alpha_i^\vee \rangle - 1) Y v_\lambda$ we see that $Y \in U(\mathfrak{n}^-)_{h\alpha}$. Conversely, if we have a $Y \in U(\mathfrak{n}^-)_{h\alpha}$ such that $Y \cdot v_\lambda$ is singular, then $\psi : M(\mu) \to M(\lambda)$ defined by $\psi(Y' \cdot v_\mu) = Y'Y \cdot v_\lambda$ will be a non-trivial element of R. So the problem reduces to finding a $Y \in U(\mathfrak{n}^-)_{h\alpha}$ such that $Y \cdot v_\lambda$ is singular. Note that this can be done by writing down a basis for $U(\mathfrak{n}^-)$ and computing a set of linear equations for Y. However, this algorithm becomes rather cumbersome if $\dim U(\mathfrak{n}^-)_{h\alpha}$ gets large. We will describe a more direct method.

2 The field of fractions

From [2], §3.6 we recall that $U(\mathfrak{n}^-)$ has a (non-commutative) field of fractions, denoted by $K(\mathfrak{n}^-)$. It consists of all elements ab^{-1} for $a \in U(\mathfrak{n}^-)$, $b \in U(\mathfrak{n}^-) \setminus \{0\}$. For the definitions of addition and multiplication in $K(\mathfrak{n}^-)$ we refer to [2], §3.6. We note that $aa^{-1} = a^{-1}a = 1$.

Let $\alpha, \beta \in \Phi^+$. If $\alpha + \beta \in \Phi^+$ then we let $N_{\alpha,\beta}$ be the scalar such that $[y_\alpha, y_\beta] = -N_{\alpha,\beta} y_{\alpha+\beta}$. Also set $P_{\alpha,\beta} = \{i\alpha + j\beta \mid i, j \geq 0\} \cap \Phi^+$. Then there are seven possibilities for $P_{\alpha,\beta}$:

(I) $P_{\alpha,\beta} = \{\alpha, \beta\}$,

(II) $P_{\alpha,\beta} = \{\alpha, \beta, \alpha + \beta\}$

(III) $P_{\alpha,\beta} = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta\}$,

(IV) $P_{\alpha,\beta} = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$,

(V) $P_{\alpha,\beta} = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}$

(VI) $P_{\alpha,\beta} = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta, 2\alpha + 3\beta\}$,

(VII) $P_{\alpha,\beta} = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, \alpha + 2\beta\}$.

Lemma 2 *In case (I) we have $y_\beta^m y_\alpha^n = y_\alpha^n y_\beta^m$ for all $m, n \in \mathbb{Z}$.*

Proof. If $n > 0$ then $y_\beta y_\alpha^n = y_\alpha^n y_\beta$. Multiplying this relation on the left and on the right by y_α^{-n} we get $y_\beta y_\alpha^{-n} = y_\alpha^{-n} y_\beta$. So we have $y_\beta y_\alpha^n = y_\alpha^n y_\beta$ for all $n \in \mathbb{Z}$. From this it follows that $y_\beta^m y_\alpha^n = y_\alpha^n y_\beta^m$ for $m > 0$, $n \in \mathbb{Z}$. If we now multiply this from the left and the right by y_β^{-m} we get the result for $m < 0$ as well. □

Note that
$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!},$$
and this defines these binomial coefficients for arbitrary $n \in \mathbb{Q}$, and $k \in \mathbb{Z}_{\geq 0}$. In fact, we see that $\binom{n}{k}$ is a polynomial of degree k in n. Note also that if $n \in \mathbb{Z}$ and $0 \leq n < k$ then the coefficient is 0.

Lemma 3 *In case (II) we have for $m \geq 0$, $n \in \mathbb{Z}$,*
$$y_\beta^m y_\alpha^n = \sum_{k=0}^{m} N_{\alpha,\beta}^k \binom{m}{k} \binom{n}{k} k! y_\alpha^{n-k} y_\beta^{m-k} y_{\alpha+\beta}^k.$$

Proof. First of all, this formula is known for $m, n \geq 0$ (see, e.g., [3]). In particular, for $n > 0$ we have $y_\beta y_\alpha^n = y_\alpha^n y_\beta + N_{\alpha,\beta} n y_\alpha^{n-1} y_{\alpha+\beta}$. If we multiply this relation on the left and the right with y_α^{-n}, and use Lemma 2, then we get it for all $n \in \mathbb{Z}$. Now the formula for $m > 1$ is proved by induction. □

Lemma 4 *In case (III) we have for $m \geq 0$, $n \in \mathbb{Z}$,*
$$y_\beta^m y_\alpha^n = \sum_{\substack{k,l \geq 0 \\ k+2l \leq m}} c_{k,l}^{m,n} y_\alpha^{n-k-l} y_\beta^{m-k-2l} y_{\alpha+\beta}^k y_{\alpha+2\beta}^l,$$
where
$$c_{k,l}^{m,n} = N_{\alpha,\beta}^{k+l} (\frac{1}{2} N_{\beta,\alpha+\beta})^l \binom{n}{k+l} \binom{m}{k+2l} \binom{k+l}{l} (k+2l)!.$$

Proof. This goes in exactly the same way as the proof of Lemma 3. □

Lemma 5 *In case (IV) we have*

$$y_\beta^m y_\alpha^n = \sum_{\substack{k,l\geq 0\\ k+l\leq m}} c_{k,l}^{m,n} y_\alpha^{n-k-2l} y_\beta^{m-k-l} y_{\alpha+\beta}^k y_{2\alpha+\beta}^l,$$

where

$$c_{k,l}^{m,n} = N_{\alpha,\beta}^{k+l}(\frac{1}{2}N_{\alpha,\alpha+\beta})^l \binom{n}{k+2l}\binom{m}{k+l}\binom{k+l}{l}(k+2l)!.$$

Proof. Again we get the formula for $m,n\geq 0$ from [3]. In this case the formula for $m=1$, $n\geq 0$ reads

$$y_\beta y_\alpha^n = y_\alpha^n y_\beta + N_{\alpha,\beta} n y_\alpha^{n-1} y_{\alpha+\beta} + N_{\alpha,\beta}N_{\alpha,\alpha+\beta}\binom{n}{2} y_\alpha^{n-2} y_{2\alpha+\beta}.$$

If we multiply this on the left and the right by y_α^{-n}, and use Lemmas 2, 3 we get the same relation with n replaced by $-n$. So the case $m=1$, $n\in\mathbb{Z}$ follows. The formula for $m>1$ now follows by induction. □

The cases (V), (VI), (VII) can only occur when the root system has a component of type G_2. We omit the formulas for these cases; they can easily be derived from those given in [3].

Now let $a = y_1^{n_1}\cdots y_s^{n_s}$ be a monomial in $U(\mathfrak{n}^-)$. For $\beta\in\Phi^+$ and $m,n\in\mathbb{Z}$ consider the element $y_\beta^m a y_\beta^{-n}$. By repeatedly applying Lemmas 2, 3, 4, 5 we see that

$$y_\beta^m a y_\beta^{-n} = \sum_{(k_1,\ldots,k_t)\in I} c(k_1,\ldots,k_t) y_\beta^{m-n-p_1k_1-\cdots-p_tk_t} a(k_1,\ldots,k_t). \tag{1}$$

Here the $a(k_1,\ldots,k_t)\in U(\mathfrak{n}^-)$, the (finite) index set I, the $p_i\in\mathbb{Z}_{>0}$ are all independent of n, they only depend on a. Only the exponents of y_β and the coefficients $c(k_1,\ldots,k_t)$ (which are polynomials in n) depend on n.

Now we take $m,n\in\mathbb{Q}$ such that $m-n\in\mathbb{Z}$. Then we define $y_\beta^m a y_\beta^{-n}$ to be the right-hand side of (1), and we say that $y_\beta^m a y_\beta^{-n}$ is an element of $K(\mathfrak{n}^-)$. More generally, if Y is a linear combination of monomials, and $m,n\in\mathbb{Q}$ such that $m-n\in\mathbb{Z}$ then $y_\beta^m Y y_\beta^{-n}$ is an element of $K(\mathfrak{n}^-)$.

In practice, when computing the right-hand side of (1), we do not compute the $c(k_1,\ldots,k_t)$ as polynomials in n. If n is not integral we apply the formulas of Lemmas 2, 3, 4, 5, where we deal with n as if it were an integer. At the end we multiply with y_β^m, and the non-integral exponent disappears.

3 Constructing singular vectors

Here we suppose that we are given a $\lambda\in\mathbb{Q}P$ and $\alpha\in\Phi^+$ with $\langle\lambda,\alpha^\vee\rangle = h\in\mathbb{Z}_{>0}$. The problem is to find a $Y\in U(\mathfrak{n}^-)_{h\alpha}$ such that $Y\cdot v_\lambda$ is a singular vector.

We recall that $l = |\Delta|$ is the rank of the root system. Let $1 \leq i \leq l$, then

$$x_i y_i^r \cdot v_\lambda = r(\langle \lambda, \alpha_i^\vee \rangle - r) y_i^{r-1} \cdot v_\lambda. \tag{2}$$

Lemma 6 *Suppose that $\alpha \in \Delta$, i.e., $\alpha = \alpha_i$, $1 \leq i \leq l$. Then $y_i^h \cdot v_\lambda$ is a singular vector.*

Proof. This follows from (2), cf. the proof of [1], Lemma 2. □

In the sequel, by an embedding $\phi : M(\mu) \hookrightarrow M(\lambda)$ we always mean an injective $U(\mathfrak{g})$-homomorphism.

Lemma 7 *Suppose that $\nu, \eta \in P$, and $\beta \in \Delta$ is such that $m = \langle \nu, \beta^\vee \rangle$ is a non-negative integer. Suppose further that we have an embedding $\psi : M(\nu) \hookrightarrow M(\eta)$ given by $\psi(v_\nu) = Y v_\eta$. Set $n = \langle \eta, \beta^\vee \rangle$. Then $y_\beta^m Y y_\beta^{-n}$ is an element of $U(\mathfrak{n}^-)$ and we have an embedding $\phi : M(s_\beta \nu) \hookrightarrow M(s_\beta \eta)$ given by $\phi(v_{s_\beta \nu}) = y_\beta^m Y y_\beta^{-n} \cdot v_{s_\beta \eta}$.*

Proof. If $n \leq 0$ then the first statement is clear. The embedding ϕ is the composition $M(s_\beta \nu) \hookrightarrow M(\nu) \hookrightarrow M(\eta) \hookrightarrow M(s_\beta \eta)$, where the first and the third maps follow from Lemma 6.

If $n > 0$, then we view $M(s_\beta \eta)$ as a submodule of $M(\eta)$. We have $v_{s_\beta \eta} = y_\beta^n v_\eta$ (Lemma 6). Set $v = y_\beta^m Y v_\eta$; then v is a singular vector (being the image of $v_{s_\beta \nu}$ under $M(s_\beta \nu) \hookrightarrow M(\nu) \hookrightarrow M(\eta)$). We claim that $v \in M(s_\beta \eta)$. Suppose that this claim is proved. Then there is a $Y' \in U(\mathfrak{n}^-)$ such that $v = Y' v_{s_\beta \eta}$. But that means that $y_\beta^m Y = Y' y_\beta^n$, and the lemma follows.

The claim above is proved in [1]. For the sake of completeness we transcribe the argument. Set $V = M(\eta)/M(s_\beta \eta)$, and let $\bar{v}_\nu$ denote the image of $\psi(v_\nu)$ in V; then $\bar{v}_\nu = X \cdot \bar{v}_\eta$, for some $X \in U(\mathfrak{n}^-)$. For $k \geq 0$ write $y_\beta^k X = X_1 y_\beta^{k_1}$. By increasing k we can get k_1 arbitraily large (cf. [2], Lemma 7.6.9; it also follows by straightforward weight considerations). By Lemma 6 we know that $y_\beta^n v_\eta \in M(s_\beta \eta)$. Therefore there is a $k > 0$ such that $y_\beta^k \bar{v}_\nu = 0$. Then by using (2) we see that the smallest such k must be equal to m. □

Proposition 8 *Let $\nu, \eta \in \mathbb{Q}P$ be such that $\nu = s_\gamma(\eta) = \eta - k\gamma$, where $\gamma \in \Phi^+$ and $k \in \mathbb{Z}_{>0}$. Let $\beta \in \Delta$, $\beta \neq \gamma$. Let $Y \in U(\mathfrak{n}^-)_{k\gamma}$ be such that $Y \cdot v_\eta$ is singular. Set $m = \langle \nu, \beta^\vee \rangle$, $n = \langle \eta, \beta^\vee \rangle$. Then $y_\beta^m Y y_\beta^{-n}$ is an element of $K(\mathfrak{n}^-)$; it is even an element of $U(\mathfrak{n}^-)$. Secondly, we have an embedding $\phi : M(s_\beta \nu) \hookrightarrow M(s_\beta \eta)$ given by $\phi(v_{s_\beta \nu}) = y_\beta^m Y y_\beta^{-n} \cdot v_{s_\beta \eta}$.*

Proof. We have that $m - n = -k\langle \gamma, \beta^\vee \rangle \in \mathbb{Z}$, so $y_\beta^m Y y_\beta^{-n}$ is an element of $K(\mathfrak{n}^-)$.

Set $V = \{\mu \in \mathbb{Q}P \mid \langle \mu, \gamma^\vee \rangle = k\}$, which is a hyperplane in $\mathbb{Q}P$. Let $\{a_1, \ldots, a_q\}$ be a basis of $U(\mathfrak{n}^-)_{k\gamma}$. Take $\mu = \sum_{i=1}^l r_i \lambda_i \in V$ and set $\tilde{\mu} = s_\gamma(\mu) = \mu - k\gamma$. Then by Theorem 1 there is a $Y_\mu = \sum_{i=1}^q \zeta_i a_i$ such that $Y_\mu \cdot v_\mu$ is singular. Here the ζ_i are polynomial functions of the r_i.

Set $p = \langle \tilde{\mu}, \beta^\vee \rangle$, $q = \langle \mu, \beta^\vee \rangle$. Then $Y' = y_\beta^p Y_\mu y_\beta^{-q} = \sum_j c_j b_j$, where the b_j are linearly independent elements of $K(\mathfrak{n}^-)$, and the c_j are coefficients that depend polynomially on the r_i. Now Lemma 7 implies that if the $r_i \in \mathbb{Z}$ and $p \geq 0$, then $Y' \in U(\mathfrak{n}^-)$. Let now j be such that $b_j \notin U(\mathfrak{n}^-)$. If the $r_i \in \mathbb{Z}$ and $p \geq 0$, then $c_j = 0$. Suppose that $\beta = \alpha_{i_0}$, the i_0-th simple root. Then the condition $p \geq 0$ amounts to $r_{i_0} \geq k\langle\gamma, \beta^\vee\rangle$. We have that $\mu \in V$ if and only if $\sum_{i=1}^l u_i r_i = k$, where the u_i are certain elements of $\mathbb{Z}$. Also, since $\beta \neq \gamma$ at least one $u_i \neq 0$ with $i \neq i_0$. We see that the requirement $r_{i_0} \geq k\langle\gamma, \beta^\vee\rangle$ cuts a half space W off V. Furthermore $V \cap P$ is an $(l-1)$-dimensional lattice in V (cf. [1]). The conclusion is that $c_j = 0$ if $\mu \in W \cap P$. Since the c_j are polynomials in the r_i, it follows that $c_j = 0$ if $\mu \in V$. In particular, $y_\beta^m Y y_\beta^{-n}$ lies in $U(\mathfrak{n}^-)$.

Finally we note that $Y' \cdot v_{s_\beta \mu}$ is singular, by the same arguments. (Indeed, $x_i \cdot (Y' \cdot v_{s_\beta\mu}) = \sum_j f_j z_j \cdot v_{s_\beta\mu}$ where the f_j are polynomials in the r_i, and the z_j are elements of $U(\mathfrak{n}^-)$. Since the f_j are zero when $\mu \in W \cap P$ we have that $f_j = 0$ when $\mu \in V$.) In particular, $y_\beta^m Y y_\beta^{-n} \cdot v_{s_\beta \eta}$ is singular. □

Example 9 Consider the Lie algebra of type A_3, with simple roots α, β, γ (with β corresponding to the middle node of the Dynkin diagram). Then $[y_\alpha, y_\beta] = y_{\alpha+\beta}$, $[y_\alpha, y_{\beta+\gamma}] = y_{\alpha+\beta+\gamma}$, $[y_\beta, y_\gamma] = y_{\beta+\gamma}$, $[y_\gamma, y_{\alpha+\beta}] = -y_{\alpha+\beta+\gamma}$. Set $a_1 = y_\alpha y_\beta y_\gamma$, $a_2 = y_\gamma y_{\alpha+\beta}$, $a_3 = y_\alpha y_{\beta+\gamma}$, $a_4 = y_{\alpha+\beta+\gamma}$. We abbreviate a weight $r_1\lambda_1 + r_2\lambda_2 + r_3\lambda_3$ by (r_1, r_2, r_3). Let $\eta = (r_1, r_2, r_3)$, be such that $r_1 + r_2 + r_3 = 1$. Set $\nu = s_{\alpha+\beta+\gamma}(\eta) = (r_1 - 1, r_2, r_3 - 1)$. Then $Y = Y_\eta = a_1 - r_1 a_2 - (r_1 + r_2)a_3 - r_1 r_3 a_4$ is such that $Y \cdot v_\eta$ is singular. Furthermore, $Y \in U(\mathfrak{n}^-)_{\alpha+\beta+\gamma}$ so that we have an embedding $\psi : M(\nu) \hookrightarrow M(\eta)$ given by $\psi(u \cdot v_\nu) = uY \cdot v_\eta$. We construct $M(s_\alpha \nu) \hookrightarrow M(s_\alpha \eta)$ by setting $Y' = y_\alpha^{r_1 - 1} Y y_\alpha^{-r_1} = y_\beta y_\gamma - (r_1 + r_2) y_{\beta+\gamma} + r_1(1 - r_1 - r_2 - r_3) y_\alpha^{-1} y_{\alpha+\beta+\gamma}$. We see that this last term vanishes.

Now we return to the situation of the beginning of the section. We have $\lambda \in \mathbb{Q}P$, $\alpha \in \Phi^+$ with $\langle\lambda, \alpha^\vee\rangle = h \in \mathbb{Z}_{>0}$. Set $\mu = s_\alpha(\lambda) = \lambda - h\alpha$. We have the following algorithm for obtaining an embedding $M(\mu) \hookrightarrow M(\lambda)$.

1. Select $\beta_1, \ldots, \beta_r \in \Delta$ and positive roots $\alpha_0, \ldots, \alpha_r$ in the following way. Set $\alpha_0 = \alpha$, and $k = 0$. If $\alpha_k \in \Delta$, then set $r = k$ and go to the next step. Otherwise, let $\beta_{k+1} \in \Delta$ be such that $\langle\alpha_k, \beta_{k+1}^\vee\rangle > 0$, and set $\alpha_{k+1} = s_{\beta_{k+1}}(\alpha_k)$, and $k := k + 1$.

2. Set $\beta = \alpha_r \in \Delta$. For $1 \leq k \leq r$ set $a_k = -\langle\mu, s_{\beta_1} \cdots s_{\beta_{k-1}}(\beta_k)^\vee\rangle$, and $b_k = \langle\lambda, s_{\beta_1} \cdots s_{\beta_{k-1}}(\beta_k)^\vee\rangle$.

3. Set $Y_0 = y_\beta^h$, and for $0 \leq k \leq r - 1$:

$$Y_{k+1} = y_{\beta_{r-k}}^{a_{r-k}} Y_k y_{\beta_{r-k}}^{b_{r-k}}.$$

Remark. Note that the β_{k+1} in 1. exists because otherwise $\langle\alpha_k, \gamma^\vee\rangle \leq 0$ for all $\gamma \in \Delta$, and this implies that the set $\Delta \cup \{\alpha_k\}$ is linearly independent (cf. [5], Chapter IV, Lemma 1), which is not possible since $\alpha_k \notin \Delta$. Also, all α_k must be positive roots because s_γ permutes the positive roots other than γ, for $\gamma \in \Delta$. Then the loop in 1. must terminate because the height of α_k decreases every step.

Proposition 10 *All Y_k are elements of $U(\mathfrak{n}^-)$ and we have an embedding $M(\mu) \hookrightarrow M(\lambda)$ given by $v_\mu \mapsto Y_r \cdot v_\lambda$.*

Proof. We write $s_i = s_{\beta_i}$. For $0 \le k \le r$ we set $w_k = s_{r-k} \cdots s_1$ (so $w_r = 1$), and $\mu_k = w_k \mu$, $\lambda_k = w_k \lambda$. We claim that there is an embedding $M(\mu_k) \hookrightarrow M(\lambda_k)$ given by $v_{\mu_k} \mapsto Y_k \cdot v_{\lambda_k}$. First we look at the case $k = 0$. Note that $s_r \cdots s_1(\alpha) = \beta \in \Delta$. Since for w in the Weyl group we have $w s_\beta w^{-1} = s_{w\beta}$ we get $s_\alpha = s_1 \cdots s_r s_\beta s_r \cdots s_1 = w_0^{-1} s_\beta w_0$. Therefore $\mu_0 = w_0 s_\alpha(\lambda) = s_\beta(\lambda_0)$, and $\langle \lambda_0, \beta^\vee \rangle = \langle \lambda, s_1 \cdots s_r(\beta)^\vee \rangle = \langle \lambda, \alpha^\vee \rangle = h$. The case $k = 0$ now follows by Lemma 6.

Now suppose we have an embedding $M(\mu_k) \hookrightarrow M(\lambda_k)$ as above. Note that $w_{k+1} = s_{r-k} w_k$ and $\alpha_k = w_k \alpha$. Also $\mu_k = w_k \mu = \lambda_k - h\alpha_k$, and $\langle \lambda_k, \alpha_k \rangle = h$, so that $\mu_k = s_{\alpha_k}(\lambda_k)$. We now apply Proposition 8 (with $\nu := \mu_k$, $\eta := \lambda_k$, $\beta := \beta_{r-k}$). We have $\beta_{r-k} \in \Delta$ and $\beta_{r-k} \neq \alpha_k$ as $\alpha_k \notin \Delta$. Furthermore, $m = \langle s_{r-k} \cdots s_1 \mu, \beta_{r-k}^\vee \rangle = -\langle \mu, s_1 \cdots s_{r-k-1}(\beta_{r-k})^\vee \rangle = a_{r-k}$. In the same way $n = \langle \lambda_k, \beta_{r-k}^\vee \rangle = -b_{r-k}$. So by Proposition 8, if we set

$$Y_{k+1} = y_{\beta_{r-k}}^{a_{r-k}} Y_k y_{\beta_{r-k}}^{b_{r-k}},$$

then we have an embedding $M(\mu_{k+1}) = M(s_{\beta_{r-k}} \mu_k) \hookrightarrow M(s_{\beta_{r-k}} \lambda_k) = M(\lambda_{k+1})$ by $v_{\mu_{k+1}} \mapsto Y_{k+1} \cdot v_{\lambda_{k+1}}$.

Finally we note that $\lambda_r = \lambda$, $\mu_r = \mu$. □

Example 11 Again we consider the root system of type A_3, and use the notation of Example 9. Let $\lambda = (\frac{1}{2}, \frac{3}{2}, -1)$ and $\mu = s_{\alpha+\beta+\gamma}(\lambda) = (-\frac{1}{2}, \frac{3}{2}, -2)$. Since $s_\beta s_\alpha(\alpha + \beta + \gamma) = \gamma \in \Delta$ we can take $\beta_1 = \alpha$, $\beta_2 = \beta$. Furthermore, $a_1 = -\langle \mu, \alpha^\vee \rangle = \frac{1}{2}$, $a_2 = -\langle \mu, s_\alpha(\beta)^\vee \rangle = -1$, $b_1 = \langle \lambda, \alpha^\vee \rangle = \frac{1}{2}$, and $b_2 = \langle \lambda, s_\alpha(\beta)^\vee \rangle = 2$. So we get $Y_0 = y_\gamma$, $Y_1 = y_\beta^{-1} Y_0 y_\beta^2 = y_\beta y_\gamma - 2y_{\beta+\gamma}$, $Y_2 = y_\alpha^{1/2} Y_1 y_\alpha^{1/2} = y_\alpha y_\beta y_\gamma - \frac{1}{2} y_\gamma y_{\alpha+\beta} - 2 y_\alpha y_{\beta+\gamma} + \frac{1}{2} y_{\alpha+\beta+\gamma}$.

4 Composition of embeddings

In this section we compare our construction with the one described in [6]. We also briefly consider the problem of obtaining an embedding that is the composition of two embeddings.

The construction described in [6] works as follows. Write $s_\alpha = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_t}}$, as a product of simple reflections. For $1 \le k \le t$ set $m_k = \langle s_{\alpha_{i_{k+1}}} \cdots s_{\alpha_{i_t}} \lambda, \alpha_{i_k}^\vee \rangle$. Then $Y = y_{i_1}^{m_1} \cdots y_{i_t}^{m_t}$ is an element of $U(\mathfrak{n}^-)$ and we have an embedding $M(\mu) \hookrightarrow M(\lambda)$ by $v_\mu \mapsto Y \cdot v_\lambda$. It is straightforward to see that our construction is a special case of this. Indeed, using the same notation as in the description of the algorithm, the expression we get is

$$y_{\beta_1}^{a_1} \cdots y_{\beta_r}^{a_r} y_\beta^h y_{\beta_r}^{b_r} \cdots y_{\beta_1}^{b_1}.$$

As remarked in the proof of Proposition 10, $s_\alpha = s_1 \cdots s_r s_\beta s_r \cdots s_1$ (where again we write $s_i = s_{\beta_i}$). Furthermore, $b_k = \langle s_{k-1} \cdots s_1 \lambda, \beta_k^\vee \rangle$, $h = \langle \lambda, \alpha^\vee \rangle = \langle s_r \cdots s_1 \lambda, \beta^\vee \rangle$, $a_k = \langle s_{k+1} \cdots s_r s_\beta s_r \cdots s_1 \lambda, \beta_k^\vee \rangle$. So we see that our method is a special case of the construction in [6]. However, the difference is that we have an explicit method to rewrite the element above to an element of $U(\mathfrak{n}^-)$. By the next lemma the expression we use for s_α is the shortest possible (so we cannot do essentially better by taking a different reduced expression).

Lemma 12 *The expression $s_\alpha = s_1 \cdots s_r s_\beta s_r \cdots s_1$ obtained by the first step of the algorithm, is reduced.*

Proof. Set $\gamma = s_1(\alpha) = \alpha - m\beta_1$, where $m > 0$. Then $s_\alpha = s_{s_1(\gamma)} = s_1 s_\gamma s_1$. By induction, the expression $s_\gamma = s_2 \cdots s_r s_\beta s_r \cdots s_2$ is reduced. We show that $\ell(s_\alpha) = \ell(s_\gamma) + 2$. For this we use the fact that the length of an element w of the Weyl group is equal to the number of positive roots that are mapped to negative roots by w. Write $\Phi^+ = A \cup \{\beta_1\}$, where $A = \Phi^+ \setminus \{\beta_1\}$. Then s_1 permutes the set A, and maps β_1 to $-\beta_1$. Note that $s_1\gamma = \alpha = \gamma + m\beta_1$. This means that $\langle \gamma, \beta_1^\vee \rangle < 0$ and hence $\langle \beta_1, \gamma^\vee \rangle < 0$. It follows that $s_\gamma(\beta_1) > 0$. Let $\delta \in A$ be such that $s_\gamma s_1 \delta < 0$. If $s_\alpha \delta > 0$, then necessarily, $s_\gamma s_1 \delta = -\beta_1$, or $s_1\delta = -s_\gamma \beta_1$ which is a negative root by what we have seen above. This is a contradiction, hence $s_\alpha \delta < 0$. There is a $\delta_0 \in \Phi$ with $s_\gamma s_1 \delta_0 = \beta_1$. We cannot have $\delta_0 < 0$, because in that case $-\delta_0 \in A$ and $s_1(-\delta_0) = -s_\gamma(\beta_1) < 0$, which is a contradiction. Hence $\delta_0 \in A$ and $s_\alpha \delta_0 = -\beta_1$. Furthermore, $s_\alpha \beta_1 = \beta_1 - n\alpha$, where $n > 0$. So $s_\alpha(\beta_1) < 0$. Set $S = \{\delta \in A \mid s_\gamma s_1 \delta < 0\} \cup \{\delta_0, \beta_1\}$. Then s_α maps all elements of S to negative roots. The cardinality of S is $\ell(\gamma) + 2$. So $\ell(\alpha) \geq \ell(\gamma) + 2$, but that means that $\ell(\alpha) = \ell(\gamma) + 2$. $\square$

Now we consider the problem of obtaining an embedding $M(\nu) \hookrightarrow M(\lambda)$, where $\nu = s_\alpha s_\beta(\lambda) = s_\alpha(\mu)$, and $\langle \lambda, \beta^\vee \rangle \in \mathbb{Z}_{>0}$, $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}_{>0}$. The obvious way of doing this is to obtain the embeddings $M(\nu) \hookrightarrow M(\mu)$, $M(\mu) \hookrightarrow M(\lambda)$ and composing them. This amounts to multiplying two elements of $U(\mathfrak{n}^-)$. This then corresponds to an expression for $s_\alpha s_\beta$, which is not necessarily reduced. The question arises whether in this case it is possible to do better, i.e., to start with a reduced exression for $s_\alpha s_\beta = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_r}}$, set $m_k = \langle s_{\alpha_{i_{k+1}}} \cdots s_{\alpha_{i_t}} \lambda, \alpha_{i_k}^\vee \rangle$, and rewrite $Y = y_{i_1}^{m_1} \cdots y_{i_t}^{m_t}$ to an element of $U(\mathfrak{n}^-)$. The next example shows that this is not always clear how to do this.

Example 13 Let Φ be of type F_4, with simple roots $\alpha_1, \ldots, \alpha_4$ and Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Let $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3$ and $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$. We abbreviate a weight $a_1\lambda_1 + \cdots + a_4\lambda_4$ by (a_1, a_2, a_3, a_4). Set $\lambda = (\frac{5}{6}, -\frac{1}{2}, \frac{2}{3}, 0)$ and $\nu = (-\frac{1}{6}, -\frac{1}{2}, -\frac{1}{3}, 2)$. Then $\nu = s_\alpha s_\beta(\lambda)$. Write $s_i = s_{\alpha_i}$. Then a reduced expression of $s_\alpha s_\beta$ is

$$s_1 s_2 s_1 s_3 s_2 s_1 s_3 s_2 s_4 s_3 s_2 s_1 s_3 s_2.$$

We get

$$Y = y_1^{\frac{1}{6}} y_2^{\frac{2}{3}} y_1^{\frac{1}{2}} y_3^{\frac{5}{3}} y_2^{\frac{3}{2}} y_1 y_3^{\frac{4}{3}} y_2^{\frac{5}{6}} y_4 y_3^{\frac{4}{3}} y_2^{\frac{1}{2}} y_1^{\frac{1}{3}} y_3^{-\frac{1}{3}} y_2^{-\frac{1}{2}}.$$

And I do not see any direct way to rewrite this as an element of $U(\mathfrak{n}^-)$.

We can obtain the embedding by composition. For this set $\mu = s_\beta(\lambda) = \lambda - \beta = (\frac{5}{6}, -\frac{3}{2}, \frac{5}{3}, 0)$. Then for the embedding $M(\mu) \hookrightarrow M(\lambda)$ we get

$$Y_1 = y_2^{\frac{2}{3}} y_3^{\frac{4}{3}} y_1^{\frac{2}{3}} y_2^{\frac{1}{2}} y_3^{-\frac{1}{3}} y_4 y_3^{\frac{4}{3}} y_2^{\frac{1}{2}} y_1^{\frac{1}{3}} y_3^{-\frac{1}{3}} y_2^{-\frac{1}{2}}.$$

For the embedding $M(\nu) \hookrightarrow M(\mu)$ we get

$$Y_2 = y_1^{\frac{1}{6}} y_3^{\frac{1}{3}} y_2 y_3^{\frac{5}{3}} y_1^{\frac{5}{6}}.$$

Then the product Y_2Y_1 will provide the embedding $M(\nu) \hookrightarrow M(\lambda)$. In this case it is clear how to rewrite Y_2Y_1 to an element of $U(\mathfrak{n}^-)$.

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