# Higher dimensional geometry from Fano to Mori and beyond

Marco Andreatta

Dipartimento di Matematica
Universitá di Trento

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# Introduction





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A large number of mathematicians, often organized in counterposed schools, in the last 50 years, starting from Fano's results, constructed theories which are among the most spectacular achievements of contemporary mathematics.

# Notation

In the lecture we consider normal projective varieties $X$ defined over $\mathbb{C}$.
If $n$ is the dimension of $X$ we sometime call $X$ and *n-fold*;
we denote by $K_X$ the *canonical sheaf*.
We assume to have good singularities such that $K_X$, or a multiple of it, is a line bundle (a Cartier divisor).

# Fano's 3folds



Let $X \subset \mathbb{P}^N$ be a projective 3-fold such that for general hyperplanes $H_1, H_2$ the curve $\Gamma := X \cap H_1 \cap H_2$ is canonically embedded (i.e. $K_\Gamma$ embeds $\Gamma$).

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Example: the quartic 3-fold in $\mathbb{P}^4$, $X_4 \subset \mathbb{P}^4$.

# Fano's 3folds



Fano noticed that for such varieties the following invariants are zero:

- $P_m(X) = h^0(X, mK_X) = 0$ for all $m \geq 1$ ($m$-th plurigenera)
  (we say that $X$ has Kodaira dimension minus infinity: $k(X) = -\infty$)
- $h^i(\mathcal{O}_X) = 0$ for all positive $i$
  (in particular the irregularity $q(X) = h^1(X, \mathcal{O}_X)$ is zero).

Varieties satisfying these two conditions were called by him
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# Non rational 3folds

Fano had the insight that among this class of varieties there are varieties which are non-rational, in spite of the fact that they have all plurigenera and irregularity equal to zero; they would provide a counterexample to a Castelnuovo type rationality criteria for 3-folds.
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The first proof of the non rationality of all $X_4 \subset \mathbb{P}^4$ is the celebrated Iskovskih and Manin's. B. Segre has constructed some unirational $X_4 \subset \mathbb{P}^4$, therefore they represents counterexamples to Luroth problem in dimension 3 (as well as to a Castelnuovo type rationality criteria).

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In the same period Clemens and Griffiths proved the non-rationality of the cubic 3-fold in $\mathbb{P}^4$.
Both papers gave rise to subsequent deep results and theories aimed to determine the rationality or not of Fano varieties.

# Fano Varieties and Fano-Mori Contractions

## Definition

A smooth projective variety $X$ is called a *Fano manifold* if $-K_X$ is ample.

If $Pic(X) = \mathbb{Z}$ then $X$ is called a *prime Fano manifold*;
if $Pic(X/Y) =< L >$ and $-K_X = rL$, $r$ is called the *index of* $X$.

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A proper surjective map between normal varieties with connected fibers, $f : X \to Y$ is a *contraction* (divisorial, small or of fiber type).

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Let $f : X \to Y$ be a contraction with $X$ smooth or with mild singularities; $f$ is called a *Fano-Mori contraction* (F-M for short) if $-K_X$ is $f$-ample.

If $Pic(X/Y) = \mathbb{Z}$ then $X$ is called a *elementary F-M contraction*;
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A Fano manifold is a Fano-Mori contraction with $dimY = 0$.
A general fiber of a Fano-Mori contraction is a Fano variety.

# Classification and MMP

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-C. Birkhar : Fields Medalist in 2018 for *the proof of the boundedness of Fano varieties and for contributions to the minimal model program*.

# Classification and MMP



According to MMP *a projective variety*, smooth (or with at most Kawamata log terminal (klt) singularities), *is birational equivalent*

- *either to a projective variety with positive (nef) canonical bundle*
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*F-M contractions are the building blocks (atoms) of the classification of projective varieties*.

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He proved that $3 \leq g \leq 12$ and $g \neq 11$ and for every such $g$ he gave a satisfactory description of the associated Fano variety.
$X_{22} \subset \mathbb{P}^{13}$ (omitted by Fano and later by Roth): the double projection from a line, $\pi_{2Z} : X_{22} \cdots > W \subset \mathbb{P}^6$, goes into $W$, a Fano 3-fold of index 2, degree 5, $Pic(W) = \mathbb{Z}$ and one singular point. $X_{22}$ is rational.

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S. Mori and S. Mukai (1981): classified Fano 3-fold with $\rho(X) \geq 2$.
At the Fano Conference in Torino (2002) they announced they have omitted the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of tridegree $(1, 1, 3)$.

# Classification of Fano n-folds



A classification of Fano manifolds of higher dimension is an Herculean task which however could be done in *finite time*.

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Fano manifold of index $r \geq n = dimX$ are simply the projective spaces and the quadrics, this was proved by Kobayashi and Ochiai.
Fano manifolds of index $(n-1)$ (del Pezzo manifolds) were intensively studied by T. Fujita, who proved the existence of a smooth divisor in the linear system $H$ generating $Pic(X)$ (that is $-K_X = (n-2)H$).
Mukai classified all Fano manifolds of index $= (n-2)$ under the assumption that $H$ has an effective smooth member. M. Mella proved later that this is true for Fano manifolds of index $= (n-2)$.

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There are several projects aiming to classify singular Fano varieties in dimension $3, 4$ and $5$. One is carried out at Imperial College-London, PI: A. Corti, title: *the periodic table of mathematical shapes*.
It is estimated that 500 million shapes can be defined algebraically in four dimensions, and a few thousand more in the fifth.

# A conjecture of Mukai

A nice conjecture of Mukai (1988), very useful for classification.

## Conjecture

*Let X be a Fano manifold and* $\rho_X = dimN_1(X)$. *Then*

$$\rho_X(r_X - 1) \leq n.$$

*More generally if*
$i_X = \min\{m \in \mathbb{N} \mid -K_X \cdot C = m, C \subset X \text{ rational curve } \}$ *is the pseudoindex of X (note that* $i_X = mr_X$*), then*

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The conjecture holds for Toric varieties (C. Casagrande) and in other special cases, for instance for $n \leq 5$ (M. Andreatta, E. Chierici, G. Occhetta).

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Wisniewski and myself (1998) classified all birational F-M contractions on a smooth 4-fold.

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One can find several results on the classification of F-M contraction of fiber type on smooth 3-folds and 4-folds: the "classical" ones on conic bundles or more recents which compared different birational models of a F-M contractions via the so called Sarkisov program (every birational morphism between two fiber type F-M contractions can be factorized via a finite number of few basic transformations).

# Classification of F-M contractions, singular case



In the eighties, immediately after the introduction of Mori theory, it appears with full evidence that the study of F-M contractions should be carried out in the singular set up. P. Francia constructed in 1985 a brilliant example of commutative diagram of F-M contractions on 3-folds which convinced everybody that a MMP can be performed only passing through singular cases. In particular he showed that even on 3-fold with mild singularities one can find small F-M contractions which need to be ”flipped”.

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Based on this I (2018) gave a characterization of birational divisorial contractions on $n$-fold with terminal singularities with nef value greater then $n-3$ = weighted blow-up of hyperquotient singularities.

# Rational Curves on Fano Manifolds

The name Fano variety is also used for some subvariety of the Grassmanian: the variety of $k$-planes contained in $X$, that is

$$F_k(X) := \{\Lambda : \Lambda \subset X\} \subset \mathbb{G}(k, n).$$

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**Theorem (S. Mori 1982)**

*Let $X$ be a Fano manifold. Through every point of $X$ there is a rational curve $D$ such that*

$$0 < -(D \cdot K_X) \leq dim X + 1.$$

# A very beautiful proof



Take any curve $C$ passing through the chosen point. By deformation theory and Riemann-Roch theorem its deformation space has dimension

$$\geq h^0(C, TX) - h^1(C, TX) - dimX = -C \cdot K_X - g(C) \cdot dimX.$$

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Mori passed to a field of positive characteristic $p$ and consider all geometric objects over this new field: $X_p$ and $C_p$.
He changed $C_p$ with its image via a number $m$ of Frobenius endomorphism; the genus of the curve remains $g(C)$ but the above estimate will be multiplied by $p^m$:

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If a curve through a point on an algebraic variety moves, staying at the point, it will "bend and break": it is algebraically equivalent to a reducible curve which one rational component through the point.

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If a curve through a point on an algebraic variety moves, staying at the point, it will "bend and break": it is algebraically equivalent to a reducible curve which one rational component through the point.

He concludes with a *general principle*, based on number theory, which says that if you have a rational curve through the point for almost all $p > 0$ then you have it also for $p = 0$.

# Uniruled and a Conjecture of Mori



A direct corollary of the Theorem is that a Fano variety is covered by rational curve, it is uniruled. A variety $X$ is *uniruled* if there is a variety $Y$ and a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$ which does not factor through the projection to $Y$).

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To be uniruled and rationally connected are birational properties.

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To be uniruled and rationally connected are birational properties.

If $X$ is uniruled (with at most canonical singularities) then
$H^0(X, mK_X) = 0$ for all $m > 0$.
Mori in 1985 conjectured that the converse is true:

**Conjecture**

*Let $X$ be a projective variety with canonical singularities, if $k(X) = -\infty$ then $X$ is uniruled.*

# Uniruled and a Conjecture of Mori



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The Conjecture is false for more general singularities, as some examples of J. Kollár show (rational varieties with ample canonical divisor).

# Conjecture of Mumford

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J. Kollár noticed that Mori’s implies Mumford’s:
via MRC fibration - Campana and Kollar-Mori-Miyaoka
and the Fibration theorem - Graber-Harris-Mazur-Starr.

# Towards Mori's Conjecture



S.Mori (1985) introduced the cone of pseudo-effective divisor, i.e. in ($\overline{Eff(X)} \subset N^1(X)$)). Non pseudoeffectivity of $K_X$ is clearly a condition in between uniruledness and negative Kodaira dimension.

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The following was conjectured by Mori in '85, then proved first by by BDPP and then by BCHM, using the bend and breaking theory of Mori.

**Theorem**

*Let $X$ be a projective variety with canonical singularities, $K_X$ is not pseudoeffective if and only if $X$ is uniruled.*

# Towards Mori's Conjecture

Another definition which is in between uniruledness and $k(X) = -\infty$. It was introduced by G. Castelnuovo and F. Enriques in the surface case and by G. Fano and U. Morin for the 3-folds.

**Definition**

(Termination of Adjunction in the classical sense) Let $X$ be a normal projective variety and let $H$ be an *effective Cartier divisor* on $X$ (or very ample) *Adjunction Terminates in the classical sense* for $H$ if there exists an integer $m_0 \geq 1$ such that

$$H^0(X, mK_X + H) = 0$$

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Together with C. Fontanari we conjectured that, if $X$ has at most canonical singularities, then A. T. for $H$ is equivalent to uniruledness. This is true for *superficie adeguatamente preparate* by a theorem of Castelnuovo-Enriques, surfaces which are final objects of a MMP.

# Non-negativity of the Tangent =Uniruledness

Miyaoka's criterium; a very general "bend and break technique".

**Definition**

$T_X$ is *generically seminegative* if for every torsion free subsheaf $E \subset T_X$ we have $c_1(E) \cdot C \leq 0$, where $C$ is a curve obtained as intersection of high multiple of $(n-1)$ ample divisors.

**Theorem**

*A normal complex projective variety $X$ is uniruled if and only if $T_X$ is not generically seminegative.*

# Positivity of the Tangent bundle

The criterium is a starting point to prove many results, including the following one [M. Andreatta-J. Wisniewski], which is the generalization of the celebrated Frenkel-Hartshorne conjecture proved by S. Mori in 1978.

**Theorem**

*Let $X$ be a projective manifold with an ample locally free subsheaf of $E \subset TX$.*
*Then $X = \mathbb{P}^n$ and $E = \mathcal{O}(1)^{\oplus r}$ or $E = T_{\mathbb{P}^n}$*

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A conjecture formulated by F. Campana and T. Peternell.

## Conjecture

*A Fano manifold with nef tangent bundle is a rational homogeneous variety.*

# VMRT and MRC



On a uniruled variety $X$ a dominating family of rational curves

$$V \subset Hom(\mathbb{P}^1, X) \quad \text{such that} \quad Locus V = X$$

having *minimal degree* with respect to some fixed ample line bundle.
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For each $x \in X$ denote by $C_x$ the subvariety of the projectivized tangent space at $x$ consisting of tangent directions to curves from $V$ passing through $x$, $V_x$; that is $C_x$ is the closure of the image of the *tangent map* $\Phi_x : V_x \to \mathbb{P}(T_xX)$.

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It has been considered first by S. Mori in his 1978 seminal paper.
Hwang and Mok studied this variety in a series of papers and called it *variety of minimal rational tangents* (VMRT) of $V$.
The tangent map and the VMRT determine the structure of many Fano manifold, for instance of the projective space and of the rational homogeneous varieties.

# VMRT and MRC

Given a genricall unsplit family, $V \subset Hom(\mathbb{P}^1, X)$ one can define a relation of rational connectedness with respect to $V$, $rcV$-relation, in the following way: $x_1, x_2 \in X$ are in the $rcV$- relation if there exists a chain of rational curves parametrized by $V$ which joins $x_1$ and $x_2$.

More generally one can consider a rationally connectedness relation with respect to all rational curves $Hom(\mathbb{P}^1, X)$: $rc$-relation.

The $rcV$ and $rc$- equivalence classes can be parametrized generically by an algebraic set: Campana and independently Kollár- Miyaoka-Mori.

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**Theorem**

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$\phi_0$ is called an *$rcV$-fibration* or a *Maximal Rationally Connected fibration* (MRC).
They are very much connected to F-M contractions and they are crucial tools for the study of uniruled varieties

# Elephants and Base Point Freeness

Consider a Fano manifold $X$ or a Fano-Mori contraction $f : X \to Y$.

**Definition**

A *general elephant* is a general element of the anticanonical system $|-K_X|$ (M. Reid terminology).

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The classification of Fano manifolds or of F-M contractions often use and *inductive procedure* on the dimension of $X$ (Apollonius method):
1. take a general elephant $D \in |-K_X|$, a variety of smaller dimension; by *adjunction formula* it has trivial canonical bundle.
2. *Lift up sections* of $(-K_X)_{|D}$ (or of other appropriate positive bundles) to sections of $-K_X$, via the long exact sequence associated to

$$0 \to \mathcal{O}_X \to -K_X \to (-K_X)_{|D} \to 0$$

(Kodaira *vanishing theorem* on a Fano manifolds implies $h^1(\mathcal{O}_X) = 0$).
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(More generally on a Fano-Mori contraction $f : X \to Y$ consider a line bundle $L$ such that $-K_X \sim_f rL$, where $r$ the nef value.
Take $D \in |L|$ and do the inductive procedure on $D$.)

# Safari Game

The procedure has classical roots and can be lead back to Fano.
Many delicate problems were solved in the last 50 years by S. Mori, V. Shokurov, Y. Kawamata, J. Kollár and others.

1. existence of a general elephant, a question unexpectedly avoided by some authors. One needs also that the singularities of the elephant are not worst than those of $X$ (if $X$ is smooth we like that also the elephant is smooth).
2. existence of enough sections of $(-K_X)_{|D}$, or of $L_D$. This is also delicate and it goes under the name of "non vanishing theorem".
3. In order to get non vanishing sections in the linear systems $|L_D|$ sometime one is forced to change slightly the line bundle $L$, introducing so called "boundary divisors" or "fractional divisors". But then the Kodaira vanishing is not sufficient and more powerful and suitable "vanishing theorems" are needed.

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This eventually lead to the proof of the existence of MMP, first in dimension 3 by S. Mori and later, under some conditions, in all dimension by Birkar-Cascini-Hacon-McKernan.

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- Smooth element in the linear system $|L|$ on a Fano of index $r$. T. Fujita (1984), $r = (n-1)$; M. Mella (1999), $r = (n-2)$.

# Kähler-Einstein metrics

The Einstein Field Equations on a Riemannian manifold $(X, g)$

$$Ric_g - \frac{1}{2} s_g g = 8\pi T$$

describe how the manifold $X$ should curve due to the existence of mass or energy, a quantity encapsulated by the stress-energy tensor $T$.
In a vacuum, where there is no mass or energy, $T = 0$ and the Einstein Field Equations simplify to $Ric_g = \lambda g$, $\lambda$ a constant.
A Riemannian manifold $(X, g)$ solving the above equation is called an *Einstein manifold*.

A Riemannian manifold with a complex structure $J$ compatible with the metric structure (i.e. $g$ preserves $J$ and $J$ is preserved by the parallel transport of the Levi-Civita connection) is called a *Kähler manifold*.

On a *Kähler-Einstein manifold* one define two $(1,1)$-forms:
$\rho(u, v) = Ric_g(Ju, v)$ and $\omega(v, u) = g(Ju, v)$, for $v, u$ vector fields. The Einstein equation becomes

$$\rho = \lambda\omega.$$

# Existence of Kähler-Einstein metrics



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- The first Chern class is zero; S.T. Yau proved the Calabi conjecture, that there is always a Kähler-Einstein metric. He was awarded with the Fields medal because of this work. That leads to the name Calabi-Yau manifolds.
- The third case, the positive or Fano case, is the hardest. In this case the manifold not always has a Kähler-Einstein metric, there is in fact a non-trivial obstruction to existence. In 2012, Chen, Donaldson, and Sun proved that in this case the existence is equivalent to an algebro-geometric criterion called *K-stability*. Their proof appeared in a series of articles in the Journal of the American Mathematical Society in 2014.

# Existence of Kähler-Einstein metrics on $X_{22}$



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in 2020 I. Cheltsov and C. Shramov proved that on any members of a one parameter family of $X_{22}$, which have a $\mathbb{C}^*$ action and are therefore denoted by $X_{22}^*$, there exist a Kähler-Einstein metric.