The Calabi complex and Killing sheaf cohomology

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Abstract

It has recently been noticed that the degeneracies of the Poisson bracket of linearized gravity on constant curvature Lorentzian manifold can be described in terms of the cohomologies of a certain complex of differential operators. This complex was first introduced by Calabi and its cohomology is known to be isomorphic to that of the (locally constant) sheaf of Killing vectors. We review the structure of the Calabi complex in a novel way, with explicit calculations based on representation theory of $GL(n)$, and also some tools for studying its cohomology in terms of of locally constant sheaves. We also conjecture how these tools would adapt to linearized gravity on other backgrounds and to other gauge theories. The presentation includes explicit formulas for the differential operators in the Calabi complex, arguments for its local exactness, discussion of generalized Poincaré duality, methods of computing the cohomology of locally constant sheaves, and example calculations of Killing sheaf cohomologies of some black hole and cosmological Lorentzian manifolds.

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1 Introduction

The Calabi complex is a differential complex that was introduced in by E. Calabi in 1961 [19]. It has an extended pre-history, though. One way to characterize it is as a canonical formal compatibility complex (the second Spencer sequence) of the Killing equation on (pseudo-)Riemannian manifolds of constant curvature. The solutions of the Killing equation are (possibly only locally defined) infinitesimal isometries. In the special context of classical linear elasticity theory, the same operator also maps between the displacement and strain fields [64, 24, 3]. It is well known that for flat spaces (zero curvature) a complete set of formal compatibility conditions for the Killing equation is given by the linearized Riemann curvature operator, also known as the Saint-Venant compatibility operator in the context of elasticity [64, 24, 3]. Subsequent compatibility conditions are furnished by the Bianchi identities. Thus, it would also be reasonable to refer to it as the Killing-Riemann-Bianchi complex.

Calabi’s interest in the eponymous complex stemmed from the isomorphism between the cohomology of its global sections and the cohomology of the sheaf of Killing vectors. Given a fine resolution of a sheaf, like one provided by a locally exact sequence of differential operators on sections of vector bundles, the general machinery of homological algebra implies that the sheaf cohomology is in fact isomorphic to the cohomology of this global sections of its resolution,
the resolution of the sheaf of locally constant functions by the de Rham complex of differential forms on a manifold being the canonical example. The bulk of Calabi’s original article was in fact spent proving that the hypotheses needed for applying this general result actually hold, thus providing a way to represent the cohomology of the Killing sheaf. It is the latter object that was of intrinsic interest, as it was in subsequent works by others [12, 89, 11], motivated by the well known interpretations of its lowest cohomology groups: in degree-0 as the Lie algebra of global isometries and in degree-1 as the space of non-trivial infinitesimal deformations of the metric under the constant curvature restriction. Later, the Calabi complex was also seen as a non-trivial example of a formally exact compatibility complex [39, 34, 35, 63, 24] constructed for the Killing operator by the methods of the formal theory of partial differential equations developed by the school of D. C. Spencer [77, 78, 79, 68, 38].

More recently, the Calabi complex resurfaced in mathematical physics, in the context of the (pre-)symplectic and Poisson structure of relativistic classical field theories. In [46, 48], the author has shown that the degeneracy subspaces of the naturally defined pre-symplectic 2-form and Poisson bivector on the infinite dimensional phase space of relativistic classical field theories with possible constraints and gauge invariance are controlled by the cohomology of some differential complexes. In the case of Maxwell-like theories [48, Sec.4.2], this role is played by the de Rham complex, while in the case of linearized gravity [48, Sec.4.4], this role is played by the formal compatibility complex of the Killing operator. In other words, for linearization backgrounds of constant curvature (important examples include Minkowski and de Sitter spaces, as well as quotients thereof), this is precisely the Calabi complex. When the linearization background is merely locally symmetric, rather than of constant curvature, the right complex to use is a slightly different one that was constructed by Gasqui and Goldschmidt [34, 35]. However, a discussion of the latter is beyond the scope of this work. The construction of similar complexes adapted to other background geometries appears to be an open problem. The degeneracy subspace of the Poisson bivector of a classical field theory is of importance because it translates almost directly into violations of a (strict) notion of locality of the corresponding quantum field theory, a subject that has recently been under intense investigation [22, 71, 10, 29, 30, 8, 41, 7].

The goal of this paper is to exploit the connection between the Calabi complex and Killing sheaf cohomologies, in a direction opposite the original one of Calabi, for the purpose of obtaining results relevant to the above mentioned applications in mathematical physics. More precisely, we consider the computation of certain sheaf cohomologies much simpler than constructing quotient spaces of kernels of complicated differential operators. Thus, the ability to equate the Calabi cohomology groups, which for us are of primary interest, with Killing sheaf cohomology groups is a significant technical simplification. Along the way, we collect some relevant facts about the Calabi complex that are either difficult or impossible to find in the existing literature, along with other little known tools from the theory of differential complexes [85] needed to prove the desired equivalence and to introduce cohomologies with compact supports. It is our hope that this treatment of the Calabi complex could serve as a model for the treatment of other differential complexes that are of importance in mathematical physics.

In Section 2, we discuss the explicit form of the Calabi complex on any constant curvature pseudo-Riemannian manifold. The tensor bundles and dif-
ferential operators between them are defined using notation and identities from the representation theory of the general linear group, which are reviewed in Appendix A.1. The differential cochain homotopy operators defined in Section 2.2 and Appendix A.5, and the relation of the formal adjoint Calabi complex to the Killing-Yano operator presented in Section 2.3 are likely new. Then, Section 3 recalls some general notions from sheaf cohomology, with emphasis on locally constant sheaves. It also covers the relation between the Calabi cohomology, with various supports, and the cohomologies of the sheaf of Killing vectors and the sheaf of Killing-Yano tensors. In Section 4 we discuss several methods for effectively computing the cohomologies of the Killing sheaf, also outside the constant curvature context. An important application of the above results is described in Section 5, which uses the Calabi cohomology to determine the degeneracy subspaces of presymplectic and Poisson structures of linearized gravity on constant curvature backgrounds. This application, and its generalizations, constitutes the main motivation for this work. Finally, Section 6 concludes with a discussion of the presented results and of how they could be generalized to other differential complexes of interest in the mathematical theory of classical and quantum gauge field theories in physics.

It should be emphasized that the Killing sheaf cohomology can be identified with the cohomology of the Calabi complex only on pseudo-Riemannian spaces of constant curvature, where the latter complex is actually defined. The Killing sheaf itself has a wider domain of definition. In terms of applications to linearized gravity, the differential complexes that are to replace the Calabi complex on other background geometries are still expected to have isomorphic cohomology to that of the Killing sheaf. So, from that perspective, the Calabi complex is a particular case study and the Killing sheaf is an object of more permanent value.

2 The Calabi complex

Below, in Sections 2.1 and 2.2, we shall explicitly describe the Calabi complex as a complex of differential operators between tensor bundles on a pseudo-Riemannian manifold \((M, g)\). Further more, we will explicitly list a corresponding sequence of differential operators that constitute a cochain homotopy from the Calabi complex to itself. The cochain maps induced by the homotopy operators will have the same principal symbol as the tensor Laplacian \(\nabla_a \nabla^a\) induced by the Levi-Civita connection of the metric tensor \(g\), though will differ from it by lower order terms. This geometric structure is very similar to that of the Hodge theory of the de Rham complex on a Riemannian manifold. This structure is used in the later Section 3.2 to show the complex’s local exactness. Finally, in Section 2.3, we will describe the formal adjoint Calabi complex, with the formal adjoint cochain maps and homotopies playing roles analogous to the original ones. It turns out that, just as the Calabi complex resolves the sheaf of Killing vectors on \((M, g)\), its formal adjoint complex resolves the sheaf of rank-\((n - 2)\) Killing-Yano tensors.

A non-negligible amount of work [12, 34, 39, 35, 24, 63], though certainly not a large one, has been done on this differential complex since the original work [19] of Calabi in 1961. Its original presentation was in terms of Cartan’s moving frame formalism and much of the subsequent work did not put a strong
emphasis on explicit formulas. Thus, it is a little difficult to find its presentation in terms of standard covariant derivatives on tensor bundles in the existing literature. We give such formulas below, together with a complete sequence of cochain homotopy operators from the complex to itself and their corresponding cochain maps. These formulas are apparently new, as their role was played by a more generic, but somewhat less natural, construction applicable to general elliptic complexes in [19, 12, 34, 39]. The advantage of our version is the connection of the homotopy and cochain maps with the equations of linearized gravity and coincidence, in low degrees, with other well known related operators, which include the Killing, linearized Riemann, Bianchi, de Donder and Ricci trace operators. One could also argue that our resulting homotopy and cochain maps are simpler, because they never exceed second differential order (in contrast to fourth differential order). Further more, we find that the tensor bundles that constitute the nodes of the complex are best described as having fibers that carry irreducible representations of GL(n), where n is the dimension of the base manifold; moreover, the principal symbols of the differential operators in the complex are GL(n) equivariant maps. Hence they are independent of the background metric, which is no longer true for subleading terms. This observation appears to have escaped the attention of earlier works, thus requiring some seemingly ad-hoc constructions [19]. A notable exception is Eastwood [24], who also identified the principal symbol complex as an instance of the general notion of BGG resolutions [13] in representation theory. Taking advantage of this connection with representation theory, we explicitly describe the tensor bundles of the complex and the equivariant principal symbol maps between them in terms of Young diagrams.

2.1 Tensor bundles and Young symmetrizers

As was mentioned in the Introduction, it is convenient to describe various tensor bundles involved in the Calabi complex, as well as various maps between them, in terms of irreducible representations (irreps) of group GL(n), where n = dim M is the dimension of the base manifold M. Irreps of GL(n) are concisely presented using Young diagrams and corresponding Young tensor symmetrizers. An excellent reference on this topic is the book [33], where we refer the reader for complete details. For an uninitiated reader, we have briefly summarized the relevant concepts and formulas in Appendix A.1. For the expert reader, it is recommended to skim the same appendix for the particulars of our notation.

Given a base manifold M of dimension n = dim M, we can construct tensor bundles over M whose fibers carry irreducible representations of GL(n). Indeed, we will consider Young symmetrized sub-bundles Y^d T^* M of the bundle of covariant k-tensors (T^*)^\otimes k M, where d is a Young diagram type with k cells.

The Calabi complex, to be introduced in the next section, is a complex of differential operators between certain tensor bundles over M. Let us denote the corresponding sequence of vector bundles by C_l M. More precisely,

C_0 = T^*, \quad C_1 = Y^{(2)} T^*, \quad C_2 = Y^{(2,2)} T^*, \quad C_l = Y^{(2,2,1^{l-2})} T^* (l > 2). \quad (1)

Note that the bundle C_1 M corresponds to symmetric 2-tensors, which we will also denote S^2 M. Also, as mentioned in the preceding section, since the bundle C_2 M corresponds to 4-tensors with the algebraic symmetries of the Riemann
Table 1: The table below lists the tensor bundles of the Calabi complex, the corresponding irreducible GL($n$) representations (labeled by Young diagrams), and their fiber ranks, for dim $M = n$. The rank is given by the famous hook formula, which is discussed in Appendix A.1.

<table>
<thead>
<tr>
<th>bundle</th>
<th>Young diagram</th>
<th>fiber rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_0M \cong T^*M$</td>
<td></td>
<td>$n$</td>
</tr>
<tr>
<td>$C_1M \cong S^2M$</td>
<td></td>
<td>$\frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>$C_2M \cong RM$</td>
<td></td>
<td>$\frac{n^2(n^2-1)}{12}$</td>
</tr>
<tr>
<td>$C_3M \cong BM$</td>
<td></td>
<td>$\frac{n^2(n^2-1)(n-2)}{24}$</td>
</tr>
<tr>
<td>$C_lM$</td>
<td>$\begin{array}{c} 1 \ 2 \ \vdots \ l \end{array}$</td>
<td>$\frac{n^2(n^2-1)(n-2)\cdots(n-l+1)}{2(l+1)(l-2)!}$</td>
</tr>
</tbody>
</table>
tensor, we will also denote it $RM$. And the bundle $C_3M$, also denoted $BM$, corresponds to 5-tensors with symmetries of the image of the Bianchi operator applied to a section or $RM$. The index $l$ essentially counts the number of rows in the corresponding Young diagram. So, for $l > n$, the number of rows exceeds the base dimension and the $C_3M$ bundles become trivial. These tensor bundles, the corresponding Young diagrams and their fiber ranks are illustrated in Table 1.

2.2 Differential operators

Below, given any $n$-dimensional pseudo-Riemannian manifold $(M, g)$ of constant curvature $k$ (normalized so that the Ricci scalar curvature\footnote{We follow [87] for conventions regarding the definitions of curvature tensors and scalars.} is equal to $k$), we give explicit formulas for the differential operators, constituting the Calabi complex, as well as formulas for the differential operators that constitute a cochain homotopy from the complex to itself and the corresponding induced cochain maps. In Calabi’s original work [19], the corresponding differential operators were constructed using an orthogonal coframe formalism. Thus, it has been difficult to find explicit formulas for these operators in the tensor formalism that is more prevalent in the physics literature on relativity. The cochain homotopy operators and the induced cochain maps coincide, in low degrees, with differential operators well known in the relativity literature. However, their explicit form in all degrees appears to be new. Furthermore, we explicitly demonstrate all the identities between these differential operators that lead to their homological algebra interpretations. We use a mixture of elementary arguments, as well as equivariance and standard GL($n$)-representation theoretic identities, unlike Calabi’s original proofs [19] that relied on a somewhat ad hoc algebraic constructions, and unlike the derivation of Gasqui and Goldschmidt [34, 35] that relied on the sophisticated theory of Spencer sequences.

First, we define a number of differential operators that will be convenient for our purposes. For homogeneous differential operators with constant coefficients, the operator is completely determined by the principal symbol. In general that is not the case, yet the presence of a preferred connection on tensor bundles (the $g$-compatible Levi-Civita connection) still allows us to specify operators by their principal symbols: the covariant derivative is applied to a tensor $k$-times, the derivative indices are full symmetrized, and the principal symbol is applied to the result.

The principal symbol of a $k$-th order differential operator between two Young symmetrized bundles $YT^*$ and $Y'T^*$ is a linear map between them that depends polynomially on a covector $p \in T^*$. If the operator (or just its principal symbol) is GL($n$)-equivariant, then the principal symbol actually corresponds to an intertwiner between the $Y^{(k)} \otimes Y$ and $Y'$ representations. Such an intertwiner is non-zero only if $Y'$ appears in the irrep decomposition of the tensor product. Moreover if $Y'$ appears with single multiplicity, the intertwiner (and hence the principal symbol) is determined uniquely up to a scalar factor. It is an old result due to Pieri [33] that, in fact, the decomposition of the product of $Y^{(k)} \otimes Y$ into irreps has only single multiplicities. Not all principal symbols of importance to us are equivariant. The main source of the lack of equivariance is the dependence on the metric $g$. However, if the metric itself is also allowed to transform, the principal symbol becomes equivariant again. For instance,
if the operator is equivariant in this way and depends linearly on the metric in covariant form, it corresponds to an intertwiner between the representations \( Y^{(2)} \otimes Y^{(k)} \otimes Y \) and \( Y' \). Because of the presence of a double tensor product, Pieri’s rule doesn’t always apply, so sometimes more information is necessary to specify the desired intertwiner unambiguously. As a rule, these ambiguities will be resolved by giving explicit formulas.

Observe that the all tensor fields defined in Section 2.1 correspond to Young diagrams with at most two columns. We shall refer to the columns as left and right. Let \( d_L \) and \( d_R \), the left and right exterior differentials, be differential operators that increase by one the number of boxes in the, respectively, left or right column. They have equivariant principal symbols. We also define several operators whose principal symbols involve the metric. Two operators of order 0 are the trace \( \text{tr} \) and the metric exterior product \( (g \odot -) \), respectively, decreasing (contracting indices between the two columns) or increasing (multiplying by \( g \) and symmetrizing) by one the number of boxes in each column. Two operators of order 1 are left and right codifferentials \( \delta_L \) and \( \delta_R \), which decrease (taking a covariant divergence and resymmetrizing, if necessary) by one the number of boxes in, respectively, the left or right column. Finally, we have the tensor Laplacian \( \Box \), a differential operator of order 2 that does not alter the Young symmetry. Explicit formulas for these operators, along with proofs that they respect the corresponding Young symmetries, are given in Appendices A.2, A.4, A.5.

The differential operators constituting the Calabi complex, as well as cochain self-homotopy and the induced cochain self-maps fit into the following diagram:

\[
\begin{array}{c}
0 \longrightarrow C_0 \xrightarrow{B_1} C_1 \xrightarrow{B_2} C_2 \xrightarrow{B_3} \cdots \xrightarrow{B_n} C_n \longrightarrow 0 \\
0 \longrightarrow C_0 \xrightarrow{P_1} C_1 \xrightarrow{P_2} C_2 \xrightarrow{P_3} \cdots \xrightarrow{P_n} C_n \longrightarrow 0
\end{array}
\]

where for simplicity we have used the symbol \( C_l \) to stand for the space of sections \( \Gamma(C^lM) \). The operators \( B_l \) constitute a complex, because \( B_{l+1} \circ B_l = 0 \). The solid arrows in the diagram commute, \( P_{l+1} \circ B_{l+1} = B_l \circ P_l \), so that the \( P_l \) are cochain maps from the complex to itself. These cochain maps, \( P_l = E_{l+1} \circ B_{l+1} + B_l \circ E_l \), are induced by the homotopy operators \( E_n \), which appear as dashed arrows. Below, we give explicit formulas for each of these operators, discuss these identities, and relate them to well known differential operators from the literature on relativity. We follow the notational conventions of Appendices A.1, A.2. In particular, we use \( : \) to separate fully antisymmetric tensor index groups belonging to different columns of the Young diagram, which characterizes the symmetry type of a given tensor. However, for simplicity, we also write \( g_{a:b} = g_{a,b} \) and \( h_{a:b} = h_{a,b} \).

\[
\begin{align*}
B_1[v]_{a:b} &= K[v]_{a:b} = \nabla_a v_b + \nabla_b v_a, \\
B_2[h]_{a:b:c:d} &= -2\tilde{R}[h]_{a:b:c:d} \\
&= (\nabla_{[a} \nabla_{c]} h_{b:d} - \nabla_{(a} \nabla_{c}) h_{b:d} - \nabla_{(a} \nabla_{d}) h_{b:c} + \nabla_{(a} \nabla_{d}) h_{b:c}) \\
&\quad + \frac{k}{n(n-1)} (g \odot h)_{a:b:c:d},
\end{align*}
\]
dependence of $\bar{B}$

It is trivial to check that $\bar{B} = 0$, since the Bianchi identity, we find that $\bar{T} = \bar{0}$. Letting $\bar{T} = \bar{0}$, we obtain the identity $\bar{B}[R[g]] = 0$.

Note that the identity $R[g] - \bar{R}[g] = 0$ holds precisely when the metric $g$ has constant curvature $k$. Finally, $B_3 = \bar{B}$ is the background Bianchi operator, which also happens to coincide with the left exterior differential $d_L$. It satisfies the well known Bianchi identity $B[R[g]] = 0$. The operators $B_l$ for $l > 3$, which we may call higher Bianchi operators, do not appear to have been studied in the literature on relativity. So, as mentioned in the Introduction, the Calabi complex might also be legitimately referred to as the Killing-Riemann-Bianchi complex.

Now we give mostly elementary arguments for the composition identities $B_{l+1} \circ B_l = 0$. Recall that if $v$ is a vector field (identified with a section of $\mathcal{C}_0 M \cong T^* M$ using the metric), then the Lie derivative of the metric along $v$ is given by the Killing operator, $\mathcal{L}_v g = K[v]$. Now, suppose that $T[g]$ is any tensor field covariantly constructed out of the metric and its derivatives. Consider its linearization $T[g + \lambda h] = T[g] + \lambda T[h] + O(\lambda^2)$. The linearization $\hat{T}$ annihilates the Killing operator if $T[g] = 0$ [83]. This fact follows from the fact that $T[g]$ itself is a tensor field, so that

$$\mathcal{L}_v T[g] = \hat{T} \mathcal{L}_v g = \hat{T} \circ K[v].$$

Letting $T[g] = R[g]$, we obtain the identity $B_2 \circ B_1 = -\Bar{B} \circ K = 0$, since $T[g] = 0$. Further, note that, since the metric is covariantly constant, $\nabla g = 0$, it is trivial to check that $\bar{B}[R[g]] = 0$, for any $g$. Combining this observation with the Bianchi identity, we find that $\bar{B}[R[g] - \bar{R}[g]] = 0$, for any $g$. Making the dependence of $\bar{B} = \bar{B}_g$ on $g$ explicit, the linearization of this identity gives

$$\bar{B}_g + \lambda \bar{B}[R[g + \lambda h] - \bar{R}[g + \lambda h]]$$

$$= \bar{B}_g[R[g] - \bar{R}[g]] + \lambda \bar{B}[R[h]] + \bar{B}[h, R[g] - \bar{R}[g]] + O(\lambda^2) = 0,$$  (9)

The same corrected curvature tensor can be obtained by linearizing the mixed form $R[g]_{abcd}$ of the Riemann tensor and then lowering all indices with the background metric. This linearized mixed Riemann tensor was previously used to isolate the gauge invariant metric perturbations on de Sitter space in [62]. That the linearized corrected Riemann tensor annihilates the Killing operator also follows from the classical analysis in [83], which noted that the linearization of any tensor built only out of the metric and vanishing on the background spacetime is invariant under linearized diffeomorphisms.
where \( B_{g+\lambda h}[T] = B[T] + \lambda B[h, T] + O(\lambda^2) \). At first order in \( \lambda \), we obtain the desired identity \( B_2 \circ B_2 = -2B \circ \hat{R} = 0 \). The remaining identities, \( B_{l+1} \circ B_l = d_I^2 = 0 \) for \( l > 2 \), follow from abstract representation theoretic reasons, described in more detail in Appendix A.4 and A.5.

\[
E_1[h]_a = D[h]_a = \nabla^b h_{ab} - \frac{1}{2} \nabla_a h, \quad (10)
\]

\[
E_2[r]_{ab} = tr[r]_{ab} = r_{acbc}, \quad (11)
\]

\[
E_3[b]_{abcd} = \nabla^e b_{abcd} + \frac{1}{2} \nabla^e (b_{abde} - b_{dabe})
- \frac{1}{2} (\nabla_a b_{cde} - \nabla_d b_{abe}c)
- \frac{1}{2} (\nabla_a b_{cde} - \nabla_d b_{abe}c)
+ \nabla_b b_{ace}d - \nabla_b b_{de}c, \quad (12)
\]

\[
E_4[b]_{abcd} = \nabla^f b_{abcd} + \frac{1}{3} \nabla^f (b_{daecf} - b_{eacdf})
+ \frac{1}{3} (\nabla_d b_{aefc} - \nabla_e b_{aefd})
+ \frac{1}{6} (\nabla_d b_{aefc} - \nabla_e b_{aefd})
+ \nabla_b b_{aefc} - \nabla_b b_{aefd}
+ \nabla_e b_{aefd}, \quad (13)
\]

\[
E_5[b]_{abcd} = \nabla^i b_{abcd} - \frac{1}{4} \nabla^i (b_{abei} - b_{faei})
- \frac{1}{4} (\nabla_a b_{dei} - \nabla_f b_{dei})
- \frac{1}{12} (\nabla_e b_{(abcd)i} - \nabla_f b_{(abcd)i}), \quad (14)
\]

\[
E_{l+1}[b]_{ai_1 \ldots ai_l} = (\delta_{L}[b] - (-1)^{l-1} d_l \circ tr[b])_{ai_1 \ldots ai_l} \quad (l \geq 2). \quad (15)
\]

The notation used in the formula for \( E_5 \) is defined in Appendix A.1. Note that \( E_1 = D \) is the de Donder operator, used as a linearized gauge fixing condition in the literature on relativity. Also, if \( R[g] \) is the Riemann tensor of the metric \( g \), then \( E_2[R[g]] = tr[R[g]] \) is the corresponding Ricci tensor. The higher homotopy operators \( E_l \) for \( l > 2 \) do not seem to have previously appeared in the literature on relativity.

\[
P_0[v]_a = \Box v_a + k \frac{1}{n} v_a, \quad (16)
\]

\[
P_1[h]_{ab} = \Box h_{ab} - k \frac{2}{n(n-1)} h_{ab} + 2k \frac{g_{ab} tr[h]}{n(n-1)}, \quad (17)
\]

\[
P_2[r]_{abcd} = \Box r_{abcd} - k \frac{2}{n} r_{abcd} + 2k \frac{(g \circ tr[r])_{abcd}}{n(n-1)}, \quad (18)
\]

\[
P_3[b]_{abcd} = \Box b_{abcd} - k \frac{(3n-7)}{n(n-1)} b_{abcd} - 2k \frac{(g \circ tr[b])_{abcd}}{n(n-1)}, \quad (19)
\]
However, we are actually free to choose any metric, say \( g \) depend on the metric \( E \) and is sometimes known as the \textit{Lichnerowicz Laplacian}.

Remark 1. Penrose wave equation known as the \textit{field} by the Riemann and Weyl tensors on any vacuum background, sometimes specified by the de Donder gauge condition \( D[h] = 0 \) in linearized gravity. The operator \( P_2 = \text{tr} \circ (-2R) + K \circ D \) is the wave-like operator of the linearized Einstein equations for gravitational perturbations \( h \) in de Donder gauge \( D[h] = 0 \). These two operators are well known and can be found (or their close analogs can) for instance in \cite[Sec.7.5]{87} and more they appeared in in \cite[41, 9]{29}. The higher cochain maps and the corresponding identities appear to be new. Though, the identity \( P_2 = E_3 \circ B - 2R \circ E_2 \) is related to the non-linear wave equations satisfied by the Riemann and Weyl tensors on any vacuum background, sometimes known as the \textit{Penrose wave equation}. For linearized fields, a related equation is sometimes known as the \textit{Lichnerowicz Laplacian}. For more details, see references \cite[Sec.1.3]{70}, \cite[Sec.7.1]{55}, \cite[Exr.15.2]{20}, \cite[Eq.35]{14}.

Note our notation \( \Box = \nabla^a \nabla_a \) for the tensor Laplacian, which is also known as the d’Alambertian in Lorentzian signature. The operator \( P_0 = D \circ K \) is gives the wave-like residual gauge condition such that the perturbation \( h = K[v] \) satisfies the de Donder gauge condition \( D[h] = 0 \) in linearized gravity. The operator \( P_1 = \text{tr} \circ (-2R) + K \circ \nabla \) is the wave-like operator of the linearized Einstein equations for gravitational perturbations \( h \) in de Donder gauge \( D[h] = 0 \). These two operators are well known and can be found (or their close analogs can) for instance in \cite[Sec.7.5]{87} and more they appeared in in \cite[41, 9]{29}. The higher cochain maps and the corresponding identities appear to be new. Though, the identity \( P_2 = E_3 \circ B - 2R \circ E_2 \) is related to the non-linear wave equations satisfied by the Riemann and Weyl tensors on any vacuum background, sometimes known as the \textit{Penrose wave equation}. For linearized fields, a related equation is sometimes known as the \textit{Lichnerowicz Laplacian}. For more details, see references \cite[Sec.1.3]{70}, \cite[Sec.7.1]{55}, \cite[Exr.15.2]{20}, \cite[Eq.35]{14}.

\[
P_4[b]_{abcd.ef} = \Box b_{abcd.ef} - k \frac{4n - 14}{n(n - 1)} b_{abcd.ef} + 2k \frac{(g \circ \text{tr}[b])_{abcd.ef}}{n(n - 1)}, \quad (20)
\]

\[
P_1[b]_{a_1 \ldots a_i；bc} = \Box b_{a_1 \ldots a_i；bc} - k \frac{(ln - l^2 + 2)}{n(n - 1)} b_{a_1 \ldots a_i；bc} + (-1)^l 2k \frac{(g \circ \text{tr}[b])_{a_1 \ldots a_i；bc}}{n(n - 1)} \quad (l \geq 3).
\]

\[
P_2 = \text{tr} \circ (-2R) + K \circ D \text{ is the wave-like operator of the linearized Einstein equations for gravitational perturbations } h \text{ in de Donder gauge } D[h] = 0. \text{ These two operators are well known and can be found (or their close analogs can) for instance in } \cite[Sec.7.5]{87} \text{ and more they appeared in in } \cite[41, 9]{29}. \text{ The higher cochain maps and the corresponding identities appear to be new. Though, the identity } P_2 = E_3 \circ B - 2R \circ E_2 \text{ is related to the non-linear wave equations satisfied by the Riemann and Weyl tensors on any vacuum background, sometimes known as the } \textit{Penrose wave equation}. \text{ For linearized fields, a related equation is sometimes known as the } \textit{Lichnerowicz Laplacian}. \text{ For more details, see references } \cite[Sec.1.3]{70}, \cite[Sec.7.1]{55}, \cite[Exr.15.2]{20}, \cite[Eq.35]{14}.\]

\[
P_0 = D \circ K \text{ is gives the wave-like residual gauge condition such that the perturbation } h = K[v] \text{ satisfies the de Donder gauge condition } D[h] = 0 \text{ in linearized gravity. The operator } P_1 = \text{tr} \circ (-2R) + K \circ \nabla \text{ is the wave-like operator of the linearized Einstein equations for gravitational perturbations } h \text{ in de Donder gauge } D[h] = 0. \text{ These two operators are well known and can be found (or their close analogs can) for instance in } \cite[Sec.7.5]{87} \text{ and more they appeared in in } \cite[41, 9]{29}. \text{ The higher cochain maps and the corresponding identities appear to be new. Though, the identity } P_2 = E_3 \circ B - 2R \circ E_2 \text{ is related to the non-linear wave equations satisfied by the Riemann and Weyl tensors on any vacuum background, sometimes known as the } \textit{Penrose wave equation}. \text{ For linearized fields, a related equation is sometimes known as the } \textit{Lichnerowicz Laplacian}. \text{ For more details, see references } \cite[Sec.1.3]{70}, \cite[Sec.7.1]{55}, \cite[Exr.15.2]{20}, \cite[Eq.35]{14}.\]

Remark 1. It is worth noting that we refer to the operators \( P_l \) as wave-like because the principal symbol of \( P_l \) has the same principal symbol as the tensor Laplacian \( \Box = \nabla^a \nabla_a \), on Lorentzian manifolds also known as the d’Alambertian or wave operator, which is a hyperbolic differential operator. Note that the principal symbol of \( P_l \) is determined only by the principal symbols of the \( B_l \) and \( E_l \). The principal symbols of \( B_l \) and \( E_l \) are metric independent, while those of \( E_l \) depend on the metric \( g \) of the background pseudo-Riemannian manifold \((M,g)\). However, we are actually free to choose any metric, say \( g' \), that is different from \( g \), to construct the cochain homotopy operators, say \( E_l' \). The principal symbol induced cochain maps \( P_l' = E_{l+1}' \circ B_{l+1} + B_l \circ E_l' \) will then still only depend on one metric, \( g' \), and be equal to the principal symbol of the tensor Laplacian \( \Box ' \), defined with respect to \( g' \). Thus, if we choose \( g' \) to be Riemannian, we can induce cochain homotopies \( P_l' \) that are elliptic. The operators \( P_l' \) will of course differ from the \( P_l \) by terms of lower differential order that would depend on both \( g \) and \( g' \). This remark will be very useful in Proposition 9 in the discussion of the local exactness of the Calabi complex.

2.3 Formal adjoint complex

Given a linear differential operator \( f: \Gamma(E) \to \Gamma(F) \), between vector bundles \( E \to M \) and \( F \to M \), its \textit{formal adjoint} is a linear differential operator \( f^* : \Gamma(F^*) \to \Gamma(E^*) \), where where we have used the notation for the bundle \( V^* \cong V^* \otimes_M \Lambda^n M \) of dual densities of a vector bundle \( V \to M \), defined as the tensor product of the its linear dual bundle \( V^* \to M \) with that of densities \( \Lambda^n M \to M \) on the base manifold if dimension \( \text{dim } M = n \). The formal adjoint operator is defined to be the unique differential operator such that a \textit{Green formula} holds,

\[
\psi \cdot f[\xi] - f^*[\psi] \cdot \xi = dG[\psi, \xi], \quad (22)
\]
for any \( \psi \in \Gamma(\tilde{F}^\ast) \), \( \xi \in \Gamma(E) \), and some bilinear bidifferential operator

\[
G : \Gamma(\tilde{F}^\ast \times_M E) \to \Gamma(\Lambda^{n-1}M).
\]  

A formal adjoint operator always exists and is unique [5, 4, 85].

In the presence of background pseudo-Riemannian metric \( g \) on \( M \), we can canonically identify the trivial bundle \( \mathbb{R} \times M \) with \( \Lambda^n M \), via multiplication by the canonical volume form \( \varepsilon_{a_1 \ldots a_n} \) with respect to \( g \) (\( \varepsilon \in \Omega^n(M) \)), and also \( V \cong V^\ast \) for any tensor bundle \( V \to M \), by lowering and raising indices with \( g \), thus also canonically identifying \( V \cong V^\ast \). Below, we will take formal adjoints with respect to this identification. Recall the identity [92]

\[
\varepsilon^{a_2 \ldots a_n} \varepsilon_{b_2 \ldots a_n} = (-1)^s(n - 1)! \delta_b^a
\]  

(24)

(where \( s \) counts the number of minuses in the signature of the metric \( g \), with \( s = 1 \) for Lorentzian metrics with mostly-plus convention) and define

\[
G^a = \frac{(-1)^s}{(n - 1)!} \varepsilon^{a_2 \ldots a_n} G_{a_2 \ldots a_n}
\]  

(25)

so that \( G_{a_2 \ldots a_n} = \varepsilon^{a_2 \ldots a_n} G^a \). The right hand side of the formal adjoint equation (22) can then be rewritten as

\[
(\text{d}G)_{a_1 \ldots a_n} = \frac{(-1)^s}{n!} \varepsilon_{a_1 \ldots a_n} \varepsilon^{b_2 \ldots b_n} \text{d} \nabla_b G_{b_2 \ldots b_n} = \varepsilon_{a_1 \ldots a_n} \nabla_a G^a,
\]  

(26)

with the whole equation becoming

\[
\psi \cdot f[\xi] - f^\ast[\psi] \cdot \xi = \nabla_a G^a[\psi, \xi],
\]  

(27)

where the dot indicates contraction of indices using the metric \( g \) between two tensors of the same index structure.

With this notation, the formal adjoint Calabi complex \((C \ast, B \ast)\) fits into the following diagram:

\[
\begin{array}{cccccccc}
0 & \leftrightarrow & C_0 & \leftrightarrow & C_1 & \leftrightarrow & C_2 & \leftrightarrow & \cdots \leftrightarrow & C_n & \leftrightarrow & 0 \\
0 & \leftrightarrow & B_1 & \leftrightarrow & B_2 & \leftrightarrow & B_3 & \leftrightarrow & \cdots \leftrightarrow & B_n & \leftrightarrow & 0 \\
\end{array}
\]

(28)

where we have identified \( \tilde{C}^a_i \cong C_i \) using the background metric. Note that all the analogous identities are satisfied, the solid arrows in the diagram commute and the dashed arrows are homotopy operators inducing the vertical cochain maps, \( P_i^a = B_{i+1}^a \circ B_i^a + E_i^a \circ B_i^a \). The main difference is that the \( B_i \) now decrease the degree index by one instead of decreasing it. The usual numbering convention can be achieved by relabelling, but we shall not do so here, expecting that no confusion will arise.

Recall that the final differential operator \( B_n \) of the Calabi complex is

\[
B_n[b]_{a_1 \ldots a_n, bc} = d_L[b]_{a_1 \ldots a_n, bc} = n \nabla_{[a_1} b_{a_2 \ldots a_n], bc},
\]  

(29)

where \( b \in \Gamma(C_{n-1}M) \). To compute its formal adjoint, let \( c \in \Gamma(C_nM) \) and consider first the identity, derived in Appendix A.6,

\[
\nabla_a (\varepsilon^{a_2 \ldots a_n} b_{a_2 \ldots a_n}) = \frac{1}{n} \varepsilon^{a_2 \ldots a_n} d_L[b]_{a_2 \ldots a_n, bc} + \delta_L[c]^{a_2 \ldots a_n, bc} b_{a_2 \ldots a_n, bc}.
\]  

(30)
Note that the operators $d_L$ and $\delta_L$ specifically produce tensors of the appropriate Young type. Therefore, the formal adjoint operator $B^*_n$ is given by the formula

$$B^*_n[c]_{a_2\cdots a_n;bc} = -\frac{1}{n} \delta_L[c]_{a_2\cdots a_n;bc} = -\frac{1}{n} \delta_L[c]_{a_2\cdots a_n;bc},$$

(31)

$$= -\frac{1}{n} \nabla^a \epsilon_{a a_2\cdots a_n;bc} - \frac{2}{n(n-1)} \nabla^a \epsilon_{b[a_2\cdots a_n;c]a} ,$$

(32)

with the Green form represented by $G^a[c,b] = \frac{1}{n} \delta^a_{a_2\cdots a_n;bc} \delta_{a_2\cdots a_n;bc}$.

While this operator $B^*_n$ may look unfamiliar, after a further local invertible transformation the equation $B^*_n[c] = 0$ becomes equivalent to the well known \textit{rank-($n-2$) Killing-Yano equation}. Let us define a rank-($n-2$) anti-symmetric tensor $y^c_{\cdots c_n}$ such that

$$c_{a_1\cdots a_n;bc} = \epsilon_{a_1\cdots a_n} y^c_{\cdots c_n} \delta_{bc} \varepsilon_{b c c_3 \cdots c_n},$$

(33)

$$y^c_{\cdots c_n} = \frac{1}{2(n-2)!(n-1)!} \epsilon^{a_1\cdots a_n} c_{a_1\cdots a_n;bc} \delta_{bc} \varepsilon_{b c c_3 \cdots c_n}.$$  

(34)

It is straightforward to check using the hook formula (Appendix A) that the tensor $c$ of Young type $(2,2,1^{n-2})$ has the same number of independent components as the tensor $y$ of Young type $(1^n-2)$. To transform the equation satisfied by $c$ into the Killing-Yano equation satisfied by $y$, we will need the following identities, which follow from the general properties of the $\epsilon$ tensor [92]:

$$\epsilon^{a a_2\cdots a_n c_{a_1\cdots a_n;bc} \delta_{b c} \varepsilon_{b c c_3 \cdots c_n}} = 2(n-2)!(n-1)! \delta^a_{n} \gamma^c_{\cdots c_n},$$

(35)

$$\epsilon^{a a_2\cdots a_n c_{b a_2\cdots a_n;ac} \delta_{b c} \varepsilon_{b c c_3 \cdots c_n}} = (n-1)! \gamma^{c_3 \cdots c_n} \delta^a_1 \delta^a_2 \cdots \delta^a_{n}.$$  

(36)

Contracting one $\epsilon$ tensor with each index group of the equation $B^*_n[c] = 0$ we get

$$0 = \epsilon^{a a_2 \cdots a_n} B^*_n[c]_{a_2 \cdots a_n;bc} \delta_{bc} \varepsilon_{b c c_3 \cdots c_n}$$

(37)

$$= -\frac{1}{n} \nabla^a \epsilon^{a a_2 \cdots a_n c_{a_1 \cdots a_n;bc} \delta_{bc} \varepsilon_{b c c_3 \cdots c_n}}$$

$$= -\frac{2}{n(n-1)} \nabla^a \epsilon^{a a_2 \cdots a_n c_{b a_2 \cdots a_n;ca} \delta_{bc} \varepsilon_{b c c_3 \cdots c_n}}$$

(38)

$$= -\frac{2}{n} (n-1)!(n-2)! \left( \nabla^a y^c_{\cdots c_n} - \nabla^a (\gamma^c_{\cdots c_n}) \right).$$

(39)

Note that the derivative $\nabla^a y^c_{\cdots c_n}$ takes values in the tensor bundle of Young type $(1)\otimes(1^{n-2})$. Using the well-known Littlewood-Richardson rules [33, 59] this representations decomposes into the direct sum $(1)^{n-1} \oplus (2, 1^{n-3})$. Note that the antisymmetrization of the above equation gives zero. Thus, the independent components of the equation satisfied by $y$ take values in a tensor bundle of Young type $(2,1^{n-3})$, which has two columns, of lengths $n-2$ (filled with indices belonging to $y$) and 1 (filled with index belonging to $\nabla$). It is also well-known that this representation can be isolated by antisymmetrizing along the columns and symmetrizing any two indices between the columns. In our case, the antisymmetrization has no effect ($y$ is already antisymmetric) and the symmetrization, after lowering all indices, gives the equation

$$KY[y]_{ac_3 \cdots c_n} = \nabla(\gamma y_{\cdots c_n}) = 0,$$  

(40)

13
which is none other than the rank-$(n − 2)$ Killing-Young equation, whose solutions are called rank-$(n − 2)$ Killing-Young tensors or Killing $(n − 2)$-forms \[82\]. We refer to the differential operator $KY$ as the Killing-Young operator. So, in the same sense that the Calabi complex constitutes the compatibility complex of the Killing equation on a constant curvature background, so does the formal adjoint Calabi complex for the rank-$(n − 2)$ Killing-Yano equation on the same background.

### 2.4 Equations of finite type, twisted de Rham complex

The Killing and Killing-Yano equations, which lie at the base of the Calabi and its formal adjoint differential complexes, are well known examples of partial differential equations of finite type \[38, 79, 63\]. That is, in any neighborhood of a point $x \in M$ they admit only a finite dimensional space of solutions. Each solution is fully determined by its value and finitely many derivatives at $x$. For the Killing and Killing-Yano equations only the first derivatives are required. This is a strong kind of unique continuation. Such equations are called regular if the dimension of the solution space in a sufficiently small neighborhood of a point $x \in M$ is independent of $x$. That number may, however, differ from the dimension of the global solution space, which can be strictly smaller in the presence of topological or geometric obstructions to continuing local solutions to global ones.

Regular equations of finite type have a very simple existence theory. Let $F \to M$ and $E \to M$ be two vector bundles, together with a differential operator $e: \Gamma(F) \to \Gamma(E)$ of order $l$ such that the equation $e[\psi] = 0$, for $\psi \in \Gamma(F)$, is finite type and regular. This means that there exists an integer $k$ such that the knowledge of $j^k\psi(x)$ for any $x \in M$ is sufficient to determine the components of all higher jets of $\psi$ at $x$. Prolongation of the equation to order $k$ (Appendix C) gives the bundle map $p^{k−l}e: J^kF \to J^{k−l}E$. By the regularity hypothesis, the map is of constant rank, so its kernel $V = \ker p^{k−l}e \subseteq J^kF$ is a vector bundle over $M$. Since all higher derivatives of a solution $\psi$ at $x$ are uniquely determined by $j^k\psi(x)$ and $j^k\psi$ only takes values in $V$, there is a unique $n$-dimensional hyperplane in $T_xV$ that is tangent to the graph of a solution $\psi$ such that $j^k\psi(x) = (x,v)$. These hyperplanes define an $n$-dimensional distribution on the total space of the bundle $V$ and it is straightforward to check that this distribution is involutive (Lie brackets of vector fields valued in the distribution remain valued in the distribution). Thus, by the theorem of Frobenius \[54\], $V$ is foliated by $n$-dimensional leaves tangent to the given hyperplane distribution. Locally, these leaves are precisely the graphs of solutions to the equation $e[\psi] = 0$. Thus the rank $rkV$ is precisely the dimension of the local solution space on any sufficiently small, connected open set in $M$.

As we have already mentioned, both the Killing and Killing-Yano operators, $K: \Gamma(T^*M) \to \Gamma(S^2M)$ and $KY: \Gamma(A^{n−2}M) \to \Gamma(Y^{(2,1^{n−1})}T^*M)$, define finite type equations. By the virtue of their covariance, they are also regular on any pseudo-Riemannian symmetric space, which includes constant curvature backgrounds. Furthermore, on constant curvature spaces, the dimensions of their local solution spaces are $rkV_K = rkV_{KY} = n(n + 1)/2$ \[82\].

The $n$-dimensional hyperplane distribution on $V$ and the resulting foliation described above can also be described in another way, namely as a flat linear connection on $V \subseteq J^kF$ \[57, Sec.2.1.3\]. The connection is linear because the
original equation $e[\psi] = 0$ is itself linear. A linear connection on $V \to M$ can alternatively be described by a first order differential operator $D : \Gamma(V) \to \Gamma(T^*M \otimes V)$ defined by the property

$$D[\omega^k \psi] = d\omega \otimes j^k \psi,$$

(41)

for any $\omega \in C^\infty(M)$ and solution $\psi \in \Gamma(F)$ of $e[\psi]$, where its $k$-jet is treated as a section $j^k \psi : M \to V$. That is, a section $\phi \in \Gamma(V) \subseteq \Gamma(J^k F)$ is constant on an open set $U \subseteq M$ iff it coincides with the $k$-jet of a solution of $e[\psi] = 0$ on $U$. So, it is clear that the equations $e[\psi] = 0$ and $D[\phi] = 0$ are equivalent (their spaces of local solutions are locally isomorphic). As discussed in Appendix C this means that there exist differential operators $f$, $f'$, $g$, $g'$, $p$ and $q$, which fit into the following diagram (again, for brevity we use the bundle symbols to stand in for their spaces of sections)

$$\begin{array}{ccc}
F & \xrightarrow{e} & E \\
\downarrow{f} & & \downarrow{f'} \\
V & \xrightarrow{D} & T^*M \otimes V \\
\downarrow{g} & & \downarrow{q}
\end{array}$$

(42)

and satisfy the following identities:

$$D \circ f = f' \circ e, \quad g \circ f = \text{id} + p \circ e, \quad e \circ g = g' \circ D, \quad f \circ g = \text{id} + q \circ D. \quad (43)$$

(44)

We have already seen that on solutions, the map $f$ simply agrees with the $k$-jet extension operator $j^k$. Thus, as a differential operator of order $k$, it can be chosen to be any projection of $J^k F$ to its subspace $V$. The choice of this projection then determines the differential operator $f'$. The differential operators $g$ and $g'$ are constructed in similar ways, making sure that $f$ and $g$ are mutual inverses on solutions. The freedom in the choice of $f$, $f'$, $g$ and $g'$ also determine the operators $p$ and $q$.

When it comes to a specific case, say the Killing or Killing-Yano equation, its equivalence to a local constancy condition with respect to a connection can be made explicit only once the solutions are themselves explicitly known. Thus this equivalence is mostly of theoretical, though non-negligible, interest.

Having defined the flat vector bundle $(V,D)$ corresponding to a regular equation of finite type, there is a standard procedure to construct a differential complex associated to it. It is called the twisted de Rham complex associated to $(V,D)$,

$$\begin{array}{cccc}
0 & \longrightarrow & V & \xrightarrow{D} & \Lambda^1 M \otimes V & \xrightarrow{D} & \Lambda^2 M \otimes V & \cdots & \Lambda^n M \otimes V & \longrightarrow & 0,
\end{array} \quad (45)$$

where $D$ has been extended to a twisted de Rham differential, defined on sections of $\Lambda^k M \otimes V$ by the condition

$$D[\omega \otimes \psi] = d\omega \otimes \psi + (-1)^k \omega \wedge D\psi,$$

(46)

for any $\omega \in \Gamma(\Lambda^k M)$ and $\psi \in \Gamma(V)$, where we recall that $D\psi$ is a section of $T^*M \otimes V = \Lambda^1 M \otimes V$ and apply the wedge product of forms in the obvious way.
Remark 2. Locally (on sufficiently small contractible open sets), this twisted de Rham complex consists $rk V$ copies of the ordinary de Rham complex. Globally, of course, if the base manifold $M$ is not simply connected, the twisted de Rham complex $(\Lambda^* M \otimes V, D)$ will differ from $rk V$ copies of the ordinary de Rham complex $(\Lambda^* M, d)$ because of the possible non-trivial bundle structure of $V \to M$ or the non-trivial monodromy $D$ (parallel transport with respect to $D$ along closed loops). The importance of the twisted de Rham complex will become clear in Section 3 where we discuss the connection between the cohomology of differential complexes and sheaf cohomology.

For later convenience, we shall denote the twisted de Rham complexes associated to the Killing and Killing-Yano equations, respectively, by $(\Lambda^* M \otimes V_K, D_K)$ and $(\Lambda^* M \otimes V_{KY}, D_{KY})$.

3 Cohomology of locally constant sheaves

The main reasons for introducing some of the general sheaf and sheaf cohomology machinery below is are two fold. First, we have made a connection between the abstract notion of sheaf cohomology and the cohomology of a differential complex. A priori, computing the cohomology of differential complex is a very hard problem, because it involves solving partial differential equations. On the other hand, because of the flexibility of the general machinery of sheaf cohomology, it may be computable in some effective way, for instance, by reducing it to a problem in finite dimensional linear algebra. The canonical example of where this connection can be leveraged is the computation of de Rham cohomology groups of a manifold $M$ using the equivalent (through sheaf theoretic machinery) computation of the simplicial (or cellular) cohomology of a finite triangulation (or cell decomposition) of $M$. The second reason is that the ideas that have been introduced give us some tools to explicitly show that the cohomologies of two different differential complexes are isomorphic as long as both complexes are formally exact, locally exact and resolve the same sheaf in degree-0 (this terminology is introduced below).

3.1 Locally constant sheaves

Recall from Section 2.4 that a regular linear differential equation of finite type has only a finite dimensional space of local solutions, with this dimension being constant over the base manifold. It so happens that, from an abstract point of view, it is convenient to view these local solutions as a locally constant sheaf of vector spaces. A sheaf $\mathcal{F}$ of vector spaces on a topological space $M$ is an assignment $U \mapsto \mathcal{F}(U)$ of a vector space (of local sections over $U$, $\mathcal{F}(\emptyset) = 0$) to each open $U \subseteq M$ satisfying the following axioms: (restriction) for any inclusion of opens $U \subseteq V$ there exist linear restriction maps $\mathcal{F}(V) \to \mathcal{F}(U)$, also written $f \mapsto f|_U$, such that $U \subseteq U$ induces the identity map and $U \subseteq V \subseteq W$ induces $\mathcal{F}(W) \to \mathcal{F}(U)$ in agreement with the composition $\mathcal{F}(W) \to \mathcal{F}(V) \to \mathcal{F}(U)$; (descent) any pair of opens $U$ and $V$ induces an exact sequence $0 \to \mathcal{F}(U \cap V) \to \mathcal{F}(U) \times \mathcal{F}(V) \to \mathcal{F}(U \cup V) \to 0$, where the first map is $f \mapsto (f|_U, f|_V)$ and the second one is $(f, g) \mapsto f|_{U \cap V} - g|_{U \cap V}$. We write $\Gamma(\mathcal{F}) = \Gamma(M, \mathcal{F}) = \mathcal{F}(M)$ for the vector space of global sections of the sheaf $\mathcal{F}$. A sheaf is called locally constant when the number $\dim \mathcal{F}_x = \max_{U \ni x} \dim \mathcal{F}(U)$, where $U$ ranges over
connected open neighborhoods of \(x \in M\), is finite and does not depend on \(x\), so we can write \(\dim F = \dim F_x\). Since \(\dim F(U)\) can only decrease for larger connected \(U\), for any \(x \in M\) there exists a connected neighborhood \(U\) of \(x\) such the vector spaces of local sections over smaller connected neighborhoods stabilize (the restriction map becomes an isomorphism), so that we can write \(\mathcal{F}(U) \cong \hat{F}\) for some fixed vector space \(\hat{F}\) that we call the stalk of \(\mathcal{F}\). Clearly, \(\dim \hat{F} = \dim \mathcal{F}\). Also, \(\mathcal{F}\) is called constant when it is locally constant and \(\Gamma(\mathcal{F}) \cong \hat{F}\).

Given a vector bundle \(F \to M\), the assignment \(\mathcal{F}(U) = \Gamma(F, U)\) of local sections of \(F\) over each open \(U \subseteq M\) defines a sheaf \(\mathcal{F}\) on \(M\), called the sheaf of (germs of) sections of \(F \to M\). Similarly, it is straightforward to check that, given another vector bundle \(E \to M\) and a linear differential operator \(e : \Gamma(F) \to \Gamma(E)\), the sets \(S_e(U) = \{\psi \in \Gamma(F, U) \mid e[\psi] = 0\}\) of solutions of the partial differential equation \(e[\psi] = 0\) also define a sheaf \(S_e\) on \(M\), called the solution sheaf of \(e : \Gamma(F) \to \Gamma(E)\). Following the preceding discussion of equations of finite type, it should be clear that solution sheaves \(\mathcal{K} = \mathcal{S}_K\) (the Killing sheaf) and \(\mathcal{KY} = \mathcal{S}_{KY}\) (the Killing-Yano sheaf) of the Killing and Killing-Yano equations are locally constant, provided the background pseudo-Riemannian manifold is chosen such that these equations are regular. Another important example is the constant sheaf \(\mathbb{R}_M = \mathcal{S}_\delta\) of locally constant functions, which solve the equation \(df = 0\), \(f \in C^\infty(M)\) and \(d\) the de Rham differential.

Sheaves are important because every sheaf \(\mathcal{F}\) (of vector spaces) on \(M\) automatically comes with an abstract notion of sheaf cohomology (vector spaces) \(H^p(M, \mathcal{F})\), called the \(p\)-th or degree-\(p\) cohomology of \(\mathcal{F}\), or of \(M\) with coefficients in \(\mathcal{F}\). Moreover, all classical cohomology theories from algebraic topology can be identified with the cohomologies of certain sheaves. Further, some superficially different looking cohomologies theories may be connected through the fact that they are both equivalent to the sheaf cohomology of the same sheaf. In particular, the classical simplicial, cellular, singular, Čech and de Rham cohomologies of a manifold \(M\) all coincide \([15, 16, 45]\) because they are each equivalent to the cohomology of \(M\) with coefficients in the sheaf \(\mathbb{R}_M\) of locally constant functions.

The intrinsic definition of sheaf cohomology is somewhat involved and not entirely intuitive (unless one is already intimately familiar with Čech cohomology and the notion of local coefficients). Fortunately, the intrinsic definition can be relegated to standard references \([16, 45]\) in favor of an equivalent but more practical definition using acyclic resolutions. To explain further, we need to introduce some terminology. A complex of sheaves of vector spaces

\[
\cdots \longrightarrow F_i \longrightarrow F_{i+1} \longrightarrow \cdots
\]  

(47)

consists of an assignment of linear maps \(F_i(U) \to F_{i+1}(U)\) to each open \(U \subseteq M\), in a way consistent with restriction maps, such that we have a complex of vector spaces of local sections (two successive maps compose to zero)

\[
\cdots \longrightarrow F_i(U) \longrightarrow F_{i+1}(U) \longrightarrow \cdots
\]  

(48)

for each open \(U \subseteq M\). A local section in \(F_i(U)\) that is in the kernel of the corresponding map is called a cocycle and a local section in \(F_i(U)\) that is in image of the corresponding map is called a coboundary. A sheaf complex is exact when, for each \(x \in M\), open neighborhood \(U \subseteq M\) of \(x\) and local section
α ∈ 𝒟(U), there exists a possibly smaller and α-dependent open neighborhood 
U ′ ⊆ U of x such that α|U ′ is a coboundary. For a complex of sheaves, like (47),
we could define its cohomology sheaves ℋ′(𝒟•) (distinct from sheaf cohomology,
to be defined later), by starting with the assignment ℋ′(𝒟•)(U) = ker(𝒟i(U) → 
𝒟i+1(U))/im(𝒟i−1(U) → 𝒟i(U)), which may not produce a sheaf but only a 
presheaf, and applying the sheafification construction to it. We will not go into 
the details of how sheafification turns presheaves into sheaves here, but they 
can be found in standard references [16, 45]. It suffices to point out that given 
a sheaf complex in non-negative degrees, 0 → 𝒟0 → 𝒟1 → · · · , the vector space 
ℋ′(𝒟•)(U) ⊆ ℋ0(U) consists of all cocycle local sections. In the sequel, we shall 
only need to refer to such cohomology sheaves in degree-0. Given a sheaf 𝒟, if 
𝒟i → 𝒟i+1 is a complex of sheaves such that 𝒟i = 0 for i < 0, ℋ′(𝒟•) = 𝒟, and 
ℋ′(𝒟•) = 0 for i > 0, we call it a resolution of the sheaf 𝒟.

In the sequel, we shall only consider sheaves of sections of vector bundles 
or of solution of some liner PDE and only complexes of sheaves where maps 
between the vector spaces of local sections are induced by restrictions of differ-
ential operators, for which the compatibility with restrictions is automatically 
satisfied.

### 3.2 Acyclic resolution by a differential complex

The de Rham complex [15] is the canonical example of a complex of sheaves 
of sections of vector bundles (differential forms on M), with maps induced by 
differential operators (de Rham differentials). The Poincaré lemma then demon-
strates that this complex of sheaves is exact. For simplicity, we shall call a dif-
fferential complex (𝒟•, 𝒥•) a sequence of vector bundles 𝒟i → M and differential 
operators 𝒥i: Γ(𝒟i−1) → Γ(𝒟i) satisfying 𝒥i ◦ 𝒥i−1 = 0, while implicitly setting 
𝒟−1 = 0 and 𝒟0 = 0. Given a differential complex, it is natural to define its 
cohomology vector spaces to be the cohomology of the cochain complex of global 
sections, 𝒥′(𝒟•, 𝒥•) = 𝒥′(Γ(𝒟•), 𝒥•), which we also refer to as the cohomology 
with unrestricted supports. Since differential operators do not increase supports,
we can equally consider the cohomology of the differential complex with compact 
supports, defined as 𝒥′′(𝒟•, 𝒥•) = 𝒥′′(Γc(𝒟•), 𝒥•). A differential complex naturally 
define a complex 0 → 𝒟0 → 𝒟1 → · · · of sheaves of sections of these bundles, 𝒟i(U) = Γ(𝒟i(U)). A differential complex is said to be locally exact if it 
defines an exact complex of sheaves. Local exactness is a very strong property 
that is crucial in the relation of the cohomology of a differential complex to 
sheaf cohomology, which we discuss next.

In general, given a complex of sheaves 𝒟i → 𝒟i+1, we call it an injective res-
olution of a sheaf 𝒟 if it is a resolution of 𝒟 (namely, 𝒟i = 0 for i < 0, it is exact except for ℋ′(𝒟•) = 𝒟), and each 𝒟i is injective. The injectivity 
condition is somewhat technical. The same can be said for the fact that every 
sheaf has an injective resolution. So we will not go into them here and defer 
to standard references instead [16, 45]. We will need these notions only for 
the following definition. The degree-i sheaf cohomology vector spaces 𝒥′(𝒟) = 
𝒥′(M, 𝒟), also called the degree-i cohomology of M with coefficients in 𝒟, as 
the cohomology vector space of the complex of global sections of any injective 
resolution 𝒟i → 𝒟i+1 of 𝒟, 𝒥′(𝒟) = 𝒥′(Γ(𝒟•)). It is important to note that 
sheaf cohomology is well defined. It does not depend on the chosen injective 
resolution, because the injectivity condition implies the existence of a homotopy

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equivalence between the complexes of global sections of any two such resolutions, thus forcing their cohomologies to be isomorphic. This is another technical fact that we shall not go into here.

Instead, we make note of yet another technical fact that provides a practical way to compute sheaf cohomology. For that, we need two more definitions. A sheaf $\mathcal{F}$ is called \textit{acyclic} if $H^i(\mathcal{F}) = 0$ for all $i > 0$, though as usual the degree-0 cohomology $H^0(\mathcal{F}) \cong \Gamma(\mathcal{F})$ is isomorphic to the vector space of global sections of $\mathcal{F}$. A sheaf $\mathcal{F}$ on $M$ is called \textit{soft} if for any closed $A \subseteq M$ the restriction maps $\mathcal{F}(M) \to \mathcal{F}(A)$ are surjective, where $\mathcal{F}(A) = \bigcap_{U \subseteq A} \mathcal{F}(U)$ with $U$ ranging over all open sets that contain the closed set $A$. In other words, given an open $U \subseteq M$ and a closed subset $A \subseteq U$, a local section on $U$ can always be extended to a global one on $M$ without modification on $A$, but possibly modified on $U \setminus A$. What is really important for us is the following

\textbf{Proposition 1.} (i) If $\mathcal{F}$ is a sheaf on $M$, and $\mathcal{F}_i \to \mathcal{F}_{i+1}$ is a resolution of $\mathcal{F}$ by acyclic sheaves (acyclic resolution), then $H^i(\mathcal{F}, \mathcal{F}) \cong H^i(\Gamma(M, \mathcal{F}))$. (ii) Any soft sheaf on $M$ is acyclic. (iii) Given a vector bundle $F \to M$, the sheaf $\mathcal{F}$ of sections of $F$ is soft.

\textit{Proof.} Any standard discussion of sheaf cohomology establishes (i) and (ii) \cite{16, 45}. On the other hand, (iii) is simply a restatement of the well known Whitney extension theorem for smooth functions \cite[Thm.2.3.6]{44}.

Note that the complex of sheaves corresponding to a differential complex then automatically consists of acyclic sheaves. The above proposition essentially tells us that, given a resolution of some sheaf $\mathcal{F}$ on a manifold $M$ by a locally exact differential complex $(\mathcal{F}_\bullet, f_\bullet)$, the sheaf cohomology of $\mathcal{F}$ and the cohomology of the differential complex will coincide, $H^i(\mathcal{F}) \cong H^i(\mathcal{F}_\bullet, f_\bullet)$. This observation will be particularly important later in Corollary 12.

Next, we discuss some conditions ensuring that the cohomologies of two given differential complexes are isomorphic. As we have now seen, local exactness is a very strong and useful property, unfortunately it can be difficult to check in practice. Two weaker notions of exactness exist that are easier to check in practice. To formulate them, we refer to the notions of \textit{jets} and \textit{jet bundles}, together with associated constructions like \textit{prolongations} and \textit{principal symbols}, all briefly recalled in Appendix C. Given a sequence of vector bundles $F_i$ and a complex of linear differential operators $f_i: F_{i-1} \to F_i$, each of order $k_i$, their prolongations define a complex of vector bundle morphisms,

$$\cdots \to j^l F_{i-1} \xrightarrow{p^l f_i} j^l F_i \xrightarrow{p^{l+1} f_{i+1}} j^{l+1} F_{i+1} \to \cdots ,$$  \hspace{1cm} (49)

with $l_i = l - k_i$ and $l_{i+1} = l - k_i - k_{i+1}$, for each sufficiently large $l$. The differential complex is said to be \textit{formally exact} if the above compositions are exact, as linear bundle maps over $M$, for any values of $l$ and $i$ for which they are defined. On the other hand, given $(x, p) \in T^* M$, the principal symbols of the differential operators $f_i$ define a complex of linear maps between the fibers of $F_i$ at $x$,

$$\cdots \to F_{i-1,x} \xrightarrow{\sigma_{x,p} f_i} F_{i,x} \xrightarrow{\sigma_{x,p} f_{i+1}} F_{i+1,x} \to \cdots .$$  \hspace{1cm} (50)

The differential complex is said to be \textit{elliptic} if the above complex is exact for every $(x, p) \in T^* M$, $p \neq 0$. These two weaker notions are distinct \cite{76}. Formal
exactness is a good hypothesis for showing that differential operators factor in certain ways. On the other hand, ellipticity is a condition that can be used to prove local exactness, via the method of parametrices and fundamental solutions. However, the general question of determining necessary and sufficient conditions for local exactness for differential complexes is a difficult and still open problem. The main conjecture is sometimes known as Spencer’s conjecture: a formally exact, elliptic complex is locally exact [79, 76, 75]. On the other hand, some supplementary sufficient conditions are known for an elliptic complex to be locally exact. A prominent condition of this kind is known as the δ-estimate [85, Sec.1.3.13], which first appeared in the works of Singer, Sweeney and MacKichan [79].

**Proposition 2.** The twisted de Rham complex associated to the flat bundle \((V, D)\) defined by a regular differential equation of finite type, defined in Equation (45), is formally exact, elliptic and locally exact.

**Proof.** As noted in Remark 2, the twisted de Rham complex is locally (on sufficiently small contractible open sets) equivalent to \(rk V\) copies of the ordinary de Rham complex. To see the equivalence, it suffices to locally choose a \(D\)-flat basis frame for \(V\). Since all of the desired properties, formal exactness, ellipticity and local exactness are purely local, it suffices to check them for the ordinary de Rham complex. It is well known that each of these properties does hold for the de Rham complex, having served as a model example for each. Formal exactness and ellipticity are discussed, for instance, in [79, 63, 85] and [43, §XIX.4]. On the other hand, local exactness is essentially the content of the Poincaré lemma [15].

There is another way to establish local exactness that bypasses the Poincaré lemma and does not require an explicit local choice of a \(D\)-flat basis frame for \(V\). In particular, as discussed for instance in the given references, local exactness and ellipticity are independent of such a choice. Then, local exactness follows provided the initial operator of the complex, the connection operator \(D: \Gamma(V) \to \Gamma(T^*M \otimes V)\), satisfies the δ-estimate. According to Example 1.3.58 of [85], any linear connection operator satisfies the δ-estimate. Hence, by Theorem 1.3.61 of [85], the twisted de Rham complex is locally exact.

As is well known in homological algebra, cochain maps and homotopies between them are important concepts, the first because they descend to cohomology, the second because equivalence up to homotopy descends to isomorphism on cohomology. When dealing with differential complexes, it becomes important to distinguish the case where the cochain maps and homotopies are defined by differential operators. The most important notion we will need is that of a *formal homotopy equivalence*. Let \((F\bullet, f\bullet)\) and \((G\bullet, g\bullet)\) be two differential complexes. They are said to be *formally homotopy equivalent* provided there exist differential operators \(e_i, h_i, u_i\) and \(v_i\) fitting into the diagram (we use the bundles to stand in for their spaces of sections)

\[
\begin{array}{c}
\cdots \longrightarrow F_{i-1} \xrightarrow{e_{i-1}} F_i \xrightarrow{f_i} F_{i+1} \xrightarrow{e_{i+1}} \cdots \\
\downarrow v_{i-1} & \downarrow u_i & \downarrow v_{i+1} \\
\cdots \longrightarrow G_{i-1} \xrightarrow{g_{i-1}} G_i \xrightarrow{h_i} G_{i+1} \xrightarrow{g_{i+1}} \cdots 
\end{array}
\]  

(51)
where the squares composed of solid arrows commute (cochain map condition on \( u_i \) and \( v_i \)) and the dashed arrows are homotopy operators with respect to which \( u_i \) and \( v_i \) are quasi-inverses, \( v_i \circ u_i - \text{id} = e_{i+1} \circ f_{i+1} + f_i \circ e_i \) and \( u_i \circ v_i - \text{id} = h_{i+1} \circ g_{i+1} + g_i \circ h_i \).

**Lemma 3.** Consider two differential complexes \((F_*, f_*)\) and \((G_*, g_*)\) that start in degree 0, also denote the corresponding complexes of sheaves of sections as \(F_i \to F_{i+1}\) and \(G_i \to G_{i+1}\). Suppose that both differential complexes are formally exact, except in degree 0. Further, suppose that the equations \( f_1[\phi] = 0 \) and \( g_1[\gamma] = 0 \), with \( \phi \in \Gamma(F_0) \) and \( \gamma \in \Gamma(G_0) \), are equivalent, or in other words the degree-0 cohomology sheaves are isomorphic to some given sheaf \( F \cong H^0(F_*) \cong H^0(G_*) \).

(i) Then there exists a formal homotopy equivalence between these differential complexes and their cohomologies are isomorphic, both with unrestricted and compact supports (or any other kind of restriction on supports):

\[
H^i(F_*, f_*) \cong H^i(G_*, g_*) \quad \text{and} \quad H^0_i(F_*, f_*) \cong H^0_i(G_*, g_*). \tag{52}
\]

(ii) If one of the differential complexes is locally exact, then both are locally exact and their cohomologies both compute the sheaf cohomology of \( F \):

\[
H^i(M, F) \cong H^i(F_*, f_*) \cong H^i(G_*, g_*). \tag{53}
\]

**Proof.** (i) Equivalence of the equations \( f_1[\phi] = 0 \) and \( g_1[\gamma] = 0 \) means (Appendix C) that there exist differential operators, say \( u_0 : \Gamma(F_0) \to \Gamma(G_0) \) and \( v_0 : \Gamma(G_0) \to \Gamma(F_0) \), such that \( u_0 \circ v_0[\phi] = 0 \) whenever \( f_1[\phi] = 0 \) and such that \( v_0 \circ u_0[\gamma] = 0 \) whenever \( g_1[\gamma] = 0 \). In other words, there exist differential operators \( \epsilon_i : \Gamma(F_i) \to \Gamma(G_i) \) and \( h_i : \Gamma(G_i) \to \Gamma(F_i) \) such that \( v_0 \circ u_0 = \epsilon_1 \circ f_1 \) and \( u_0 \circ v_0 = h_1 \circ g_1 \). These differential operators are the initial step in establishing the desired formal homotopy equivalence.

We proceed by a standard induction argument from homological algebra (in fact, a version of this argument proves the independence of sheaf cohomology from the injective resolution used to compute it). Assume that all the desired differential operators have been defined up to \( \epsilon_i, h_i, u_{i-1} \) and \( v_{i-1} \), which also satisfy the desired identities. We can easily verify the identities

\[
(g_i \circ u_{i-1}) \circ f_{i-1} = (g_i \circ g_{i-1}) \circ u_{i-2} = 0, \tag{54}
\]

\[
(f_i \circ v_{i-1}) \circ g_{i-1} = (f_i \circ f_{i-1}) \circ v_{i-2} = 0, \tag{55}
\]

which together with the formal exactness of the compositions \( f_i \circ f_{i-1} = 0 \) and \( g_i \circ g_{i-1} = 0 \) imply the factorizations \( g_i \circ u_{i-1} = u_i \circ f_i \) and \( f_i \circ v_{i-1} = v_i \circ g_i \), for some differential operators \( u_i : \Gamma(F_i) \to \Gamma(G_i) \) and \( v_i : \Gamma(G_i) \to \Gamma(F_i) \) (see Appendix C). Further, we can also verify the identities

\[
(v_i \circ u_i - \text{id} - f_i \circ e_i) \circ f_i = (v_i \circ g_i) \circ u_{i-1} - f_i - f_i \circ e_i \circ f_i = (v_i \circ (v_{i-1} \circ u_{i-1}) - f_i - f_i \circ e_i \circ f_i = 0, \tag{56}
\]

\[
(v_i \circ v_i - \text{id} - g_i \circ h_i) \circ g_i = (u_i \circ f_i) \circ v_{i-1} - g_i - g_i \circ h_i \circ g_i = g_i \circ (u_{i-1} \circ v_{i-1}) - g_i - g_i \circ h_i \circ g_i = 0, \tag{57}
\]

\[
(u_i \circ u_i - \text{id} - f_i \circ e_i \circ f_i \circ e_i \circ f_i = (v_i \circ f_i) \circ v_{i-1} - u_i \circ id - g_i \circ h_i = h_{i+1} \circ g_{i+1}, \quad \text{for some differential}
\]

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operators $e_{i+1} : \Gamma(F_{i+1}) \to \Gamma(F_i)$ and $h_{i+1} : \Gamma(G_{i+1}) \to \Gamma(G_i)$. This concludes the inductive step.

Now, let us consider the cohomology of these complexes, $H^i(F_s, f_s) = H^i(\Gamma(F_s), f_s)$ and $H^i(G_s, g_s) = H^i(\Gamma(G_s), g_s)$. As is well known from homological algebra, a homotopy equivalence (of which a formal homotopy equivalence is a special kind) induces an isomorphism in cohomology: $H^i(F_s, f_s) \cong H^i(G_s, g_s)$. However, if the operators implementing the homotopy equivalence are differential operators, as in this case, we can replace unrestricted sections $\Gamma(\cdot)$ by sections with compact supports $\Gamma_c(\cdot)$, so that $H^i_c(F_s, f_s) = H^i(\Gamma_c(F_s), f_s)$ and $H^i_c(G_s, g_s) = H^i(\Gamma_c(G_s), g_s)$. The homotopy equivalence of the resulting complexes still holds because differential operators do not increase supports, and so we still have an isomorphism in cohomology: $H^i_c(F_s, f_s) \cong H^i_c(G_s, g_s)$. Incidentally, instead of compact supports, any other family of supports would do as well.

(ii) By the local exactness hypothesis, both differential complexes provide resolutions of the sheaf $F$ (which happens to be isomorphic to the solution sheaves $S_{F_i} = H^0(F_s)$ and $S_{G_i} = H^0(G_s)$). Then, by Proposition 1, these resolutions are acyclic and hence the corresponding cohomologies with unrestricted supports compute the sheaf cohomology of $F$. This concludes the proof.

\[\square\]

3.3 Generalized Poincaré duality

In Section 3.2, we discussed how the cohomology $H^i(F_s, f_s)$ of a differential complex can, under optimal conditions, be equated with the cohomology $H^i(F)$ of the sheaf resolved by $(F_s, f_s)$. However, even under optimal conditions, this connection breaks down if we consider cohomology $H^i_c(F_s, f_s)$ with compact (or some other family of) supports instead of unrestricted ones. What we discuss below is a way to relate cohomology with compact supports to that with unrestricted supports, a kind of Poincaré duality.

For the de Rham complex on a manifold $M$, $\dim M = n$, a well known formulation of Poincaré duality is the isomorphism $H^p(M) \cong H^{n-p}_c(M)$ [15, Rmk.5.7] between the linear dual of cohomology in degree-$p$ and compactly supported cohomology in degree-$(n-p)$. This isomorphism is induced by the existence of a non-degenerate natural pairing between $p$-forms and $(n-p)$-forms on $M$ and its non-degenerate descent to cohomology. The goal of this section is to leverage the properties of the Calabi complex and its formal adjoint complex that were discussed in the preceding section to demonstrate a generalized version of Poincaré duality, which effectively computes the cohomology with compact supports in terms of sheaf cohomology.

There are two ways to establish generalized Poincaré duality for a differential complex $(F_s, f_s)$ that would be applicable to the case of the Calabi complex and its formal adjoint. One of them, discussed in Section 3.3.1, relies on the fact that the corresponding complex of sheaves resolves the sheaf of solutions of a regular differential equation of finite type (a locally constant sheaf). This method is somewhat more elementary. The other, discussed in Section 3.3.2, works for any elliptic complex, but requires some results from functional analysis and distribution theory. Either of these results, as will be shown in Section 3.4, can be applied to prove generalized Poincaré duality for the Calabi complex and its formal adjoint complex.
3.3.1 Twisted de Rham complex

First, we will discuss the twisted de Rham complex, as introduced in Section 2.4. The results will then apply to the Calabi complex and its formal adjoint by virtue of Lemma 3. The strategy is straightforward and reproduces the logic of the proofs of the ordinary Poincaré duality, cf. [15, §5], [80, Ch.11], or [40, Sec.V.4]. First, generalized Poincaré duality is shown to hold on contractible open patches. Then, given a “good cover” of the manifold consisting of such patches, we use a version of the Mayer-Vietoris exact sequence as an inductive step to conclude that generalized Poincaré duality also holds on the entire manifold.

First, recall that we denote the fiber of the vector bundle $V \to M$ by $\bar{V}$. Then, $\bar{V}^*$ is the fiber of the dual vector bundle $V^* \to M$. We are interested in the relation between the cohomology of the twisted de Rham complex $H^i(\Lambda^p M \otimes V, D)$ and the compactly supported cohomology of the formal adjoint complex, which happens to be $(\Lambda^p M \otimes V^*, D)$, where the connection $D$ has been extended to $V^* \to M$ by the rule $d(\xi \cdot \psi) = (D\xi) \cdot \psi + \xi \cdot (D\psi)$, with $\xi \in \Gamma(V^*)$ and $\psi \in \Gamma(V)$. Presuming that $M$ is oriented, which is a prerequisite for integrating top-degree forms, there is a duality pairing between elements of $\Gamma(\Lambda^p M \otimes V)$ and $\Gamma_c(\Lambda^n - p M \otimes V^*)$ given by the formula

$$\langle \xi, \psi \rangle = \int_M \langle \xi \wedge \psi \rangle,$$

(60)

where $\langle (\alpha \otimes \xi) \wedge (\beta \otimes \psi) \rangle = (\alpha \wedge \beta) \otimes \langle \xi \cdot \psi \rangle$. The formal adjoint relation is established (up to signs) for $\xi \in \Gamma(\Lambda^{n-p-1} M \otimes V^*)$ and $\psi \in \Gamma(\Lambda^p M \otimes V)$ by the identity

$$d(\xi \wedge \psi) = \langle (D\xi) \wedge \psi \rangle - (-1)^{n-p} \langle \xi \wedge (D\psi) \rangle.$$

(61)

**Lemma 4.** Let $U \subseteq M$ be an oriented contractible open set. Then, generalized Poincaré duality holds, $H^p(\Lambda^p M \otimes V|_U, D) \cong H^{n-p}_c(\Lambda^p M \otimes V^*|_U, D)^*$, because all of the cohomology spaces vanish except $H^0(\Lambda^p M \otimes V, D) \cong \bar{V}$ and $H^0_c(\Lambda^p M \otimes V^*, D) \cong \bar{V}^*$.

**Proof.** As we have already noted in the proof of Proposition 2, a choice of a locally $D$-flat basis frame for $V$ over $U \subseteq M$ identifies the twisted de Rham complex with $\text{rk} V$ copies of the usual de Rham complex. Since $U$ is contractible, such a choice is always possible. Moreover, the pairing (60) reduces to the usual pairing between forms and compactly supported forms of complementary degrees on an oriented manifold. Thus, we can easily conclude that

$$H^p(\Lambda^p M \otimes V|_U, D) = H^p(U) \otimes \bar{V},$$

(62)

$$H^{n-p}_c(\Lambda^p M \otimes V^*|_U, D) = H^{n-p}_c(U) \otimes \bar{V}^*.$$  

(63)

Recalling that, for contractible $U$, $H^0(U) = 0$ except for $H^0(U) = \mathbb{R}$ and $H^{n-p}_c(U) = 0$ except for $H^{n-p}_c(U) = \mathbb{R}$, concludes the proof. □

**Lemma 5** (Mayer-Vietoris). Consider two open subsets $U, W \subseteq M$. We have the following long exact sequences in cohomology with unrestricted and compact supports, which we shall for brevity denote as $H^i(\bar{\cdot}) = H^i(\Lambda^p M \otimes V|_\cdot, D)$ and
Thus the initial step of the inductive argument.

Proof. Both long exact sequences in cohomology follow from short exact sequences of cochain complexes. These short exact sequences, where for brevity we write \( \Gamma^i(-) = \Gamma^i(\Lambda^*M \otimes V^*|\cdot, D) \) and \( \Gamma^i = \Gamma^i(\Lambda^*M \otimes V^*|\cdot) \) are

\[
0 \longrightarrow \Gamma^i(U \cup W) \longrightarrow \Gamma^i(U) \oplus \Gamma^i(W) \longrightarrow \Gamma^i(U \cap W) \longrightarrow 0, \tag{66}
\]

\[
0 \longrightarrow \Gamma^i(U \cap W) \longrightarrow \Gamma^i(U) \oplus \Gamma^i(W) \longrightarrow \Gamma^i(U \cup W) \longrightarrow 0. \tag{67}
\]

In the first sequence, the maps are restrictions, \( \alpha \mapsto (\alpha|_U, \alpha|_W) \) and \( (\alpha, \beta) \mapsto (\alpha|_{U \cap W} - \beta|_{U \cap W}) \). The exactness follows from the usual ability to restrict and glue together smooth sections over open regions, also known as their sheaf property. In the second sequence, the maps are extensions by zero, \( \alpha \mapsto (\alpha|_U, \alpha|_W) \) and \( (\alpha, \beta) \mapsto \alpha|_{U \cup W} - \beta|_{U \cup W} \). The exactness follows from the existence of a smooth partition of unity adapted to the cover of \( U \cup W \) by \( U \) and \( W \).

These maps are clearly compatible with the connection differential operator \( D \) and so are cochain maps. The general connection between short exact sequences of cochain complexes and long exact sequences in cohomology (Appendix B) gives the desired long exact sequences and concludes the proof. \( \square \)

**Proposition 6.** Given a flat vector bundle \((V, D)\) on an oriented \( n \)-dimensional orientable manifold \( M \), the unrestricted cohomology \( H^p = H^p(\Lambda^\bullet M \otimes V, D) \) of the associated twisted de Rham complex and the compactly supported cohomology \( H^p_{c} = H^p_{c}(\Lambda^\bullet M \otimes V^*, D) \) of its formal adjoint complex satisfy generalized Poincaré duality:

\[
H^p \cong (H^p_{c})^*. \tag{68}
\]

Note the asymmetry of the isomorphism. The reverse identity \((H^p)^* \cong H^p_{c}\) also holds when the cohomology vector spaces are finite dimensional, but in general may not when they are infinite dimensional.

**Proof.** In this proof, we shall use induction over a special kind of open cover of \( M \). An open cover \((U_k)\) of \( M \) is called good if it is locally finite, every nonempty finite intersection \( U_{k_1} \cap \cdots \cap U_{k_m} \) is diffeomorphic to \( \mathbb{R}^n \), and it is closed under finite intersections. In particular, each of the \( U_k \) is itself diffeomorphic to \( \mathbb{R}^n \) and thus contractible. Good covers are known to exist for any manifold [15, Thm.5.1]. Inducing an orientation on each element of the cover from the orientation on \( M \), Lemma 4 establishes the desired duality relation for any \( U_k \) and thus the initial step of the inductive argument.
Next, we show, provided the desired duality relation holds on any finite union \( U_k \cup \cdots \cup U_{k_m-1} \) of \( m \) sets, that it also holds on any finite union \( U_k \cup \cdots \cup U_{k_m-1} \cup \cdots \cup U_{k_m-1} \) of \( m + 1 \) sets as well. Of course, we take all such unions to be oriented in a way compatible with the global orientation on \( M \). Let \( U = U_{k_m}, W = U_k \cup \cdots \cup U_{k_m-1} \) and notice that both \( W \) and \( W \cap U \) are finite unions of \( m \) sets from the cover (recall that the cover is closed under intersections). The fact that the pairing (60), well defined on a given oriented, open \( U \subseteq M \), descends to cohomology means that we always have a mapping \( H^p(U) \to H^{n-p}(U)^* \), which may or may not be an isomorphism. It is in fact an isomorphism on \( U \) and, by the inductive hypothesis, also on \( W \) and \( W \cap U \). Combining the long exact sequences of Lemma 5 for \( W \) and \( U \) together with these maps and isomorphisms, we obtain the following diagram (notice the arrow reversal by linear duality in the second row):

\[
\begin{array}{cccccc}
H^p(W \cup U) & \to & H^p(W) \oplus H^p(U) & \to & H^p(U \cap U) & \to & H^{p+1}(W \cup U) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{n-p}(W \cup U)^* & \to & H^{n-p}(W)^* \oplus H^{n-p}(U)^* & \to & H^{n-p}(U \cap U)^* & \to & H^{n-p+1}(W \cup U)^* \\
\end{array}
\]

Thus, by the 5-lemma (Appendix B), the map in the center of the diagram is also an isomorphism and the inductive step is established.

The only problem remaining is that a good cover is not always finite (though it can be chosen to be finite for compact manifolds). There is a way around that, however. Using a similar argument, one can show that the desired duality holds also on disjoint countable unions of finite unions of covering sets. It is at this stage that the asymmetry between the cohomologies with unrestricted and compact supports appears. Then, provided the manifold is second countable, one can choose a much coarser, yet finite, cover \( (U_k') \). The key property of this cover is that each of the non-empty finite intersections \( U_k' \cap \cdots \cap U_m' \) is itself either a finite union of sets from \( (U_k) \) or a disjoint countable union of those. The same 5-lemma argument then shows that the desired generalized Poincaré duality relation \( H^p \cong (H^{n-p})^* \) holds on all of \( M \). The technical details of this argument can be found in [40, Sec.V.4].

3.3.2 Elliptic complexes and Serre duality

Now we will discuss generic elliptic complexes, of which both the Calabi and the twisted de Rham complexes are special cases. The result is essentially the same, though clearly more general. The arguments are somewhat less elementary and rely on some background in functional analysis and an result originally due to Serre [74]. The Serre duality method also gives some more information. Namely, that the cohomology does not change if we replace smooth functions by distributions with the same supports. Serre’s original work was in the context of the Dolbeault complex in the theory of several complex variables. A good exposition of this result in the setting of general elliptic complexes can be found in [85].

At this point it is convenient to recall some basic facts of distribution theory [72, 86, 69]. Recall that, for any vector bundle \( F \to M \), we can interpret \( \Gamma(F) \) and \( \Gamma_c(F) \) as locally convex topological vector spaces, with the Whitney weak Fréchet topology for the former and an inductive limit over supports of similar Fréchet topologies for the latter, with the limit topology still locally
convex but no longer Fréchet (metrizable). These are the usual topologies used in the theory of distributions. The spaces of distributional sections \( \Gamma'(F) \) and \( \Gamma'_c(F) \) of \( F \), with respectively compact and unrestricted supports, are defined as topological duals endowed with the strong topology (the usual distributional topology), \( \Gamma'(F) = \Gamma(\tilde{F}^*)^* \) and \( \Gamma'_c(F) = \Gamma_c(\tilde{F}^*)^* \). Recall that \( \tilde{F}^* = \Lambda^n M \otimes F^* \) is the densitized dual bundle; the densitized dual of the densitized dual is the original bundle. It so happens that, if we stick with the strong topology for dual spaces, the topological dual of \( \Gamma'(F) \) is again \( \Gamma(\tilde{F}^*) \) and that of \( \Gamma'_c(F) \) is \( \Gamma_c(\tilde{F}^*) \). So the spaces of smooth and distributional sections are reflexive (with respect to the strong topology). Using the natural pairing

\[
\langle \psi, \alpha \rangle = \int_M \psi \cdot \alpha
\]  

(69)

between \( \psi \in \Gamma(F) \) and \( \alpha \in \Gamma_c(\tilde{F}^*) \), well-defined provided \( M \) is oriented, we have the natural inclusions \( \Gamma(F) \subset \Gamma'_c(F) \) and \( \Gamma'_c(F) \subset \Gamma'(F) \). By the Schwartz kernel theorem, the continuous maps \( G: \Gamma_c(F_1) \to \Gamma'_c(F) \) are in bijection with bidistributions, elements \( G \in \Gamma'_c(F_2 \boxtimes \tilde{F}^*_2) \), where \( F_2 \boxtimes \tilde{F}^*_2 \to M \times M \) is the bundle with total space \( F_2 \times \tilde{F}^*_2 \) and the obvious projection onto its base, by the formula

\[
(G\psi)(x) = \int_M G(x,y) \cdot \psi(y).
\]  

(70)

Let \( \pi_1(x,y) = y \) and \( \pi_2(x,y) = x \) denote the two projections \( M \times M \to M \). We say that a bidistribution \( G \in \Gamma'_c(F_2 \boxtimes \tilde{F}^*_2) \) is properly supported if \( \pi_1: \text{supp} G \to M \) is a proper map (the preimage of a compact set is compact). Differential operators define properly supported bidistributions, because their support lies on the diagonal of \( M \times M \) by the crucial property that differential operators preserve supports. On the other hand, properly supported bidistributions need not preserve supports, though they still map compactly supported sections to compactly supported distributions. The amount by which the support of the image grows depends on the size of the support of the bidistribution in \( M \times M \).

Once we have introduced distributional sections, we can extend to them many operators that were previously defined only on smooth functions. For instance, any linear differential operator \( f: \Gamma'(F) \to \Gamma'(E) \) between vector bundles \( F \to M \) and \( E \to M \) can be extended to act on distributions, \( f: \Gamma'(F) \to \Gamma'(E) \) or even \( f: \Gamma'_c(F) \to \Gamma'_c(E) \), according to the following formula:

\[
[f(\alpha), \psi] = -\langle \alpha, f^*[\psi] \rangle,
\]  

(71)

for any \( \psi \in \Gamma_c(\tilde{F}^*) \) and \( \alpha \in \Gamma'_c(F) \), where \( f^* \) is the formal adjoint of \( f \) and \( \langle -, - \rangle \) is the natural dual pairing between sections and distributions. Since this natural pairing is non-degenerate, it suffices to define \( f \) on the larger domain. Any other operator defined on smooth sections for which the above formula applies can also be extended to distributions, possibly with a restriction on their supports.

In particular, the operators of a differential complex \((\tilde{F}_\bullet, f_\bullet)\) can be extended to distributional sections. Then we can consider the cohomology of the complex in distributional sections, \( H^i(\Gamma'_c(\tilde{F}_\bullet), f_\bullet) \), which may a priori be different from its cohomology in smooth sections \( H^i(\tilde{F}_\bullet, f_\bullet) = H^i(\Gamma(\tilde{F}_\bullet), f_\bullet) \), and similarly with compact supports. Below we shall see some sufficient conditions for the cohomologies in smooth and distributional sections to coincide.
A crucial concept in the general theory of differential complexes is that of a \textit{parametrix} \cite[Ch.2]{85}. Let the vector bundles $F_i$ with differential operators $f_i: \Gamma(F_{i-1}) \to \Gamma(F_i)$ constitute a differential complex $(F_\bullet, f_\bullet)$ on $M$. Then, a \textit{parametrix} is a sequence of bidistributions $G_i \in \Gamma'_c(F_{i-1} \otimes \bar{F}_i)$ such that
\begin{equation}
\text{id}_i - Q_i = G_{i+1} \circ f_{i+1} + f_i \circ G_i,
\end{equation}
where $\text{id}_i: \Gamma_i(F_i) \to \Gamma_i(F_i)$ is the identity map and $Q_i \in \Gamma(F_{i+1} \otimes \bar{F}_i^*) \subset \Gamma'_c(F_{i+1} \otimes \bar{F}_i^*)$ is a smooth bidistribution. We say that the parametrix is \textit{properly supported} if each $G_i$ is a properly supported bidistribution. Obviously, if each $G_i$ is properly supported, then so is each $Q_i$.

**Proposition 7.** Let $(F_\bullet, f_\bullet)$ be an elliptic complex on an oriented manifold $M$. (i) Then, for any open neighborhood $U \subseteq M \times M$ of the diagonal $M \subset M \times M$, there exists a properly supported parametrix $G_i \in \Gamma'_c(F_{i-1} \otimes \bar{F}_i^*)$ with support $\text{supp} \, G_i \subseteq U$. (ii) Then also, the cohomologies of smooth and distributional sections are isomorphic:
\begin{equation}
H^i(\Gamma'_c(F_\bullet), f_\bullet) \cong H^i(F_\bullet, f_\bullet) \quad \text{and} \quad H^i(\Gamma(F_\bullet), f_\bullet) \cong H^i(F_\bullet, f_\bullet).
\end{equation}

\textbf{Proof.} (i) The existence of a parametrix for any elliptic complex follows from Corollary 2.1.11 and Theorem 2.1.12 of \cite{85}. The support of an existing parametrix can be restricted arbitrarily close to the diagonal since $G_i^\chi[\psi] = \chi G_i[\psi]$, is a parametrix as long as $G_i$ is a parametrix and $\chi \in C^\infty(M \times M)$ is properly supported with $\chi \equiv 1$ on a neighborhood of the diagonal.

(ii) By the defining Equation (72), they are cochain homotopic to the identity operator, with respect to the cochain homotopy $G_i$. Further, being by hypothesis smooth and by (i) properly supported, they define smoothing operators, $Q_i: \Gamma'_c(F_i) \to \Gamma_i(F_i)$ and $Q_i: \Gamma'_c(F_i) \to \Gamma(F_i)$, when extended to distributions. It is then straightforward to see that the $Q_i$ and the inclusions of smooth sections in distributional ones (well defined because $M$ is oriented) constitute a homotopy equivalence between the complexes of smooth $(\Gamma(F_\bullet), f_\bullet)$ and distributional $(\Gamma'_c(F_\bullet), f_\bullet)$ sections, and similarly for compact supports. Thus, as desired, these complexes have isomorphic cohomologies. \hfill \Box

**Proposition 8** (Serre, Tarkhanov). Given a differential complex $(F_\bullet, f_\bullet)$, that is not necessarily elliptic, on an oriented manifold $M$ that is countable at infinity (there exists an exhaustion by a countable sequence of compact sets), let $(\bar{F}_\bullet, \bar{f}_\bullet)$ be its formal adjoint complex. The following are algebraic (the topologies may not agree) isomorphisms of vector spaces
\begin{align}
H^i(F_\bullet, f_\bullet)^* &\cong H^i(\Gamma'(\bar{F}_\bullet), \bar{f}_\bullet)^*, \quad H^i(F_\bullet, f_\bullet) &\cong H^i(\Gamma'(\bar{F}_\bullet), \bar{f}_\bullet), \quad (74) \\
H^i_c(F_\bullet, f_\bullet)^* &\cong H^i(\Gamma'_c(\bar{F}_\bullet), \bar{f}_\bullet)^*, \quad H^i_c(F_\bullet, f_\bullet) &\cong H^i(\Gamma'_c(\bar{F}_\bullet), \bar{f}_\bullet)^*, \quad (75)
\end{align}
where the cohomology vector spaces are endowed with the natural Hausdorff locally convex topology of a quotient of a subspace of the corresponding space of sections (be it smooth or distributional) and the topological duals are taken with the strong topology.

\textbf{Proof.} The original result of Serre \cite{74} appeared in the context of the Dolbeault differential complex in the theory of several complex variables. A detailed discussion and proof of the result for general differential complexes can be found
in Sections 5.1.1 and 5.1.2 of [85]. In particular, the desired conclusion can be found in Remark 5.1.9 thereof. Further conditions under which some of the duality isomorphisms are also continuous, and not merely algebraic, can be found there as well.

Combining the two preceding propositions, it is easy to see that for any elliptic complex (subject to a countability condition on \(M\)) we have the Poincaré-Serre duality relation \(H^i(F_\bullet, f_\bullet) = H^i_\text{c}(\tilde{F}_\bullet, f_\bullet)\).

### 3.4 The Calabi cohomology and homology

Below, we finally make use of the background information summarized in Sections 3.1, 3.2, and 3.3 and its consequences for the Calabi and its formal adjoint complexes, \((C_\bullet, B_\bullet)\) and \((C_\bullet, B^*_\bullet)\), which were introduced in 2. Namely, we make precise the identification between their cohomologies and the sheaf cohomologies of the Killing and Killing-Yano sheaves, \(\mathcal{K}\) and \(\mathcal{K}_\text{Y}\), introduced in Section 3.1. The hope created by this identification is that the difficult problem of solving systems of differential equations, which appear in these complexes, can be replaced by the equivalent and potentially easier problem of computing sheaf cohomologies. The latter problem is potentially easier because of the many available methods of computing sheaf cohomology. Some of which will be discussed in Section 4.

First, we introduce the basic definitions of Calabi cohomology and homology. Let us denote the cohomology of the Calabi complex (Calabi cohomology) on a pseudo-Riemannian manifold \((M, g)\) of constant curvature as

\[
HC^i(M, g) = H^i(C_\bullet, B_\bullet) = H^i(\Gamma(C_\bullet), B_\bullet). \tag{76}
\]

Let us also denote the cohomology of the formal adjoint Calabi complex with compact supports (Calabi homology)

\[
HC_i(M, g) = H^i_c(C_\bullet, B^*_\bullet) = H^i(\Gamma_c(C_\bullet), B^*_\bullet). \tag{77}
\]

The naming convention will be justified later by the generalized Poincaré duality relation in Corollary 11. Similarly, we define the cohomology of the Calabi complex with compact supports (Calabi cohomology with compact supports) as

\[
HC^i_c(M, g) = H^i_c(C_\bullet, B_\bullet) = H^i(\Gamma_c(C_\bullet), B_\bullet), \tag{78}
\]

and the cohomology of the formal adjoint Calabi complex (locally finite Calabi homology) as

\[
HC^\text{lf}_i(M, g) = H^i(\Gamma_c(C_\bullet), B^*_\bullet) \tag{79}
\]

The following proposition is the main technical tool that we use to establish all other results in this section.

**Proposition 9.** Consider a pseudo-Riemannian manifold \((M, g)\) of constant curvature and dimension \(n\). The corresponding Calabi complex \((C_\bullet, B_\bullet)\) is elliptic, formally exact and locally exact (except in degree 0). The same is true for its formal adjoint complex \((C_\bullet, B^*_\bullet)\) (except in degree \(n\)).
Proof. In principle, we would need quite a bit of machinery for a full proof. Instead, we give a sketch of the main ideas and refer to the literature for technical details. The Calabi complex is actually an instance of a second Spencer sequence construction \cite{68, 38, 79, 63} applied to the Killing operator $B_1 = K$. This fact is the demonstrated in the papers \cite{34, 35, 39}. These papers make use of the general construction and properties of the differential complex constituting a second Spencer sequence demonstrated in \cite{68, 38}. In fact, the resulting differential complex gives a formally exact compatibility complex for the Killing operator, which is also an elliptic complex. This holds since the Killing operator $K$ is itself elliptic (has injective symbol, which follows from the property of being of finite type, cf. Section 2.4) and formally integrable (contains all of its integrability conditions) on a constant curvature background.

A more elementary argument for ellipticity can be made on representation theoretic grounds (Appendix A.1). The fibers of the tensor bundles $C_i M$ carry irreducible representations of $GL(n)$. Further, as mentioned in Remark 1, the principal symbols of the differential operators $B_i$ are all $GL(n)$-equivariant maps $\sigma B_i : Y^{(k_i)} T^* \otimes C_{i-1} \to C_i$ or equivalently $\sigma_p B_i : C_{i-1} \to C_i$, for $p \in T^*$. By Schur’s lemma, the symbol map $\sigma B_i$ is then an isomorphism when restricted to an irreducible summand of the tensor product representation. The well-known Littlewood-Richardson rules \cite{33, 59} for tensor products of $GL(n)$ representations then show that the $C_i$ irreps have been chosen precisely such that the symbol sequence $\sigma p B_i$ is exact for $p \neq 0$. This representation-theoretic line of argument is a special case of the construction of what are known as BGG resolutions \cite{13}.

Finally, local exactness (except in degree 0) can be established by checking, for the Killing operator, a sufficient condition known as the \(\delta\)-estimate \cite[Sec.1.3.13]{85}. Equivalently, we can simply invoke Proposition 2, since, being of finite type, the Killing operator is equivalent to a flat covariant operator (Section 2.4).

A more elementary proof of local exactness was given in the original article by Calabi \cite{19}. He relied on the well known local exactness of the de Rham complex and its relation to the simplified form of the complex in the flat (zero curvature) case. The non-zero curvature case was handled by embedding it in a flat space and then restricting and extending the relevant sheaves with respect to this embedding. Unfortunately, unlike the more sophisticated argument above, this simpler argument is unlikely to generalize, when the Calabi complex is replaced by a more general one.

To finish the proof, we note that the properties of formal exactness and ellipticity are obviously preserved by taking formal adjoints, so that they apply equally well to the formal adjoint Calabi complex $(C^*_\bullet, B^*_\bullet)$. The formal adjoint complex then serves as the formally exact compatibility complex for the Killing-Yano operator $B^*_n = KY$, which is also regular and of finite type on constant curvature backgrounds, as discussed in Section 2.4. Thus, repeating the same arguments as above establishes local exactness (except this time in degree $n$) for the adjoint complex as well.

\textbf{Corollary 10.} There is a formal homotopy equivalence between the Calabi complex $(C^*_\bullet, B^*_\bullet)$ and the twisted de Rham complex $(\Lambda^* M \otimes V_K, D_K)$ resolving the Killing sheaf, $H^0(C^*_\bullet, B^*_\bullet) = K$. The same is true (up to a trivial renumbering) of the formal adjoint complex and the twisted de Rham complex $(\Lambda^* M \otimes V_{KY}, D_{KY})$. \hfill \qed
resolving the Killing-Yano sheaf, $H^n(C_\bullet, B_\bullet^*) = KY$.

Proof. We already know that both the Calabi and twisted de Rham complex associated to the Killing operator are formally exact, locally exact (Propositions 9 and 2) and both resolve the Killing sheaf, since the operators $K$ and $D_K$ are equivalent (Section 2.4). Thus, by Lemma 3, there exists a formal homotopy equivalence (realized by differential operators) between the two complexes. Noting that the exact same argument (with trivial changes) applies to the formal adjoint Calabi complex and the Killing-Yano sheaf concludes the proof. □

Corollary 11. Provided the manifold $M$ is countable at infinity (there is an exhaustion by a countable sequence of compact sets) or is of finite type (has a finite “good cover”), we have the following generalized Poincaré duality isomorphisms

$$HC_i(M, g) \cong HC_i(M, g)^*, \quad HC_c^i(M, g) \cong HC_c^{i\dagger}(M, g),$$

$$HC^i(M, g)^* \cong HC_i(M, g), \quad HC_c^i(M, g) \cong HC_c^{i\dagger}(M, g)^*,$$

where isomorphisms are taken in the algebraic sense and duality is meant in the topological sense, as described in Proposition 8.

Note that in the case when all cohomology vector spaces are finite dimensional, the distinction between algebraic or topological isomorphisms and duals is irrelevant.

Proof. There are two ways to establish the desired duality isomorphisms, each relying on slightly different conditions on $M$, reflected in the hypotheses. We should note that both require an orientation on $M$. The existence of a non-degenerate metric on $M$ implies that it is orientable. We then simply fix an orientation arbitrarily.

The Mayer-Vietoris argument (Proposition 6) establishes the duality isomorphisms

$$H^i(\Lambda^\bullet M \otimes V_K, D_K) \cong H_c^i(\Lambda^\bullet M \otimes V_K, D_K)^*,$$

$$H_c^i(M, g)^* \cong HC_i(M, g), \quad HC_c^i(M, g) \cong HC_c^{i\dagger}(M, g)^*,$$

where isomorphisms are taken in the algebraic sense and duality is meant in the topological sense, as described in Proposition 8.

Under the finite type condition on $M$, an easy modification to the Mayer-Vietoris argument (Propositions 5.3.1 and 5.3.2 of [15]) also shows that each of these cohomology groups is finite dimensional, so the reverse duality isomorphisms hold as well. Finally, the formal homotopy equivalence of Corollary 10 translates these isomorphisms into the desired duality relations for Calabi cohomology and homology.

The Poincaré-Serre argument, applies by virtue of the ellipticity of the Calabi and its formal adjoint complexes (Proposition 9) and the hypothesis of countability at infinity. Combining the results of Propositions 7 and 8, easily establishes the desired duality isomorphisms directly. □

Corollary 12. Assume the same hypotheses on $M$ as in Corollary 11. The Calabi cohomology and homology, assuming they are finite dimensional, the following identities hold (with respect to algebraic duals):

$$HC_i(M, g) \cong H^i(K), \quad HC_c^i(M, g) \cong H^{n-i}(KY)^*,$$

$$HC_i(M, g) \cong H^i(K)^*, \quad HC_c^{i\dagger}(M, g) \cong H^{n-i}(KY).$$
Note that we do expect the relevant cohomology and homology spaces to be finite dimensional in most applications. If the cohomology vector spaces happen to be infinite dimensional, then the correct (topological and algebraic) isomorphisms can be deduced from Corollary 11 and Proposition 8.

**Proof.** By Proposition 9 and Corollary 10 we already know that the Calabi and its formal adjoint complexes are locally exact differential complexes that respectively resolve the Killing and Killing-Yano sheaves, $\mathcal{K}$ and $\mathcal{KY}$. Then, Lemma 3 establishes the isomorphisms $HC^i(M,g) \cong H^i(\mathcal{K})$ and $HC^i_{lf}(M,g) \cong H^{n-i}(\mathcal{KY})$. Finally, the duality isomorphisms of Corollary 11 establish the rest of the desired identities. Note that we have added the finite dimensionality hypothesis only to avoid explicitly specifying a topology on the relevant cohomology vector spaces, so that the topological and algebraic duals coincide. □

4 The Killing sheaf and its cohomology

In this section we concentrate on possible effective ways of computing the Killing sheaf cohomology (or rather the cohomology of any locally constant sheaf) of a pseudo-Riemannian manifold $(M,g)$ of constant curvature. For us, effective is used somewhat loosely and we take it to mean roughly to either consist of finitely many steps involving only finite-dimensional linear algebra or to reduce to calculation that has already been done in the literature. In particular, any such method would be more effective than the brute force approach of trying to solve the systems of differential equations appearing in the Calabi complex. Since the interest in the cohomology of the Killing sheaf may extend beyond the constant curvature context, we always discuss the more general situation, specializing to the constant curvature case when necessary.

There are two main possibilities, either the manifold $M$ is simply connected or it is not. They are discussed respectively in Sections 4.1 and 4.2. In the simply connected case, the sheaf cohomology can be expressed completely in terms of the de Rham cohomology. The non-simply connected case is more complicated, where several complementary but potentially overlapping methods may be used. None of them, unfortunately, gives a complete solution.

Crucial to the discussion that follows (see Appendix D for relevant notation and concepts related to $G$-bundles) is the notion of the monodromy representation of the fundamental group $\pi = \pi_1(M)$ of a manifold with respect to a flat connection $D$ on a vector bundle $V \to M$ (cf. Section 2.4). Let us identify $\pi_1(M) = \pi_1(M,x)$ for some $x \in M$. The connection $D$ gives rise to a notion of parallel transport on $V$. Since the connection is flat, the parallel transport along a curve connecting $x,y \in M$ depends only on the homotopy class of the path with its endpoints fixed. Therefore, since parallel transport acts linearly, parallel transport along loops based at $x \in M$ induces a representation $\rho_x : \pi \to \text{GL}(\mathcal{V})$, where $\mathcal{V} \cong V_x$ is the typical fiber of $V \to M$, called the monodromy representation. Another common term is the holonomy representation. However, we reserve the term holonomy for the same concept associated specifically to the $g$-compatible Levi-Civita connection on $M$. If $V \to M$ is a vector $G$-bundle, then there necessarily is an associated representation of the structure group on $\mathcal{V}$, $\sigma_V : G \to \text{GL}(\mathcal{V})$. When the connection $D$ preserves the $G$-bundle structure, parallel transport and hence monodromy factors through the assci-
ated representation. Hence $\rho_V = \sigma_V \circ \rho$, where $\rho: \pi \to G$ is the monodromy representation of $\pi$ in the structure group.

Recall also that for any manifold $M$ there exists a unique (up to diffeomorphism) connected, simply connected universal cover $\tilde{M} \to M$, where the projection map is a surjective local diffeomorphism. In fact, $\tilde{M} \to M$ is a $\pi$-principal bundle over $M$. The principal bundle action of $\pi$ on $\tilde{M}$ by is called action by deck transformations. Note that $M \cong \tilde{M}/\pi$. Deck transformations, being diffeomorphisms, commute with the de Rham differential. Hence the action by deck transformations descends to de Rham cohomology. We call it $\rho$. Hence $\rho$ acts on $\tilde{M}$ by pulling back the bundle $\tilde{V}$ to $\tilde{M}$ and the connection $\tilde{D}$ to $\tilde{D}$. Since the universal cover is simply connected, the pulled back bundle trivializes, $\tilde{V} \cong \tilde{V} \times M$. Therefore, we have the isomorphism $H^i(\Lambda^*\tilde{M} \otimes \tilde{V}, \tilde{D}) \cong H^i(\tilde{M}) \otimes \tilde{V}$. It is not hard to see that the two sides are isomorphic not only as vector spaces but also as representations of the fundamental group $\pi$, with the right side transforming as the tensor product of the deck and monodromy representations $\Delta^i \otimes \rho_V$.

Let us fix the assumptions that $(M,g)$ is connected and that its Killing sheaf $\mathcal{K}_g$ is locally constant, then concretize the above ideas to this case. Recall from Section 2.4 that $\mathcal{K}_g$ is then resolved by the twisted de Rham complex associated to the flat vector bundle $(V_K, D_K)$. The typical fiber $\tilde{V}_K$ of $V_K \to M$ consists of the germs of local Killing vector fields. Each local Killing vector field extends to a global one, and hence to an infinitesimal isometry, on the universal cover $(\tilde{M}, \tilde{g})$. Thus, we can identify $\tilde{V}$ with the Lie algebra $\mathfrak{g}$ of the Lie group $G = \text{Isom}(\tilde{M}, \tilde{g})$ of isometries of $(\tilde{M}, \tilde{g})$.

Infinitesimal isometries act on each other by the formula $\mathcal{L}_v u = [u, v]$, which corresponds to the infinitesimal adjoint representation $\text{ad}: \mathfrak{g} \to \text{End}(\mathfrak{g})$. This representation integrates to the adjoint representation $\text{Ad}: G \to \text{GL}(\mathfrak{g})$, which is how finite isometries act on Killing vector fields. Also, it is clear by construction that deck transformations act on $(\tilde{M}, \tilde{g})$ by isometries. Let us denote this representation of the fundamental group $\pi = \pi_1(M)$ by isometries as $\rho: \pi \to G$. As described in Section D.3, this information is equivalent to specifying a flat principal $G$-bundle $P \to M$ with monodromy representation $\rho$ of $\pi$ in $G$. Further, it is clear that $V_K \cong \mathfrak{g}_P$ is the vector $G$-bundle over $M$ associated to $P$ with respect to the adjoint action $\text{Ad}$ of $G$ on $\mathfrak{g}$ and that $D_K$ is the connection associated to the flat principal connection on $P$. The monodromy representation of $\pi$ on $V_K$ is then the composite adjoint monodromy representation $\rho_V = \text{Ad}_\rho = \text{Ad} \circ \rho$.

4.1 Simply connected case

The simplest case is when the manifold $M$ is simply connected, that is, its fundamental group $\pi = \pi_1(M)$ is trivial. Let the locally constant sheaf $\mathcal{F}$ have stalk $\tilde{F}$ so that it defines a flat vector bundle $(F, D)$, with $\tilde{F}$ the typical fiber of $F \to M$ (Sections 3.1 and 2.4). We know that the twisted de Rham differential complex $(\Lambda^*M \otimes F, D)$ is an acyclic resolution of $\mathcal{F}$. Hence their cohomologies agree. On the other hand, since $M$ is simply connected, we can choose a global $D$-flat basis frame for $F$ and identify the twisted de Rham complex with $\text{rk} F = \dim \tilde{F}$ copies of the standard de Rham complex. This argument proves

**Theorem 13.** Let $(M,g)$ be a connected, simply connected pseudo-Riemannian
manifold with locally constant Killing sheaf $\mathcal{K}_g$, resolved by the twisted de Rham complex $(\Lambda^\bullet M \otimes V_K, D_K)$. Let $\mathfrak{g} \cong V_K$ be the Lie algebra of isometries of $(M, g)$. Then the following isomorphisms hold:

$$H^i(K_g) \cong H^i(\Lambda^\bullet M \otimes V_K, D_K) \cong H^i(M) \otimes \mathfrak{g}. \quad (86)$$

In particular $H^0(K_g) \cong \mathfrak{g}$ and $H^1(K_g) = 0$.

4.2 Non-simply connected case

The non-simply connected case is of course more complicated and we can offer only partial results, which we summarize in this paragraph. The simplest sub-case is when the fundamental group $\pi = \pi_1(M)$ of the pseudo-Riemannian manifold $(M, g)$ is finite (Section 4.2.1). The Killing sheaf cohomology is then the $\pi$-invariant subspace of the de Rham cohomology of the universal covering space. If the fundamental group is not necessarily finite, we still have the following general result for the degree-1 cohomology of constant curvature spaces. We can equate $\dim H^1(K_g)$ to the dimension of the space of possible infinitesimal deformations of the metric that preserve the constant curvature condition as well as the value of the scalar curvature itself. That observation was already made in the original work of Calabi [19] and in fact prompted his interest in a resolution of the Killing sheaf $\mathcal{K}_g$. This space of infinitesimal deformations can also be computed as the degree-1 group cohomology of $\pi$ with coefficients in a certain representation on the Lie algebra of isometries of the universal cover of $M$ (Section 4.2.2). Another result helps compute higher degree cohomology groups. The Killing sheaf, being locally constant, defines a local system or a system of local coefficients on $M$, a concept well known in algebraic topology. A general result from the theory of local systems is that the aforementioned group cohomology computes higher Killing sheaf cohomology groups up to the degree of asphericity of $M$ (Section 4.2.3). Finally, there is a general method for completely computing the Killing sheaf cohomology based on a presentation of the manifold $M$ as a finite simplicial set (Section 4.2.4).

4.2.1 Finite fundamental group

The basic idea here is to take advantage of the complete decomposability of representations of a finite group and then apply Schur’s lemma. As will be clear from the proof, it is the complete decomposability that is important not the finiteness of $\pi$. So the same result actually holds under suitably weaker hypotheses.

**Theorem 14.** Let $(M, g)$ be a connected pseudo-Riemannian manifold with fundamental group $\pi = \pi_1(M)$ and Killing sheaf $\mathcal{K}_g$, resolved by the twisted de Rham complex $(\Lambda^\bullet M \otimes V_K, D_K)$. Let $\mathfrak{g} \cong V_K$ be the Lie algebra of isometries of the universal cover $(\tilde{M}, \tilde{g})$. If $\pi$ is finite, we have the following isomorphisms:

$$H^i(K_g) \cong (H^i(\tilde{M}) \otimes \mathfrak{g})^\pi, \quad (87)$$

where the superscript $\pi$ denotes the $\pi$-invariant subspace with respect to the representation $\Delta^i \otimes \text{Ad}_\rho$, the tensor product of the deck and composite adjoint monodromy representations. In particular $H^0(K_g) \cong \mathfrak{g}^\pi$. 

33
Proof. Consider the spaces of sections $\Omega_i = \Gamma(\Lambda^i \check{M} \otimes \check{V}_K)$, where $\check{V}_K \to \check{M}$ is the pullback of $V_K \to M$ along the universal covering projection $\check{M} \to M$. Let $\check{D}_K$ denote the pullback of the $D_K$. As we have already discussed at the top of Section 4, this pulled back bundle is trivial, $\check{V}_K \cong g \times M$. Moreover, by simple connectedness of $\check{M}$ and Theorem 13, we have the isomorphism $H^1 = H^1(\Omega_i, \check{D}_K) \cong H^1(\check{M}) \otimes g$.

As also discussed at the top of Section 4, the spaces $\Omega_i$ carry representations of the fundamental group $\pi$, which also descends to the cohomologies $H^1$. Since $\pi$ is finite, it is well known that any representation thereof is completely decomposable [93], that is, any subrepresentation has a direct sum complement subrepresentation. So, the subspace $\Omega_i \subset \Omega$ invariant under the action $\pi$ (every element of $\pi$ acting as the identity operator) has a direct sum complement $\Omega_i$, so that $\Omega_i \cong \Omega_i \oplus \Omega_i$. This direct sum induces the short exact sequence

$$0 \longrightarrow \Omega_i \longrightarrow \Omega_i \longrightarrow \Omega_i \longrightarrow 0. \quad (88)$$

It is straightforward to note that, by construction of the universal cover $\check{M} \to M$, the $\pi$-invariant subcomplex $(\Omega_i, \check{D}_K)$ on $\check{M}$ is in fact cochain isomorphic to the complex $(\Gamma(\Lambda^i \check{M} \otimes \check{V}_K), D_K)$ on $M$. Therefore the desired cohomology groups are $H^1(\Lambda^i \check{M} \otimes \check{V}_K, D_K) \cong H^1(\Omega_i, \check{D}_K)$.

The complement $\Omega_i$ naturally does not contain any non-zero vectors invariant under the action of $\pi$. In representation theoretic terminology, these two complementary subspaces are disjoint. By Schur’s lemma [94], the only equivariant map (intertwiner) between any two disjoint representations is the zero map. Note that the differentials $\check{D}_K$ and the maps in the short exact sequence (88) are in fact $\pi$-equivariant. By the general machinery of homological algebra (Appendix B) the short exact sequence (88) induces the long exact sequence

$$0 \longrightarrow H^0 \longrightarrow H^0 \longrightarrow H^0 \longrightarrow \cdots \quad (89)$$

where $H^0 = H^0(\Omega_i, \check{D}_K)$, $H^0 = H^0(\Omega_i, \check{D}_K)$ and all the maps are also $\pi$-equivariant. It is clear that the representations carried by $H^0$ and $H^0$ are also disjoint. Therefore, the maps connecting the rows of diagram (89) are all zero. In other words, each of the rows becomes a short exact sequence on its own. Invoking again complete decomposability of representations of $\pi$, we can write $H^1 \cong H^1 \oplus H^1$ and hence identify $H^1 \cong (H^1)^\pi$ with the subspace of $H^1$ on which $\pi$ acts trivially.

Collecting the above arguments together, while recalling the sheaf cohomology identity $H^1(K_g) \cong H^1(\Lambda^i \check{M} \otimes \check{V}_K, D_K)$, we obtain the isomorphism $H^1(K_g) \cong (H^1(\check{M}) \otimes g)^\pi$. Noting the special cases $H^0(M) = \mathbb{R}$ and $H^1(\check{M}) = 0$, as in Theorem 13, concludes the proof. \qed

4.2.2 Degree-1 cohomology

Consider a 1-parameter family of $n$-dimensional pseudo-Riemannian manifolds $(M, g(t))$ where each $g(t)$, for $t$ in some neighborhood of zero, has constant curvature, with scalar curvature independent of $t$: Riemann tensor equal to
\( \frac{k}{n(n-1)} g(t) \odot g(t) \). Let \( g(0) = g \) and \( \dot{g}(0) = h \). Then the linearization of the identity \( R[g(t)] - \frac{k}{n(n-1)} g(t) \odot g(t) = 0 \) at \( t = 0 \) will give (cf. Section 2.2)

\[
\dot{R}[h] - \frac{2}{n(n-1)} (g \odot h) = -\frac{1}{2} C_2[h] = 0. \tag{90}
\]

In other words, \( h \) is a Calabi 1-cocycle. It is possible that not every Calabi 1-cocycle gives rise to an actual 1-parameter family of deformations, since there may be algebraic obstructions\(^3\) to solving for higher order terms in the expansion parameter \( t \). However, at the infinitesimal level, there are no other conditions and we can identify infinitesimal deformations with Calabi 1-cocycles. If the deformation family \( g(t) \) is trivial, induced by a 1-parameter family of diffeomorphisms of the manifold \( M \), then it is well known that \( h = K[v] \) for some 1-form \( v \) (vector field generating the diffeomorphism family, with index lowered by the metric \( g \)), in other words a Calabi 1-coboundary. It is easy to see that Calabi 1-coboundaries can be identified with infinitesimal trivial deformations. Therefore, the Calabi cohomology vector space \( HC^1(M, g) \), and hence the Killing cohomology vector space \( H^1(K_g) \) isomorphic to it, is in bijective correspondence with the space of infinitesimal deformations of the metric \( g \) within the space of constant curvature metrics of scalar curvature \( k \), modulo infinitesimal diffeomorphisms.

There is another way to look at this infinitesimal deformation space. It is well known that the only geodesically complete, simply connected, constant curvature spaces are the pseudo-Euclidean \( (k = 0) \), pseudo-spherical \( (k > 0) \) and pseudo-hyperbolic \( (k < 0) \) spaces [95, Sec.2.4]. In Riemannian signature, these are respectively the ordinary Euclidean, spherical and hyperbolic spaces. In Lorentzian signature, these are respectively the Minkowski, de Sitter and anti-de Sitter spaces. Thus, the elements of a family \( (M, g(t)) \) of geodesically complete, constant curvature, pseudo-Riemannian manifolds of fixed scalar curvature \( k \) all have isometric universal covers \( (\tilde{M}, \tilde{g}) \). Moreover, since the action of the fundamental group \( \pi = \pi_1(M) \) on its universal cover via deck transformations is by isometries, there is an injective group homomorphism \( \pi \to G = \text{Isom}(\tilde{M}, \tilde{g}) \), so that we have a subgroup \( \rho(\pi) \subseteq G \) that acts on \( \tilde{M} \) properly and discontinuously [95, Sec.1.8]. Conversely, for any subgroup of \( \pi' \subseteq G \) that acts on \( M \) properly and discontinuously the quotient \( (\tilde{M}', \tilde{g}') = (\tilde{M}, \tilde{g})/\pi' \) will be a manifold of the same constant curvature, but with fundamental group \( \pi' = \pi_1(M') \). So, we have already noticed that all \( (M, g) \) with constant curvature arise in this way. Of course, any two subgroups \( \pi', \pi'' \subseteq G \) that are conjugate, \( \pi'' = a\pi'a^{-1} \) for some \( a \in G \), give rise to isometric quotients. In fact, we have just argued that the infinitesimal deformations of the representation \( \rho: \pi \to G \), up to conjugation by \( G \), are in bijection with infinitesimal constant curvature deformations of the metric \( (M, g) \). It is well known that the deformations of the representation \( \rho \) are in bijection with certain degree-1 group cohomology of the fundamental group \( \pi \). On the other hand, we have already seen that deformations of the constant curvature spaces are parametrized by the Killing sheaf cohomology \( H^1(K_g) \). Thus, computing the group cohomology may be an effective way of computing the Killing sheaf cohomology, at least in degree-1. The details of the definition of the representation \( \rho \) are described

\(^3\)The study of these obstructions follows the general ideas outlined by Kodaira and Spencer [77, 78]. See also the related phenomenon of linearization instabilities [49].
This connection between the degree-1 Killing sheaf cohomology, deformations of the geometry and group cohomology of the fundamental group \( \pi \) extends far beyond the case of manifolds of constant curvature. We base what follows on the remark at the top of Section 4 and the contents of Appendix D. If \((\tilde{M}, \tilde{g})\) is the universal cover of \((M, g)\) and \(G = \text{Isom}(\tilde{M}, \tilde{g})\) with Lie algebra \(g\), then there is a naturally defined flat principal \(G\)-bundle \(P \to M\). Then, the infinitesimal deformations of this flat principal \(G\)-bundle are in bijections with \(H^1(K_g)\), the degree-1 Killing sheaf cohomology group. That is because the flat vector bundle \((V_K, D_K)\), whose twisted de Rham complex resolves the Killing sheaf, is isomorphic to the associated bundle \(g_P \to M\) with connection \(D\) induced by the flat principal connection on \(P\). Recall that the fibers of \(g_P\) transform under the adjoint representation \(\text{Ad} : G \to \text{GL}(g)\) and that parallel transport with respect to the flat connection on \(P\) defines a representation \(\rho : \pi \to G\) of the fundamental group \(\pi = \pi_1(M)\). Their composition \(\text{Ad}_\rho = \text{Ad} \circ \rho\), as already mentioned at the top of Section 4, is known as the composite adjoint monodromy representation. In the case of spaces of constant curvature, the infinitesimal deformations of the flat principal bundle \(P \to M\) are the same thing as the infinitesimal deformations of the given constant curvature metric, fixing the value of the curvature.

**Theorem 15.** Given the notations and hypotheses of the above paragraph, the following isomorphisms between the Killing sheaf cohomology and group cohomology of \(\pi\) with coefficients in \(\text{Ad}_\rho\) hold:

\[
H^0(K_g) \cong H^0(\pi, \text{Ad}_\rho) \cong g^\pi, \quad H^1(K_g) \cong H^1(\pi, \text{Ad}_\rho).
\]

(91)

This result is a direct consequence of Proposition 17 of Appendix D. Unfortunately, we cannot use the same methods to establish isomorphisms between the group and sheaf cohomologies in higher degrees. See, however, Section 4.2.3. The connection between group cohomology of \(\pi\) and deformations of a flat principal bundle is well known, cf. \[37\]. The connection between, specifically, the cohomology of the Killing sheaf, infinitesimal deformations of the corresponding principal bundle, and group cohomology seems to be less well known, but is mentioned explicitly in \[2, \text{App.A.2}\].

**4.2.3 Cohomology with local coefficients**

We have just noted, in Section 4.2.2, a geometric relation between degree-1 locally constant sheaf cohomology and cohomology of the fundamental group. A more general connection between the cohomology of a locally constant sheaf, or equivalently cohomology with coefficients in a local system \[90, \text{Ch.VI}\], and group cohomology of the fundamental group has also been noticed in pure algebraic topology. In fact, that is how the notion of group cohomology first arose.

The original goal was to calculate the cohomology of a space (with or without coefficients in a non-trivial local system) in terms of data specifying its homotopy type. Following some early work by Hurewicz, Hopf and Eilenberg, Eilenberg and Maclane \[25\] introduced what are now known as \(K(1, \pi)\) spaces (topological spaces with \(\pi_1 = \pi\) and \(\pi_i = 0\) for all \(i > 0\)) and computed all of
their cohomology groups by introducing an algebraic construction based on the knowledge of the group \( \pi \). We now call this construction group cohomology [91]. They further showed that the same construction works also for any topological space \( M \), not just a \( K(1, \pi) \), for the cohomologies in degree-\( i \), with \( 0 < i \leq p \), as long as the space \( M \) is \( p \)-aspherical, \( \pi_i = 0 \) for \( 0 < i \leq p \). This result, applied to the Killing sheaf gives the following

**Proposition 16.** Let \((M, g)\) be a connected pseudo-Riemannian manifold with locally constant Killing sheaf \( K_g \) and universal cover \((\tilde{M}, \tilde{g})\). Denote \( G = \text{Isom}(\tilde{M}, \tilde{g}) \) the group of isometries of the universal cover and let \( g \) be its Lie algebra. The fundamental group \( \pi = \pi_1(M) \) acts on \( g \) via the composite adjoint monodromy representation \( \text{Ad}_\rho : \pi \rightarrow \text{GL}(g) \). If the manifold \( M \) is \( p \)-aspherical, meaning \( \pi_i(M) = 0 \) for \( 0 < i \leq p \), then we have the following isomorphisms:

\[
H^i(K_g) = H^i(\pi, \text{Ad}_\rho) \quad \text{for} \quad 0 \leq i \leq p.
\]

(92)

For higher degree cohomology there are other contributions to the homology groups. There is still a homomorphism \( H^i(\pi, \text{Ad}_\rho) \rightarrow H^i(K_g) \), but it need no longer be an isomorphism [57, Sec.1.4.2].

Later, Postnikov [65, 66] proposed a full solution for algebraically determining all the cohomology groups of a space based on its homotopy type. Postnikov’s method encodes the full homotopy type of a space in terms of their homotopy groups and certain additional algebraic data known as a Postnikov system or tower. This construction is currently more commonly known in its topological form [90, Ch.IX]. For \( p \)-aspherical spaces and cohomology in degree \( i \), with \( 0 < i \leq p \), Postnikov’s construction coincides with group cohomology. In general, the two constructions do differ in degrees higher than the degree of asphericity.

Unfortunately, both Postnikov’s encoding of the homotopy type and his algebraic reconstruction of the cohomology are rather complicated, do not appear to have gained much popularity. They seem to be fully described only in his original monograph [66] or its translation [67], both being rather obscure references. At the moment, it is not clear to us what is the modern state of the art in terms of reconstructing the cohomology of a space with coefficients in a local system in terms of the space’s homotopy type.

### 4.2.4 Simplicial set cohomology

The last mathematical tool, which we will discuss, that can aid in the computation of the cohomologies of a locally constant sheaf is simplicial cohomology with local coefficients. The idea is to substitute the underlying manifold \( M \) with a combinatorial structure like a simplicial complex or a simplicial set. Then, provided the combinatorial model is finite, the corresponding cohomology theory reduces to the computation of the cohomology of a finite dimensional cochain complex, and thus to finite dimensional linear algebra. We defer to the discussion in [36, Sec.1.4.7–10] for technical details.

A disadvantage of this method is that finite combinatorial models only cover the case of compact manifolds. Non-compact manifolds require either an infinite combinatorial model or a non-trivial extension of the formalism. Another inconvenience, besides the need for an explicit decomposition of \( M \) into simplices, is the need to define a discrete analog of parallel transport on the simplicial model.
to reproduce the composite adjoint monodromy representation $\text{Ad}_\rho$. That is usually done by associating a copy of $\mathfrak{g}$ to each vertex of the simplicial model for $M$ and explicitly assigning a coherent set of linear isomorphisms between these copies to the edges connecting them, such that the composition of the isomorphisms of the edges along a closed loop is equal to the $\text{Ad}_\rho$ action of the corresponding element of $\pi$. These choices may be simplified if all vertices could be collapsed into a single one, which is allowed for simplicial sets. Such a construction is always possible when $M$ is compact and results in a so-called reduced simplicial set [58].

5 Application to linearized gravity

Recently, the symplectic and Poisson structure of linear classical field theories has been studied by the author within a very general framework [48, 46] (see also [31, 17, 42] for related work), which admits in particular any linear field theory whose gauge fixed equations of motion can be formulated as a hyperbolic PDE system with possible constraints and residual gauge freedom. Certain sufficient geometric conditions need to be satisfied for a field theory to fit into that framework. The framework can then precisely characterize the degeneracies of the presymplectic and Poisson tensors on the solution space of the theory. These sufficient conditions require the gauge generator and the constraint operator to fit into differential complexes and the degeneracies of the presymplectic and Poisson tensors are then characterized using the cohomology of these complexes. Once known, these presymplectic and Poisson degeneracies are known to be of importance in classifying the charges, locality, superselection sectors and quantization of the corresponding classical theory.

The well known examples of Maxwell electromagnetism and Maxwell $p$-forms [71, 10, 8] fit into this framework [48, Sec.4.2], invoking the well known de Rham complex. Linearized gravity on a constant curvature Lorentzian manifold also fits into this framework, with the role of the de Rham complex replaced by the Calabi complex or, as appropriate, the formal adjoint Calabi complex. For linearized gravity on an arbitrary background, we would need to make use of different differential complexes. The Calabi complex would be replaced by complexes defined by the property that they (at least formally) resolve the sheaf of Killing vectors on the given background (cf. Section 3.2). The corresponding formal adjoint complexes would play a role as well. This connection to the Killing sheaf, even without explicitly knowing the needed differential complexes, shows that the Killing sheaf cohomology plays a similar role both in the constant curvature context and more generally. Thus the ability to compute the Killing sheaf cohomology in as many circumstances as possible (as discussed in Section 4) should take us a large part of the way towards understanding the presymplectic and Poisson degeneracies of linearized gravity on general backgrounds. Unfortunately, about half the desired information would still be missing, since it is not clear which sheaf cohomology theory would control the cohomology of the formal adjoint differential complex. In the case of constant curvature, we were able to identify it as the cohomology of the sheaf of rank-$(n-2)$ Killing-Yano tensors, which is resolved by the formal adjoint Calabi complex (Section 2.3). It is currently not clear how to identify its analog in the case of a general background, without knowing the full differential complex that
(formally) resolves the sheaf of Killing vectors.

Now, specialized to the case of linearized gravity on a constant curvature background \((M, g)\), the analysis of \([48, 46]\) concludes that the presymplectic and Poisson tensors are actually non-degenerate (with spacelike compact support for solutions and compact support for smeared observables) if and only if the following two conditions are satisfied: (global recognizability) a certain bilinear pairing between degree-1 Calabi cohomology with spacelike compact supports and degree-1 Calabi homology, (global parametrizability) a certain bilinear pairing between on-shell degree-1 Calabi cohomology with spacelike compact supports and on-shell degree-1 timelike finite Calabi homology.

The descriptions of off-shell or on-shell Calabi cohomologies with spacelike compact supports, \(HC_{sc}^{1}\) or \(HC_{\square, sc}^{1}\), and of off-shell or on-shell timelike finite Calabi homology, \(HC_{tf}^{1}\) or \(HC_{\square, tf}^{1}\) go beyond the scope of the current work. However, they are defined and studied in detail in \([47]\) (similar ideas appear also in \([8]\)). In fact, the results of \([47]\) show how to express these non-standard cohomologies in terms of the standard ones with unrestricted or compact supports, and similarly for homology. Recall also (Section 3.4) that the latter are isomorphic to appropriate cohomologies (or their linear duals) of the Killing or Killing-Yano sheaves, \(K_g\) or \(K_Y_g\). Using all of these results we are able to translate the non-degeneracy requirements as follows: (global recognizability) a certain bilinear pairing between

\[
HC_{sc}^{1}(M, g) \cong H^{n-2}(M, K_Y g)^* \\
\text{and} \quad HC_{1}(M, g) \cong H^{1}(M, K_g)^*
\]

is non-degenerate, (global parametrizability) a certain bilinear pairing between

\[
HC_{\square, sc}^{1}(M, g) \cong H^{n-1}(M, K_Y g)^* \oplus H^{n-2}(M, K_Y g)^* \\
\text{and} \quad HC_{\square, tf}^{1}(M, g) \cong H^{1}(M, K_g)^* \oplus H^{0}(M, K_g)^*
\]

is non-degenerate. Notice that we have succeeded in expressing the vector spaces on which these pairings are defined purely in terms of Killing and Killing-Yano sheaf cohomologies.

Checking non-degeneracy of course requires an explicit expression for the required bilinear pairings. Such expressions can be obtained from the general framework of \([48, 46]\). However, there are two cases were we do not need such detailed information, and these are the ones we shall content ourselves here. For instance, if all the relevant cohomology vector spaces are trivial, then the only possible, trivial bilinear pairing is automatically non-degenerate. On the other hand, if the paired vector spaces have different dimensions, then every possible pairing between them must be degenerate.

We conclude this section by listing several well known Lorentzian backgrounds for which the methods of Section 4 allow us to determine all or a few of the cohomologies of the Killing sheaf. For the reasons discussed above, we make note of the Killing-Yano sheaf cohomologies only for constant curvature backgrounds.

The easiest case is that of simply connected spacetimes. Then, the Killing sheaf cohomology is just the de Rham cohomology tensored with the Lie algebra of global isometries (Section 4.1), with an analogous result for any other locally constant sheaf. Many of the well known exact solutions are in fact defined
A list of some well known, simply connected solutions of (cosmological) vacuum Einstein equations, together with their topology and non-vanishing dimensions of Killing or Killing-Yano sheaf cohomologies. Note that $b^0$ always counts the number of independent global Killing vectors, and similarly for $c^0$. The Tangherlini solutions generalize the Schwarzschild one to higher dimensions and the Meyers-Perry solutions do the same for Kerr [26]. For the latter, $N$ counts the number of rotational symmetries, which varies depending on the variant of the solution. We only consider the exterior regions for black hole solutions.

<table>
<thead>
<tr>
<th>Spacetime</th>
<th>Topology</th>
<th>$b^i = \dim H^i(K)$</th>
<th>$c^0 = \dim H^0(KY)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minkowski</td>
<td>$\mathbb{R}^n$</td>
<td>$b^0 = \frac{n(n+1)}{2}$</td>
<td>$c^0 = \frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>Open FLRW</td>
<td>$\mathbb{R}^n$</td>
<td>$b^0 = \frac{(n-1)n}{2}$</td>
<td>$c^0 = \frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>De Sitter</td>
<td>$\mathbb{R} \times S^{n-1}$</td>
<td>$b^0 = \frac{n(n+1)}{2}$</td>
<td>$c^0 = \frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>Closed FLRW</td>
<td>$\mathbb{R} \times S^{n-1}$</td>
<td>$b^0 = \frac{n(n+1)}{2}$</td>
<td>$c^0 = \frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>Schwarzschild</td>
<td>$\mathbb{R}^2 \times S^2$</td>
<td>$b^0 = b^2 = 4$</td>
<td>$c^0 = \frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>Tangherlini</td>
<td>$\mathbb{R}^2 \times S^{n-2}$</td>
<td>$b^0 = \frac{n(n+1)}{2}$</td>
<td>$c^0 = \frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>Kerr</td>
<td>$\mathbb{R}^2 \times S^2$</td>
<td>$b^0 = b^2 = 2$</td>
<td>$c^0 = \frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>Meyers-Perry</td>
<td>$\mathbb{R}^2 \times S^{n-2}$</td>
<td>$b^0 = b^{n-2} = 1 + N$</td>
<td>$c^0 = \frac{n(n+1)}{2}$</td>
</tr>
</tbody>
</table>

Table 2: A list of some exact solutions of Einstein’s equations indexed by spacetime topology. Note that only the Minkowski and de Sitter spaces are of constant curvature, so that the Calabi complex could be defined on them. For these backgrounds, it makes sense to also compute the Killing-Yano sheaf cohomologies $H^i(KY)$. However, since we know that the number of linearly independent rank-$(n-2)$ Killing-Yano tensors on these spaces is the same as the number of linearly independent Killing vectors (Section 2.3), the cohomology vector spaces are isomorphic, $H^i(KY) \cong H^i(K)$.

In the non-simply connected case, we can rely on the results of Sections 4.2.1, 4.2.2 and 4.2.3, according to which we can equate the Killing sheaf cohomologies with the group cohomology of the fundamental group with coefficients in the composite adjoint monodromy representation, at least up to the degree of asphericity of the underlying spacetime manifold. Unfortunately, there does not seem to exist a comprehensive list of exact solutions of Einstein’s equations indexed by spacetime topology. So it takes some effort to find explicit examples of exact solutions on non-simply connected spacetimes. A rich source of examples comes from quotients of simply connected spacetimes (such as those mentioned in the preceding paragraph) by a discrete, freely acting subgroup $\pi$ of the isometry group. The quotient is a manifold because the action of $\pi$ is free and the metric descends to the quotient because the action of $\pi$ on the original spacetime is by isometries. The group $\pi$ then becomes the fundamental group of the quotient.

An nearly exhaustive study of possible quotients of 4-dimensional cosmological solutions (meaning spatially homogeneous ones) has been carried out in [52, 84, 51]. A complete presentation of the results is rather complicated and is relegated to the original references. A particular cosmological solution
Table 3: Known values of $b^i = \dim H^i(K_g)$ for a generic spatially homogeneous spacetime $(M,g)$ with given topology and symmetry properties. See text for more details.

\[
\begin{array}{cccccc}
M \times T^3 & \pi_1(M) & \text{Bianchi sym.} & \text{additional sym.} & b^0 & b^1 \\
\mathbb{R} \times T^3 & \mathbb{Z}^3 & \mathbb{R}^3 & & 1 & 3 & 6 \\
\mathbb{R} \times T^3 & \mathbb{Z}^3 & \text{VII(0)} & & 1 & 2 & 4 \\
\mathbb{R} \times T^3 & \mathbb{Z}^3 & \mathbb{R}^3 & SO(2) & 3 & 6 \\
\mathbb{R} \times T^3 & \mathbb{Z}^3 & \mathbb{R}^3 & SO(3) & 3 & 5 \\
\end{array}
\]

$(M,g)$ is identified by (a) the topology of the spacetime $M$, (b) the topology and isometry group of the universal cover $(\tilde{M}, \tilde{g})$, (c) a number of continuous metric parameters specifying $\tilde{g}$, and (d) a number of continuous moduli (or Teichmüller parameters) specifying the quotient class. There may also be additional discrete parameters, but we ignore them here, since they do not affect the number of continuous parameters. According to the discussion of Section 4.2.2, the number of moduli (denoted by $N_m$ in [51]) is in fact equal$^4$ to $b^1 = \dim H^1(K_g)$. We shall not specify any metric parameters, since, as long as they take generic values they do not affect the number of moduli. For simplicity, we only consider the examples with toroidal spatial topology $M = \mathbb{R} \times T^3$, where $T^3 = S^1 \times S^1 \times S^1$. Hence, the fundamental group is $\pi_1(M) = \mathbb{Z}^3$ and the universal cover is $\mathbb{R}^4$. The (identity component) of the isometry group of $(\tilde{M}, \tilde{g})$ is then a semidirect product of a 3-dimensional transitive Bianchi group and an additional connected Lie group. Let us concentrate on the cases of either Bianchi type I $\cong \mathbb{R}^3$ or VII(0). Under these conditions, we can read off all the remaining possibilities and information form Table IV of [51]. They are summarized in Table 3. Note that $\dim H^0(K_g)$ counts the number of independent global Killing vectors on $(M,g)$. The number of independent Killing vectors on $(\tilde{M}, \tilde{g})$ counts the dimension of the Bianchi group (always 3) and the dimension of the additional symmetry group. The number of independent Killing vectors not broken by compactification to $T^3$ can be deduced from the explicit presentation of the isometry groups $\text{Isom}(\tilde{M}, \tilde{g})$ and the discrete subgroups effecting the compactification, which for the examples given in Table 3 in [51, Sec.3]. Many more examples can be read off from Tables IV, VII and Section 5.3 of [51].

It appears difficult to locate literature on explicit calculations that are equivalent to computing higher Killing sheaf cohomologies for other non-simply connected spacetimes.

6 Discussion and generalizations

We have reviewed in detail the algebraic, geometric and analytical properties of the Calabi differential complex [19].

$^4$The space of moduli may not always be a smooth manifold, but may have algebraic singularities. Still, the number of moduli is the dimension of the generic subset of the moduli space, which is a smooth manifold. This dimension is also equal to $b^1 = \dim H^1(K)$. At singular points of the moduli space, $b^1$ may actually exceed the number of moduli, so at these points a more careful analysis is needed.
In Section 2 we have defined the nodes of the complex in terms of Young symmetrized tensor bundles and given explicit formulas for the differential operators between them, verifying through explicit calculations that they in fact constitute a complex (Appendix A). Such explicit formulas are otherwise difficult to extract from the existing literature, especially in terms of tensor variables, as opposed to moving coframe variables used in Calabi’s original work. Further, our formulas work for pseudo-Riemannian backgrounds of any signature, generalizing from the standard purely Riemannian context. We have also identified a differential operator cochain homotopy (Equations (2), (10)–(15)) that generates a cochain map from the complex to itself with a Laplacian-like principal symbol. This cochain homotopy map may be new. However, its lower order terms coincide with well known geometric operators known from the theory of linearized gravity (General Relativity). Another interesting and likely novel observation involved the formal adjoint complex (Section 2.3), whose initial differential operator turned out to be equivalent to the rank-$(n-2)$ Killing-Yano operator, in analogy with the Killing operator in the original complex.

In Sections 3 and 4 we showed that the Calabi complex is elliptic and locally exact. Hence, it resolves the sheaf of Killing vectors on the given constant curvature pseudo-Riemannian manifold. The same is true for the formal adjoint complex and the sheaf of rank-$(n-2)$ Killing-Yano tensors. Thus the cohomology of the Calabi complex could be expressed in terms the Killing sheaf cohomology, while that of its formal adjoint in terms of the Killing-Yano sheaf cohomology. When a sheaf is locally constant (covering the relevant cases on constant curvature pseudo-Riemannian manifolds), its cohomology can be effectively computed in many circumstances using tools from algebraic topology, thus enabling effective computation of the Calabi cohomology. These methods were reviewed in Section 4, specialized to the Killing sheaf.

Finally, in Section 5, we discussed a physical application that motivated this work. Jointly, the results collected in this work, together with those of [47, 48, 46] imply that knowledge of Killing and Killing-Yano sheaf cohomologies allows some degree of control over the degeneracy subspaces of the presymplectic and Poisson structures within the classical field theory of linearized gravity on constant curvature backgrounds.

Unfortunately, the above results do not apply directly to linearized gravity on arbitrary Lorentzian manifolds, only those that have constant curvature and where the Calabi complex is defined. However, the Calabi complex serves as a case study for the more general situation and the same results partially generalize to general backgrounds. In particular, we can already make the following conclusions. In general, the Calabi complex will have to be replaced by a different differential complex, which will likely depend on some of the algebraic characteristics of the Lorentzian manifold (such as its isometries and the algebraic type of the curvature tensor and its derivatives). This complex would be identified, as was the Calabi complex [34, 35], by the property of being a formally exact compatibility complex of the Killing operator. Such a complex is known to exist under general conditions and also have the property of being elliptic, since the Killing operator is itself elliptic [68, 38]. Further, under a generic condition, it can be shown to be locally exact (Section 3.2). The local exactness property connects the cohomology of this complex to that of the Killing sheaf, which can be effectively computed, at least in many circumstances, when the sheaf is locally constant. Unfortunately, one piece of the puzzle remains incom-
The connection between the cohomology of the formal adjoint complex and sheaf cohomology depends on the knowledge of the initial operator in that differential complex, which is the adjoint of the final operator of the differential complex resolving the Killing sheaf. In the Calabi case it is equivalent to the Killing-Yano operator. However, since the differential complex is expected to change depending on the Lorentzian manifold, so is this initial operator. Thus, it is not clear which sheaf cohomology will replace the Killing-Yano sheaf in the general case.

Hence, in future work, it would be very interesting to investigate these differential complex resolutions of the Killing sheaf, especially computing their differential operators explicitly. Besides the general existence results [68, 38], such a complex has already been constructed for locally symmetric spaces ($\nabla_a R_{bcde} = 0$) [34, 35]. Also, heuristic arguments suggest that they could be partially constructed by linearizing the so-called ‘ideal’ characterizations of certain exact families of solutions of Einstein’s equations. These include Schwarzschild [27], Kerr [28] and some perfect fluid [21] solutions. An ‘ideal’ characterization consists of a number of tensor fields, locally and covariantly defined using the metric and its derivatives, which vanish iff the given metric is locally isometric to a particular geometry from the desired family. For instance, the vanishing of the Riemann tensor $R$ is an ideal characterization of the flat geometry, while the vanishing of the corrected Riemann tensor $R - \bar{R}$ (Section 2.2) does the same for a constant curvature geometry. It should be clear from these examples, that the linearization of the tensors that constitute such an ideal characterization gives an operator whose composition with the Killing operator is formally exact. At the moment it is not completely clear what geometric interpretation can be given to subsequent differential operators in the desired formally exact differential complex.

Finally, one can easily imagine situations where the number of independent solutions to the Killing equations changes over the background pseudo-Riemannian manifold. The Killing sheaf is then no longer locally constant and many of the techniques described in this work are no longer applicable. In those cases, perhaps some insight can be gained from the theory of constructible sheaves [23, Ch.4], [45, Ch.VIII], which are allowed to deviate from being locally constant in a controlled way.

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A Young tableaux and irreducible $GL(n)$ representations

A.1 Basic background

A Young diagram of type $(r_1, r_2, \ldots)$ with $k$ cells consists of a number of rows of cells of non-increasing lengths $r_i$, $r_{i+1} \leq r_i$, such that $\sum_i r_i = k$. Example:

\[
\text{type (3,3,1) or (3^2,1), diagram} \quad \begin{array}{c}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & & \\
\end{array}
\end{array}
\]

Given a Young diagram with $k$ cells, an instance of the corresponding $GL(n)$ irrep can be realized as the image of the space of covariant $k$-tensors after two projections: assign an independent tensor index to each cell of the diagram, symmetrize over each row, anti-symmetrize over each column. The composition of these operations is called a Young symmetrizer, which we will denote by $Y^d$, where $d = (r_1, r_2, \ldots)$ is the type of the Young diagram. It will be convenient for us to group the indices of a symmetrized tensor by the columns of the corresponding diagram, separating them by a colon. For instance, we write $b_{abcd:e}$ corresponding to the filling

\[
\begin{array}{c}
\begin{array}{cc}
a & d \\
b & c \\
\end{array}
\end{array}
\]

Here’s an example of a simple Young symmetrizer:

\[
Y^{(2,1)}[t]_{abc} = \frac{1}{4}(t_{abc} + t_{cba} - t_{bac} - t_{abc}).
\]

Different permutations of tensor indices filling a Young diagram create distinct Young symmetrizers, unless the permutation preserves the columns. The images of the Young symmetrizer for given diagram type with $k$-cells are all isomorphic as $GL(n)$ representations, but are not necessarily all identical as subspaces of the space of covariant $k$-tensors. The reason for this observation is that the space of covariant $k$-tensors is a reducible $GL(n)$ representation that decomposes into a sum of irreps corresponding to all possible diagram types with $k$-cells, but with, in general, non-trivial multiplicities. Both the dimension and the multiplicity of each occurring irrep can be computed with the so-called hook formulas. The hook length for a given cell is the number of cells constituting a hook with vertex at the given cell, extending to the right and down. Multiplicity: $k!$ divided by the product of the hook lengths for each cell. Dimension: the product of shifted dimensions for each cell, divided by the product of hook lengths for each cell; the shifted dimensions of the cells are obtained by placing $n$ in the top left cell, then always increasing by 1 to the right and decreasing by
1 down. Example:

\[
\begin{array}{ccc}
5 & 3 & 2 \\
4 & 2 & 1 \\
1 & & \\
\end{array}
\]

hook lengths, shifted dimensions \((n = 4)\)

\[
\begin{array}{ccc}
4 & 5 & 6 \\
3 & 4 & 5 \\
2 & & \\
\end{array}
\]

multiplicity: \(\frac{7!}{(5 \cdot 3 \cdot 2)(4 \cdot 2 \cdot 1)(1)} = 21\), dimension: \(\frac{(4 \cdot 5 \cdot 6)(3 \cdot 4 \cdot 5)(2)}{(5 \cdot 3 \cdot 2)(4 \cdot 2 \cdot 1)(1)} = 60\).

Note that when the number of rows exceeds \(n\), the corresponding representation becomes zero-dimensional. This clearly follows from the dimension formula and from the more elementary observation that there do not exist non-trivial fully antisymmetric tensors of rank greater than \(n\), the dimension of the fundamental representation of \(GL(n)\).

By construction, it is clear that every Young symmetrized subspace of covariant \(k\)-tensors is fully antisymmetric in the indices corresponding to each column of its Young diagram. However, this subspace will actually be even smaller and thus satisfy more identities. A complete set of identities selecting an irreducible \(GL(n)\) sub-representation of the space of covariant \(k\)-tensors identified by a diagram of type \((r_1, \ldots, r_t)\) filled with indices \(a^i_k\) \((k \text{ being the row number and } i \text{ the column number})\), consists of (i) **intracolumn** exchange identities and (ii) **intercolumn** exchange identities. The exchange of any two indices within a column changes the tensor by a sign. All such exchanges constitute the **intracolumn** identities. Let us define a two-column exchange as follows. Fix two columns \(i < j\) and select the top \(k\) indices of column \(i\). A two-column-exchange consists of a swap between a set of \(k\) indices from column \(i\) and the top \(k\) indices of column \(j\), without altering the internal order the substituted set of indices. For a fixed choice of such \(i, j, k\) an intercolumn identity consists of the equality of the tensor with unpermuted indices with the sum over all corresponding two-column exchanges. All such exchange identities with consistent choices of \(i, j, k\) constitute the **intercolumn** identities.

There already exists a special notation for antisymmetrization of a group of indices: inclusion in square brackets, \([a_1^1 a_2^2 \cdots]\). Let us introduce a special notation for the sum over all two column exchanges: fixing integers \(i < j\) and \(k\), we shall enclose the indices of column \(i\) in curly braces, \(\{a_1^1 a_2^2 \cdots\}\), as well as the top \(k\) indices of column \(j\), \(\{a_1^k a_2^k \cdots\}\). We give explicit examples of intracolumn and intercolumn identities for Young diagrams of type \((2, 2)\) and \((2, 2, 1)\):

\[
\begin{align*}
raabcd &= r[ab];cd = \frac{1}{2}(raabcd - rba;cd), \\
raabcd &= ra_{bc};d = \frac{1}{2}(raabcd - ra_{bd};c), \\
raabcd &= r(\{ab\};\{c\}d) = r_{eb;ad} + ra_{ac;bd}, \\
raabcd &= r(\{ab\};\{cd\}) = r_{cd;ab}, \\
babc;de &= b_{[abc];de} = \frac{1}{3}(ba_{[bc];de} + ba_{[ca];de} + ba_{[ab];de}), \\
babc;de &= b_{abc;[de]}, \\
babc;de &= b_{((abc);\{d\}e} = b_{d[bc];ae} + b_{d[dc];be} + b_{d(ab);ce}, \\
babc;de &= b_{((abc);\{de\}) = b_{d[dc];ab} + b_{d[bc];ae} + b_{d[de];be}, \end{align*}
\]
It is remarkable, upon noticing the identity \( r_{ab:cd} - r_{(ab)\cdot(c)d} = 3r_{(ab;cd)} \), that according to Equations (96)–(99) a tensor \( r_{ab:cd} \) with Young symmetry type (2, 2) has the same algebraic symmetries as a Riemann curvature tensor (anti-symmetry in \( ab \) and \( cd \), interchange of \( ab \) with \( cd \), and the algebraic Bianchi identity). This fact is well-known [32], but not often mentioned in textbooks on relativity.

### A.2 Special algebraic and differential operators

Now, suppose that we are working on an \( n \)-dimensional pseudo-Riemannian manifold \((M, g)\), with Levi-Civita connection \( \nabla \). As in Section 2.1, the Young symmetrizes introduced above define vector bundles of Young symmetrized covariant tensors \( Y^d T^* M \rightarrow M \), where \( d \) stands for a Young diagram. We define special linear algebraic and differential operators, already briefly discussed in Section 2.2, between these Young symmetrized tensor bundles occurring in the Calabi complex. Each of the corresponding Young diagrams has at most two columns, where the first column usually has at most \( n \) cells and the second column has at most two cells. The operator \( \text{trace} \) \((\text{tr})\) removes one row of cells, \( \text{metric exterior product} \) \((g \odot -)\) adds one row of cells, \( \text{left or right exterior derivative} \) \((d_L \) and \( d_R)\) adds one cell to the left or right column respectively, and \( \text{left or right divergence} \) \((\delta_L \) and \( \delta_R)\) removes one cell from the left or right column respectively. The name of each of these operators should be suggestive of their form, with the main complication being to maintain appropriate Young symmetry.

In principle, the Littlewood-Richardson decomposition rules uniquely fix the principal symbols of each of these operators up to a scalar multiple, with the Levi-Civita operator canonically converting a first order principal symbol into a first order operator. In practice, it takes a bit of work to find explicit formulas for them, given that a naive application a Young symmetrizer produces unmanageably large expressions. Moreover, the existence of the intracolumn and intercolumn symmetrization identities introduces non-uniqueness into possible explicit expressions. Below, we give explicit formulas for these operators. In case of ambiguity, the choice was dictated by practical convenience. Then, in Secs. A.3 and A.4 we show by explicit calculation that they satisfy the required symmetrization identities and thus carry the correct Young type.

\[
\text{tr}[b]_{a_1\cdots a_l;b} = b_{a_1\cdots a_{l-1}c}^c, \quad \text{(104)}
\]
\[
(g \odot t)_{a_1\cdots a_l;bc} = \left( g_{b[a_1}^b t_{a_2\cdots a_l]c}^c - g_{c[a_1} t_{a_2\cdots a_l]b}^b \right), \quad \text{(105)}
\]
\[
d_L[b]_{a_1\cdots a_l;bc} = \left( \nabla b_{a_1b_2\cdots a_l;bc} \right), \quad \text{(106)}
\]
\[
\delta_L[b]_{a_1\cdots a_l;bc} = \nabla b_{a_1b_2\cdots a_l;bc} + 2l^{-1} \nabla b_{a_1\cdots a_l;c}^c, \quad \text{(107)}
\]
\[
d_R[t]_{a_1\cdots a_l;bc} = 2\nabla b_{a_1b_2\cdots a_l;c} + 2(l-1)^{-1} \nabla [b]_{a_1\cdots a_l;c}^c, \quad \text{(108)}
\]
\[
\delta_R[b]_{a_1\cdots a_l;b} = \nabla b_{a_1b_2\cdots a_l;bc}. \quad \text{(109)}
\]

Let us give an explicit example of (105) for \( l = 2 \), which appears in the formulas for the constant curvature Riemann tensor (123) and for the linearized Riemann...
curvature operator (7):

\[
\begin{align*}
(g \odot h)_{a_1a_2b;c} &= g_{a_1b}h_{a_2c} - g_{a_2b}h_{a_1c} - g_{a_1c}h_{a_2b} + g_{a_2c}g_{a_1b}, \\
(g \odot g)_{a_1a_2b;c} &= 2(g_{a_1b}g_{a_2c} - g_{a_2b}g_{a_1c}),
\end{align*}
\]

(110)

\[
(\nabla \nabla \odot h)_{a_1a_2b;c} = \nabla \nabla h_{a_1b}h_{a_2c} - \nabla \nabla h_{a_2b}h_{a_1c} - \nabla \nabla h_{a_1c}h_{a_2b} + \nabla \nabla h_{a_2c}g_{a_1b},
\]

where \( \nabla \nabla h_{ab} = \nabla (a \nabla b) = \frac{1}{2} (\nabla a \nabla b + \nabla b \nabla a). \)

In the last equation we used the \( \odot \) operation to define another differential operator of definite Young type. This property follows directly from that of (105).

### A.3 Preservation of Young type

Each of the operators (104)–(109) maps tensors of one Young type into another one, as is indicated by the index notation described in Sec. A.1. Below we explicitly demonstrate that, by showing that the result of applying one of these operators to a tensor of a given Young type always satisfies the required intracolumn and intercolumn identities.

First, we list some key identities satisfied by the idempotent antisymmetrization and column exchange operations. They follow from straightforward, though possibly lengthy, application of their definitions. Here, a tensor \( \iota_{a_1 \cdots a_l} \) is assumed to be fully antisymmetric. Also, to simplify the notation for nested operations, we use the notation \( \{ \cdots \}^k \), where the braces necessarily enclose the indices \( a_k, a_{k+1}, \ldots, a_l \), though perhaps also others, to mean that we apply the appropriate column exchange operation to these \( a_i \) indices as if they appeared in the order \( \{ a_k \cdots a_l \} \).

\[
(l + 1)p_{\{a_l a_{l+1} \cdots a_i \}} = p_{a_l} \iota_{a_1 \cdots a_i} = p_{\{a_l \} \iota_{a_1 \cdots a_i}},
\]

(113)

\[
p_{\{b\}} \iota_{\{a_1 \cdots a_{l-1}\}} = p_{a_1} \iota_{a_{l-1} a_1} + p_{\{b\}} \iota_{a_1 a_{l-1}},
\]

(114)

\[
p_{\{a_l \}} \iota_{\{a_1 \cdots a_{l-1}\}} = -p_{\{a_l \}} \iota_{a_1 a_{l-1}},
\]

(115)

\[
p_{(bqc)} \iota_{\{a_1 \cdots a_{l-1}\}} = p_{a_q} q_{(c)} \iota_{b_{a_1 a_{l-1}}} + p_{\{bqc\}} \iota_{a_1 a_{l-1}},
\]

(116)

\[
(p \iota_{\{a_1 a_{l-1}\}})[c'] \delta_{[b]c'}^{a} = p_{\{c\}} \iota_{\{a_1 \cdots a_{l-1}\};c},
\]

(117)

\[
(l' \iota_{\{a_1 \cdots a_{l-1}\}})[c'] \delta_{[a]c'}^{a} = -2(l - 1)p_{\{a_l \}} \iota_{a_{l-1} a_1},
\]

(118)

\[
p_{\{b\}} \iota_{\{a_1 \cdots a_{l-1}\}} = l p_{\{a_l \}} \iota_{a_{l-1} a_1},
\]

(119)

\[
p_{\{b\}} q_{\{c\}} \iota_{\{a_1 \cdots a_{l-1}\}} = p_{\{c\}} q_{\{b\}} \iota_{\{a_1 \cdots a_{l-1}\}} + (p_{\{b\}} q_{\{c\}} - q_{\{b\}} p_{\{c\}}) \iota_{\{a_1 \cdots a_{l-1}\}},
\]

(120)

\[
\left(p_{\{b\}} \iota_{\{a_1 \cdots a_{l-1}\}}\right)_{[c'] \delta_{[a]c'}^{a} \iota_{\{a_1 \cdots a_{l-1}\}}} = 2(l - 1)p_{\{a_l \}} \iota_{a_{l-1} a_1} - 2(l - 2)p_{\{a_l \}} \iota_{a_{l-1} a_1},
\]

(121)

\[
p_{\{b\}} \iota_{\{a_1 \cdots a_{l-1}\}} \iota_{\{b\} \iota_{\{a_1 \cdots a_{l-1}\}}} = p_{\{a_l \}} \iota_{\{a_1 \cdots a_{l-1}\}} \iota_{\{a_1 \cdots a_{l-1}\}},
\]

(122)
use of the symmetrization properties of the Young type tensors on which the operations are being performed.

For the trace (104), the intracolumn identities are obvious, so there is only one intercolumn identity to check:

\[
\text{tr}[b](a_{1 \cdots a_1} \cdot \{b\}) = b(a_{1 \cdots a_1} \cdot \{b\}) - b(a_{1 \cdots a_1} \cdot \{b\})_{:c} = b(a_{1 \cdots a_1} \cdot b)_{:c} = \text{tr}[b]_{a_1 \cdots a_1}.b.
\]

For the metric exterior product (105), the intracolumn identities are obvious, so there are two intercolumn identities to check:

\[
(g \otimes l)(a_{1 \cdots a_1} \cdot \{b\})_{:c} = l(g(b)(a_{1 \cdot a_2 \cdots a_1})x - g_c(\{a_{1 \cdot a_2 \cdots a_1}\} \cdot \{b\})) = -l(l + 1)(g(b)(a_{1 \cdot a_2 \cdots a_1})x - g_c(\{a_{1 \cdot a_2 \cdots a_1}\} \cdot \{b\}) + l(g(b)(a_{1 \cdot a_2 \cdots a_1})x - g_c(\{a_{1 \cdot a_2 \cdots a_1}\} \cdot \{b\}) = (g \otimes l)_{a_1 \cdots a_1}.bc,
\]

\[
(g \otimes l)(a_{1 \cdots a_1} \cdot \{bc\}) = l(g(b)(a_{1 \cdot a_2 \cdots a_1})x - g_c(\{a_{1 \cdot a_2 \cdots a_1}\} \cdot \{bc\}) \delta^a_b \delta^a_c = (g \otimes l)(a_{1 \cdots a_1} \cdot \{bc\}) = (g \otimes l)_{a_1 \cdots a_1}.bc.
\]

The double anti-symmetrizations vanished because of the identities \(g_{ab} = 0\) and \(t_{a_2 \cdots a_1} = 0\), with the latter following from a combination of (113) and an intercolumn identity. Also, we have used the fact that \(p_{a_1 t_{a_2 \cdots a_1}}\) is a tensor of the corresponding Young type, which follows from the identities in the paragraph below.

For the left exterior derivative (106), the intracolumn identities are obvious, so there are two intercolumn identities to check:

\[
d_L[b](a_{1 \cdots a_1} \cdot \{b\})_{:c} = l(\nabla(\{a_{1}b_{a_2 \cdots a_1}\}) \cdot \{b\})_{:c} = -l(l + 1)\nabla(\{a_{1}b_{a_2 \cdots a_1}\})_{:bc} = d_L[b](a_{1 \cdots a_1} \cdot \{bc\} = \nabla(\{a_{1}b_{a_2 \cdots a_1}\})_{:bc} = \nabla_{a_1} b_{a_2 \cdots a_1} + \nabla_{b} b_{a_2 \cdots a_1} a_1 \cdot \{c\} - \nabla_{a_1} b_{a_2 \cdots a_1} \cdot \{bc\} - \nabla_{b} b_{a_2 \cdots a_1} \cdot \{c\} = (\nabla(\{a_{1}b_{a_2 \cdots a_1}\})_{:c} - \nabla_{a_1} b_{a_2 \cdots a_1} \cdot \{b\} - \nabla_{b} b_{a_2 \cdots a_1} \cdot \{c\}) = (\nabla(\{a_{1}b_{a_2 \cdots a_1}\})_{:c} - \nabla_{a_1} b_{a_2 \cdots a_1} \cdot \{b\} - \nabla_{b} b_{a_2 \cdots a_1} \cdot \{c\}).
\]

For the left divergence (107), the intracolumn identities are obvious. It
remains to check the two intercolumn identities:

\[ \delta_L[b_{(a_1 \cdots a_l)}:(b) c] = \nabla^a(b_{[a_1 \cdots a_l]}c) + l^{-1}b_{(b) \{a_1 \cdots a_l\} c a} - l^{-1}b_{c \{a_1 \cdots a_l\} \{b\} a} \]
\[ = \nabla^a(b_{[a_1 \cdots a_l]}c) + b_{ba_1 \cdots ca} \quad (114) \]
\[ - l^{-1}(l + 1)b_{[ba_1 \cdots ca] c a} + l^{-1}b_{ba_1 \cdots ca} \quad (113) \]
\[ - l^{-1}b_{\{ca_1 \cdots a_l\} \{b\} a} + l^{-1}b_{ba_1 \cdots ca} \quad (114) \]
\[ = \nabla^a(b_{ba_1 \cdots c a} + 2l^{-1}b_{[a_1 \cdots a_l c] a}) = \delta_L[b_{(a_1 \cdots a_l b c)}] \]

\[ \delta_L[b_{(a_1 \cdots a_l)}:(bc) c] = \nabla^a(b_{[a_1 \cdots a_l]}c) \quad (116) \]
\[ + l^{-1}(b_{a' \{a_1 \cdots a_l\} c'} a - b_{c' \{a_1 \cdots a_l\} b a}) \delta^a_{c'a'} \quad (118) \]
\[ = \nabla^a(b_{[a_1 \cdots a_l]}c) + b_{a_1 \cdots a_l \{c\} a} \quad (114) \]
\[ - l^{-1}(l - 1)(b_{ba_1 \cdots ca} - b_{ca_1 \cdots ba}) \]
\[ = \nabla^a(b_{ba_1 \cdots c a} + b_{ca_1 \cdots ba} - b_{ca_1 \cdots ba}) \]
\[ = \delta_L[b_{(a_1 \cdots a_l b c)}] \]

For the right exterior derivative (108), the following rewriting makes the intracolumn identities obvious:

\[ d_R[b_{(a_1 \cdots a_l)}] c = \nabla b_{ba_1 \cdots a_l c} - \nabla c b_{ba_1 \cdots a_l b} \]
\[ + (l - 1)^{-1}(\nabla b_{(a_1 \cdots a_l)} c - \nabla c b_{(a_1 \cdots a_l) \{b\}}) \quad (113) \]
\[ = \nabla b_{ba_1 \cdots a_l c} - \nabla b_{ba_1 \cdots a_l b} \]
\[ + (l - 1)^{-1}(\nabla b_{ba_1 \cdots a_l c} - \nabla b_{ba_1 \cdots a_l b}) \]
\[ - (l - 1)^{-1}(l + 1)(\nabla b_{ba_1 \cdots a_l c} - \nabla b_{ba_1 \cdots a_l b}) \]
\[ + (l - 1)^{-1}(\nabla b_{ba_1 \cdots a_l c} - \nabla b_{ba_1 \cdots a_l b}) \]
\[ = d_R[b_{(a_1 \cdots a_l)}] c \]

There are also two intercolumn identities to check:

\[ d_R[b_{(a_1 \cdots a_l)}] c = \nabla b_{ba_1 \cdots a_l c} - \nabla c b_{ba_1 \cdots a_l b} \]
\[ + (l - 1)^{-1}(\nabla b_{(a_1 \cdots a_l)} c - \nabla c b_{(a_1 \cdots a_l) \{b\}}) \quad (113) \]
\[ = \nabla b_{ba_1 \cdots a_l c} - \nabla b_{ba_1 \cdots a_l b} \]
\[ + (l - 1)^{-1}(\nabla b_{ba_1 \cdots a_l c} - \nabla b_{ba_1 \cdots a_l b}) \]
\[ - (l - 1)^{-1}(\nabla b_{ba_1 \cdots a_l c} - \nabla b_{ba_1 \cdots a_l b}) \]
\[ = d_R[b_{(a_1 \cdots a_l)}] c \]
\[ d_R[b_{(a_1 \cdots a_l)}] c = \nabla b_{ba_1 \cdots a_l c} - \nabla c b_{ba_1 \cdots a_l b} \]
\[ + (l - 1)^{-1}(\nabla b_{ba_1 \cdots a_l c} - \nabla c b_{ba_1 \cdots a_l b}) \]
\[ + (l - 1)^{-1}(l + 2)(\nabla b_{ba_1 \cdots a_l c} - \nabla c b_{ba_1 \cdots a_l b}) \]

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For the right divergence (109), the intracolumn identities are obvious, so there is only one intercolumn identity to check:
\[
\delta R[b]_{a_1 \ldots a_l ; b} = \nabla c b_{a_1 \ldots a_l ; c} = \nabla c b_{a_1 \ldots a_l ; bc} = \delta R[b]_{a_1 \ldots a_l ; b}.
\]

A.4 Composition identities

Below, we list identities between some possible compositions of the operators (104)–(109). These will be instrumental in the following Sec. A.5, where they will be used to explicitly define the operators involved in the Calabi complex (2) and the necessary identities between them. We do not show the necessary explicit calculations, as they are lengthy but straightforward. It suffices to make use of the key identities (113)–(122), as explicitly illustrated in Sec. A.3.

Recall that \( \nabla \) denotes the Levi-Civita connection on a pseudo-Riemannian space of constant curvature with metric \( g \) and dimension \( n \). The Riemann tensor on this space is defined by the convention \( 2 \nabla [a \nabla b] \omega_c = R_{ab;c} d \omega_d \) and is explicitly equal to
\[
R_{ab;c} d = \frac{\lambda}{2}(g \otimes g)_{ab;cd} = \lambda (g_{ac} g_{bd} - g_{ad} g_{bc}), \quad \text{with} \quad \lambda = \frac{k}{n(n-1)}, \quad (123)
\]
such that \( k = g^{ac} R_{ab;c} b \) is the curvature constant.

The simplest composition identity is of two left exterior derivatives \( (l \geq 4) \):
\[
d_L \circ d_L [b]_{a_1 \ldots a_l ; bc} = 0. \quad (124)
\]
The principal symbols of these operators augment the left index column of the argument and antisymmetrize over it, thus composing to zero, as in the case of the de Rham differential. This means that, at worst, the result of the composition of the operators is of order zero and proportional to the background curvature \( R \) given in (123). Now, note that the background curvature is \( GL(n) \)-equivariantly composed only out of the metric and the composition \( d_L \circ d_L \) is also an equivariant operator (taking into account the transformation properties of the covariant derivative and the metric). Then the result of the composition (Young type \( (2, 2, 1^{l-2}) \)) must be equivariantly composed only out of the metric \( g \) (Young type \( (2) \)) and the argument \( b \) (Young type \( (2, 2, 1^{l-4}) \)). However, according the Littlewood-Richardson rules \([33, 59]\), there is no non-trivial combination of that kind. Therefore, the composition of these operators must vanish.

Next, we show the relation between the compositions \( \delta_L \circ d_L \) and \( d_L \circ \delta_L \), along with some auxiliary identities involving the curvature. These formulas hold when the length of the left index column of the output is \( l > 2 \).

\[
2 \nabla [a \nabla b]_{a_1 \ldots a_l ; cd} = (R \cdot b)_{ab a_1 \ldots a_l ; cd}, \quad (125)
\]
\[
(R \cdot b)_{ab a_1 \ldots a_l ; cd} = R_{ab;c} e_{b_{a_1 \ldots a_l ; cd}} + R_{ab;c} e_{b_{a_1 \ldots a_l ; cd}} + R_{ab;d} e_{b_{a_1 \ldots a_l ; cd}}
\]
\[
= \lambda (g_{a[b} - g_{b[a}) b_{a_1 \ldots a_l ; cd}
\]
\[
+ \lambda (g_{ac} b_{a_1 \ldots a_l ; bd} - g_{bc} b_{a_1 \ldots a_l ; ad})
\]
\[
- \lambda (g_{ad} b_{a_1 \ldots a_l ; bc} - g_{bd} b_{a_1 \ldots a_l ; ac}); \quad (126)
\]
(l + 1)(\bar{R} \cdot b)_{a[a_1 \cdots a_l]bc} \\
= -l(l + 1)\lambda g_{a[a_1 \cdots a_l]bc} \\
- (l + 1)\lambda(g_{a[b_a[a_1 \cdots a_l]ad} - g_d[b_a[a_1 \cdots a_l]ad]),
\\
(l + 1)(\bar{R} \cdot b)^{a[a_1 \cdots a_l]bc} \\
= -(l(n - l) + 2)\lambda b_{a_1 \cdots a_lbc} + (-)^l\lambda(g \circ \text{tr}[b])_{a_1 \cdots a_lbc},
\\
(l + 1)(\bar{R} \cdot b)^{a[b_{a_1 \cdots a_l}]ca} - (l + 1)(\bar{R} \cdot b)^{a[c_{a_1 \cdots a_l}]ba} \\
= 0;
\\
\delta_L \circ d_L[b]_{a_1 \cdots a_lbc} = \Box b_{a_1 \cdots a_lbc} - d_L \circ \delta_L[b]_{a_1 \cdots a_lbc} + l^{-1}d_R \circ \delta_R[b]_{a_1 \cdots a_lbc}
\\
- (l(n - l) + 2)\lambda b_{a_1 \cdots a_lbc} + (-)^l\lambda(g \circ \text{tr}[b])_{a_1 \cdots a_lbc}. \quad (127)

The main identity that will be useful in Section A.5 is the following:
\\
(\delta_L \circ d_L + d_L \circ \delta_L)[b]_{a_1 \cdots a_lbc} = \Box b_{a_1 \cdots a_lbc} - d_L \circ \delta_L[b]_{a_1 \cdots a_lbc}
\\
- (l(n - l) + 2)\lambda b_{a_1 \cdots a_lbc} + (-)^l\lambda(g \circ \text{tr}[b])_{a_1 \cdots a_lbc}. \quad (128)

The composition \(\delta_L \circ d_L\) has a special form when the length of the left index column of the output is \(l = 2\):
\\
\nabla_{(a}\dot{r}_{(a_1 a_2):bc} = -\nabla_{(c}r_{(a_1 a_2):ba} + \nabla_{(b}r_{(a_1 a_2):ca}; \quad (129)
\\
(\bar{R} \cdot r)^{a_{b_{a_1 a_2}:ca} = -(n - 1)\lambda r_{a_1 a_2bc} + \lambda g_{b_{a_1 \text{tr}[r]_{a_2}c} - g_{b_{a_2 \text{tr}[r]_{a_1}c}};
\\
(\bar{R} \cdot r)^{a_{b_{a_1 a_2}:ca} = -(\bar{R} \cdot r)^{a_{b_{a_1 a_2}:ca}} - 2(n - 1)\lambda r_{a_1 a_2bc} + \lambda(g \circ \text{tr}[r])_{a_1 a_2bc},
\\
(\bar{R} \cdot r)^{a_{b_{a_1 a_2}:ca} = \{\bar{R} \cdot r\}^{a_{b_{a_1 a_2}:ba}} = 0;
\\
\\
\delta_L \circ d_L[r]_{a_1 a_2bc} = \Box r_{a_1 a_2bc} + \frac{1}{2}d_R \circ \delta_R[r]_{a_1 a_2bc}
\\
- 2(n - 1)\lambda r_{a_1 a_2bc} + \lambda(g \circ \text{tr}[r])_{a_1 a_2bc}. \quad (130)

Next, we show the relation between the compositions \(\text{tr} \circ d_L\) and \(d_L \circ \text{tr}\):
\\
\text{tr} \circ d_L[b]_{a_1 \cdots a_l:b} = d_L \circ \text{tr}[b]_{a_1 \cdots a_l:b} + (-)^l\delta_R[b]_{a_1 \cdots a_l:b}. \quad (131)

Next, we show the relations between the compositions \(d_L \circ d_R\), \(d_R \circ d_L\) and the operator \((\nabla \nabla \circ \text{tr})\), along with some auxiliary identities involving the curvature. Note that below we make use of the notation \(\cdot \cdot \cdot \) which denotes idempotent antisymmetrization of the indices \(a_1 \cdots a_l\) as if they were given in that position and ignoring any other indices appearing within the same brackets.
\\
2\nabla_{(a}\nabla_{b}b_{a_1 \cdots a_l:c} = (\bar{R} \cdot b)_{ab_{a_1 \cdots a_l:c}}, \quad (132)
\\
(\bar{R} \cdot b)_{ab_{a_1 \cdots a_l:c} = \bar{R}_{ab{d_{a_1 \cdots a_l}}:c} + \bar{R}_{ab{c_{a_1 \cdots a_l}}:d}
\\
= \lambda g_{b_{a_1 \cdot \cdot \cdot a_l:b}} - g_{b_{a_1 \cdot \cdot \cdot a_l:a}}; \quad (133)
The main identity that will be useful in Section A.5 is the following:

\[ l(\tilde{R} \cdot b)_{[a_1 a_2 \cdots a_l]}^e = -l^2 \lambda g_{[a_1 b_2 a_3] c} + \lambda(g \circ b)_{a_1 a_2 a_3} \]
\[ 2l(\tilde{R} \cdot b)_{[a_1 a_2 \cdots a_l]}^e = -(l-2) \lambda(g \circ b)_{a_1 a_2 a_3} \]
\[ l(\tilde{R} \cdot b)_{(a_1 a_2 a_3)}^e = -l^2 \lambda g_{[a_1 b_2 a_3] c} + \lambda(g \circ b)_{a_1 a_2 a_3} \]
\[ 2l(\tilde{R} \cdot b)_{(a_1 a_2 a_3)}^e = -(l-2) \lambda(g \circ b)_{a_1 a_2 a_3} \]
\[ Q_{b \cdot a_1 a_2 a_3} = (\tilde{R} \cdot b)_{a_1 a_2 a_3} \]
\[ l(\tilde{R} \cdot b)_{[a_1 a_2 a_3]}^e = l(\tilde{R} \cdot b)_{(a_1 a_2 a_3)}^e \]
\[ l(\tilde{R} \cdot b)_{(a_1 a_2 a_3)}^e = 2(\lambda(g \circ b)_{a_1 a_2 a_3}) \]

The main identity that will be useful in Section A.5 is the following:

\[ l^{-1} d_R \circ d_L - (l-1)^{-1} d_L \circ d_R = -\lambda(g \circ b)_{a_1 a_2 a_3} \]

A.5 Calabi complex and its homotopy formulas

Below, we use the special differential operators introduced earlier in Section A.2 to explicitly define the differential operators \( B_1, E_l \) and \( P_l \) that make up the Calabi complex and its homotopy formulas, as discussed in more detail in Section 2.2:

\[ B_1[v]_{a,b} = \nabla_a v_b + \nabla_b v_a \]
\[ B_2[h]_{a_1 a_2 b c} = (\nabla \nabla \circ h)_{a_1 a_2 b c} + \lambda(g \circ h)_{a_1 a_2 b c} \]
\[ B_l[b]_{a_1 a_2 a_3} = d_L[b]_{a_1 a_2 a_3} \quad (l \geq 3) \]
\[ E_1[h]_{a} = \nabla^b h_{a b} - \frac{1}{2} \nabla_a \tr[h] \]
\[ E_2[h]_{a b} = \tr[h]_{a b} \]
\[ E_{l+1}[b]_{a_1 a_2 a_3} = (d_L - (-)^l l^{-1} d_R \circ \tr) [b]_{a_1 a_2 a_3} \quad (l \geq 2) \]

Explicit formulas for \( B_l \) and \( E_l \) with low \( l \) have been given in Section 2.2.
Further, we make use of the identities given in Section A.4 to show that these operators satisfy the required identities, namely $B_{i+1} \circ B_i = 0$. The identities $B_2 \circ B_1 = 0$ and $B_3 \circ B_2 = 0$ have already been shown to follow in Section 2.2 from the usual transformation properties of the Riemann curvature tensor under diffeomorphisms and from its Bianchi identities. The identities $B_{i+1} \circ B_l = 0$ for $l > 2$ then follow directly from the composition identity $d_L \circ d_L = 0$ in Equation (124).

Again, appealing to the identities of Section A.4, we give the homotopy formulas $P_l = E_{l+1} \circ B_{l+1} + B_l \circ E_l$ for $l \leq 2$:

$$E_1 \circ B_1[v]_{a:b} = \nabla^b(\nabla_a v_b + \nabla_b v_a) - \frac{1}{2} \nabla_a(2\nabla^b v_b)$$

$$P_0 = \Box v_a + \lambda(n - 1)v_a,$$

$$(E_2 \circ B_2 + B_1 \circ E_1)[h]_{a:b}$$

$$= (\nabla \nabla \circ h)_{ac:b}^e + \lambda (g \circ h)_{ac:b}^e$$

$$+ \nabla_a \left( \nabla^c h_{bc:a} - \frac{1}{2} \nabla_b \text{tr}[h] \right) + \nabla_b \left( \nabla^c h_{ac:b} - \frac{1}{2} \nabla_a \text{tr}[h] \right)$$

$$P_1 = \Box h_{ab} - 2\lambda h_{ab} + 2\lambda g_{ab} \text{tr}[h],$$

$$(E_3 \circ B_3 + B_2 \circ E_2)[r]_{a_1a_2:b:c}$$

$$= \delta_L \circ d_L[r]_{a_1a_2:b:c} - \frac{1}{2} d_R \circ \text{tr} \circ d_L[r]_{a_1a_2:b:c}$$

$$+ (\nabla \nabla \circ \text{tr}[r])_{a_1a_2:b:c} + \lambda (g \circ \text{tr}[r])_{a_1a_2:b:c}$$

$$P_2 = \Box r_{a_1a_2:b:c} - 2(n - 1)\lambda r_{a_1a_2:b:c} + 2\lambda (g \circ \text{tr}[r])_{a_1a_2:b:c}.$$ 

Finally, the same set of identities also implies the following formulas for $P_l$ with $l > 2$:

$$(E_{l+1} \circ B_{l+1} + B_l \circ E_l)[b]_{a_1 \cdots a_l:b:c}$$

$$= \delta_L \circ d_L + d_L \circ \delta_L[b]_{a_1 \cdots a_l:b:c}$$

$$- (-1)^l (l^{-1}d_R \circ \text{tr} \circ d_L - (l - 1)^{-1}d_L \circ d_R \circ \text{tr}) [b]_{a_1 \cdots a_l:b:c}$$

$$P_l = \Box h_{a_1 \cdots a_l:b:c} - \left((n - l) + 2\lambda h_{a_1 \cdots a_l:b:c} + (-1)^l 2\lambda (g \circ \text{tr}[b])_{a_1 \cdots a_l:b:c}\right).$$

Explicit formulas for $P_l$ with low $l$ have also been given in Section 2.2. Recall that, as in Equation (123), we have used the notation $\lambda = \frac{k}{n(n-1)}$.

### A.6 An adjoint operator

Here we derive Equation (30), which according to the general formula (22) implies that $-n^{-1}\delta_L$ is the formal adjoint of $d_L$ when acting on tensors of Young type $c_{a_2 \cdots a_n:b:c}$. 

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\[
\n\nabla_a (c_{a_2 \cdots a_n} b_{a_2 \cdots a_n}) = c_{a_2 \cdots a_n} \nabla a b_{a_2 \cdots a_n} + (\nabla c_{a_2 \cdots a_n} b_{a_2 \cdots a_n}) - \frac{1}{n-1} (\nabla c_{a_2 \cdots a_n} c_{a_2 \cdots a_n} - \nabla \delta L[c] c_{a_2 \cdots a_n} b_{a_2 \cdots a_n}) \\
= \frac{1}{n} (c_{a_2 \cdots a_n} b_{a_2 \cdots a_n} - \delta L[c] c_{a_2 \cdots a_n} b_{a_2 \cdots a_n}) (113)
\]

We have simply used the definitions of the \(d_L\) and \(\delta_L\) differential operators as well as the fact that the contraction of two tensors, one of which being totally anti-symmetric in a subset of indices, allows the insertion of an anti-symmetrization over the corresponding indices of the second tensor. Finally, some of the anti-symmetrizations annihilated the corresponding tensors, due to their intercolumn identities and the application of the identity (113).

### B Homological algebra

Below we introduce some basic notions from homological algebra. A standard text on the subject is [88], where more details can be found along with complete proofs.

Let \(A_i\), also denoted \(A_i\), be a sequence of vector spaces (real vector spaces, for our purposes) with linear maps \(A_i \to A_{i+1}\) between them. If each successive pair of maps \(A_{i-1} \to A_i \to A_{i+1}\) composes to zero, this sequence is called a complex (of vector spaces) or a cochain complex, with an element \(a \in A_i\) being referred to as a cochain (of degree \(i\)), and the maps \(A_i \to A_{i+1}\) referred to as cochain differentials. Any complex gives rise to cohomologies

\[
H^i(A_\bullet) = \ker(A_i \to A_{i+1}) / \text{im}(A_{i-1} \to A_i). \quad (138)
\]

If all the cohomologies vanish, \(H^i(A_\bullet) = 0\) or the image of each map is equal to the kernel of the subsequent map, the complex is called exact or an exact sequence. Given two complexes \(A_\bullet\) and \(B_\bullet\), the vertical maps in the diagram

\[
\begin{array}{ccccc}
\cdots & \longrightarrow & A_i & \longrightarrow & A_{i+1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & B_i & \longrightarrow & B_{i+1} & \longrightarrow & \cdots
\end{array}
\]

are called cochain maps provided they make the diagram commute. Furthermore, the diagonal maps in a diagram like

\[
\begin{array}{ccccc}
\cdots & \overset{d}{\longrightarrow} & A_i & \overset{d}{\longrightarrow} & A_{i+1} & \overset{d}{\longrightarrow} & \cdots \\
\overset{\delta}{\downarrow} & \overset{h}{\delta} & \downarrow & \overset{h}{\delta} & \downarrow & \overset{h}{\delta} & \cdots \\
\cdots & \overset{d}{\longrightarrow} & B_i & \overset{d}{\longrightarrow} & B_{i+1} & \overset{d}{\longrightarrow} & \cdots
\end{array}
\]

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are called **cochain homotopies**. The homotopy maps induce vertical cochain maps by the formula $h = d\delta + \delta d$. It is a basic fact that cochain maps $A_* \to B_*$ naturally induce maps in cohomology $H^i(A_*) \to H^i(B_*)$. Of course, identity chain maps induce identity maps in cohomology and zero chain maps induces zero maps in cohomology. Also, two cochain maps induce the same map in cohomology when their difference is induced by a cochain homotopy.

A **short exact sequence**

$$
\begin{array}{ccc}
0 & \longrightarrow & A_* \\
& \longrightarrow & B_* \\
& \longrightarrow & C_* \\
& \longrightarrow & 0 \\
\end{array}
$$

(141)

between complexes $A_*$, $B_*$ and $C_*$ consists of cochain maps between them such that each instance of

$$
\begin{array}{ccc}
0 & \longrightarrow & A_i \\
& \longrightarrow & B_i \\
& \longrightarrow & C_i \\
& \longrightarrow & 0 \\
\end{array}
$$

(142)

is an exact sequence of vector spaces. Another basic fact of homological algebra is that a short exact sequence of complexes induces the following **long exact sequence** in cohomology

$$
\cdots \longrightarrow H^i(A_*) \longrightarrow H^i(B_*) \longrightarrow H^i(C_*) \longrightarrow H^{i+1}(A_*) \longrightarrow \cdots
$$

(143)

where the maps $H^i(A_*) \to H^i(B_*)$ and $H^i(B_*) \to H^i(C_*)$ are induced by the cochain maps from the short exact sequence and the **connecting maps** $H^i(C_*) \to H^{i+1}(A_*)$ are induced by the cochain differential.

Finally, another standard result is the so-called 5-**lemma** (or a simple variant thereof). It states that the central vertical map in the commutative diagram

$$
\begin{array}{cccccc}
A_{-2} & \longrightarrow & A_{-1} & \longrightarrow & A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 \\
\cong & & \cong & & \cong & & \cong & & \cong \\
B_{-2} & \longrightarrow & B_{-1} & \longrightarrow & B_0 & \longrightarrow & B_1 & \longrightarrow & B_2 \\
\end{array}
$$

(144)

is an isomorphism, provided that the top and bottom rows are exact sequences and all the other vertical maps are isomorphisms themselves.

### C  Jets and jet bundles

In this appendix, we briefly introduce jet bundles, fix the relevant notation and discuss differential operators in the context of jets. For simplicity, we restrict ourselves to fields taking values in vector bundles. However, the discussion could be straightforwardly generalized to general smooth bundles. More details, as well as a coordinate independent definition, can be found in the standard literature [61, 53].

Given a vector bundle $F \to M$ over a connected $n$-dimensional smooth manifold $M$, the $k$-**jet bundle** $J^k F \to M$ is a vector bundle whose defining characteristic is that for any (possibly non-linear) differential operator $f : \Gamma(F) \to \Gamma(F')$ of order $k$, there exists a canonical factorization $f[u] = f \circ j^k u$ for any section...
\( u: M \to F \), where the \( k \)-jet prolongation \( j^k: \Gamma(F) \to \Gamma(J^k F) \) is composed with a smooth bundle map \( f: J^k F \to F' \), which by a slight abuse of notation we denote using the same symbol as the original differential operator. Composing the differential operator \( f \) with an \( l \)-jet prolongation canonically defines a new differential operator \( p_l f: J^{l+k} F \to J^l F' \) called its \( l \)-prolongation, \( j^l f[u] = p_l f \circ j^k u \).

Given a trivializable restriction \( F_U \to U \) of \( F \) to a chart \( U \subset M \) with local coordinates \((x')\) and fiber-adapted local coordinates \((x^i, u^a)\), there is a corresponding adapted chart \( J^k F_U \subset J^k F \) with adapted local coordinates \((x', u^a)\), where \( I = i_1 \cdots i_r \) runs through multi-indices of orders \( |I| = l = 0, \ldots, k \).

In these coordinates, the \( k \)-jet prolongation is given by \( j^k u(x) = (x', \partial_I u^a(x)) \), while the \( l \)-prolongation is given by \( p_l f[u](x) = (x', \partial_I f^b[u](x)) \), where \( f[u](x) = (x', f^b[u](x)) \) in fiber-adapted local coordinates \((x^i, v^a)\) on \( F' \). For any \( l > k \), discarding the information about all derivatives of order \( > k \) defines a natural projection \( J^l F \to J^k F \). The projective limit \( J^\infty F := \lim_{k \to \infty} J^k F \) defines the \( \infty \)-jet bundle. The \( \infty \)-jet prolongation \( j^\infty \) and \( \infty \)-prolongation \( p_\infty \) are defined in the obvious way. By composing with the natural projection \( J^\infty F \to J^k F \), the differential operator \( f \) also canonically defines the smooth bundle map \( f: J^\infty F \to J^k F \xrightarrow{j^k} F' \), which is again denoted by the same symbol \( f \).

Conversely, due to the projective limit construction, any smooth bundle map \( f: J^\infty F \to F' \) can only depend on finitely many coordinates of its domain, which means that there exists a \( k \geq 0 \) such that this bundle map canonically factors as \( f: J^\infty F \to J^k F \xrightarrow{j^k} F' \), with the smallest such \( k \) being the order of \( f \).

Given vector bundles \( F \to M, E \to M \) and a differential operator \( c: \Gamma(F) \to \Gamma(E) \), we write down the partial differential equation (PDE) \( c[\psi] = 0 \), with \( \psi \in \Gamma(F) \). Sometimes it is convenient to refer to \( F \to M \) as the field bundle and to \( E \to M \) as the equation bundle. We will only consider linear PDEs below, where the differential operator \( c \) is linear. We denote the local spaces of solutions by \( S_\psi(U) \), where \( U \subset M \) is open and \( \psi \in \Gamma(F|_U) \) belongs to \( S_\psi(U) \) iff \( c[\psi] = 0 \) on \( U \). The PDE is said to be of order \( k \) if it can be written as \( c[\psi] = c(j^k \psi) \), where on the right-hand side we have a (linear) bundle map \( c: J^k F \to E \).

In adapted coordinates \((x', u^a)\) on \( F \), the PDE \( c[\psi] = 0 \) has the form \( e^l(x) \partial_I \psi(x) = 0 \). When the PDE is of order \( k \), the coefficients \( e^l(x) \) vanish for multi-indices with \(|I| > k \). The coefficients of the highest order derivatives, \( e^I(x) \) with \(|I| = k \), in fact transform as a tensor under coordinate changes and define a linear bundle map \( \sigma_e: F \cong S^k T^* M \to E \) called the principal symbol of \( e \). If we fix \((x, p) \in T^* M \), then the corresponding linear map \( \sigma_{e(x, p)} = \sigma_e(x) \cdot \tilde{p}^{\otimes k}: F_x M \to E_x M \) can also be referred to as the value of the principal symbol of \( e \) at \((x, p)\).

The PDE \( c[\psi] = 0 \) is equivalent to the PDE \( c'[\psi'] = 0 \), with \( e': J^l F' \to F' \), if they have isomorphic solution spaces. That is \( c[\psi] = 0 \) implies that \( c'[\psi'] = 0 \) and \( c'[\psi'] = 0 \) implies that \( c[f'(\psi')] = 0 \), for some differential operators \( f \) and \( f' \). In fact, it can be shown that the two PDEs are equivalent precisely when they fit into the following diagram, where arrows are differential operators and
the bundle labels stand in for the corresponding spaces of sections,

\[
\begin{array}{ccc}
F & \xrightarrow{e} & E \\
\downarrow{f} & & \downarrow{g} \\
F' & \xrightarrow{f'} & E' \\
\end{array}
\]

where differential operators satisfy the following identities:

\[
\begin{align*}
\text{for } & e', f, g, q, e' \in \mathbb{F} \\
(145) \quad \ & e' \circ f = g \circ e, & f' \circ f = \text{id} + q \circ e, \\
& e \circ f' = g' \circ e', & f \circ f' = \text{id} + q' \circ e'.
\end{align*}
\]

The reason we can express equivalence in this way, at least when all the differential operators are linear, follows from linear algebra on jets. If we replace the operators \(e, e', f, f'\) by the corresponding jet bundle maps, prolonged to the appropriate order, it follows from basic linear algebra that there exist jet bundle maps that ostensibly correspond to the operators \(g, g', q, q'\). It then follows from a deeper analysis of the properties of linear PDEs \([38, 18, 73]\) that once these differential operators are defined using prolongations of sufficiently high order, the appropriate identities hold at all higher orders. As a simple example, note that the equation \(e[\psi] = 0\) and its prolongation \(p_k e[\psi] = 0\) are equivalent, with \(f = f' = \text{id}\).

Consider vector bundles \(E, F, G \to M\) and linear differential operators

\[
\begin{align*}
f : \Gamma(G) & \to \Gamma(F) \quad \text{and} \quad e : \Gamma(F) & \to \Gamma(E), \\
(148) & &
\end{align*}
\]

of respective orders \(k\) and \(l\), such that \(e \circ f = 0\). We say that the composition of \(e\) and \(f\) is formally exact if the composition \(p^{k+m}e \circ p^m f\) of jet bundle maps is exact in the usual linear algebra sense (the image of \(p^mf\) is equal to the kernel of \(p^{k+m}e\)). Formal exactness is a powerful hypothesis. For instance, it implies that certain differential operators factorise through either \(e\) or \(f\) \([38, 63]\). Namely, if \(g\) is any differential operator such that \(g \circ f = 0\), then there must exist another differential operator \(g'\) such that \(g = g' \circ e\). Similarly, if \(g\) is any differential operator such that \(e \circ g = 0\), then there must exist another differential operator \(g'\) such that \(g = f \circ g'\).

## D Deformations of flat principal bundles

The material below requires some familiarity with the theory of \(G\)-principal bundles \([81, 50, 57, 6]\). Its main point is to show how one can reduce the computation of the degree-1 cohomology space of a certain locally constant sheaf on a manifold \(M\) to the computation of the degree-1 group cohomology of the fundamental group \(\pi = \pi_1(M)\) with coefficients in a certain corresponding representation. This reformulation is a significant simplification because group cohomology calculations can often be reduced to finite dimensional linear algebra and many explicit calculations of that sort have already been performed and are available in the literature. The connection between these sheaf and group cohomologies is established by noticing that both of them describe equivalence
classes of infinitesimal deformations of flat principal bundles. Unfortunately, this argument is not sufficient to establish an isomorphism between these sheaf and group cohomologies in higher degrees, but degree-1 is already interesting because it is the one relevant in the physical application we have in mind (Section 5).

We briefly recall some basic facts about principal $G$-bundles \[81, 50, 57, 6\]. The total space of the principal bundle $P \to M$ has fibers that are right principal homogeneous spaces of the group $G$. A right principal homogeneous space is defined by the possession a free, transitive action of $G$. Thus, any principal homogeneous space is diffeomorphic to the manifold underlying the Lie group $G$ and, if any particular point is identified with the unit element of $G$, the action of $G$ coincides with right-multiplication. The fiber-wise right action of $G$ on $P$ allows us to construct so-called associated bundles. If $F$ is a left $G$-space, with action $\rho: G \to \text{Aut}(F)$, then we define the corresponding associated bundle, denoted sometimes $F \rho$ or $F \rho P$, as $P \times \rho F \sim = (P \times F) / G$, where the quotient identifies the points $(pg, f) = (p, gf), p \in P, f \in F, g \in G$. In particular, we can define the associated bundles $G_P = P \times _{Ad} G$ and $g_P = P \times _{Ad} g$, where $Ad$ denotes respectively the adjoint action of the Lie group on itself and its Lie algebra, $Ad(b) a = bab^{-1}$ and $Ad(b) \alpha = bab^{-1}$, with $a, b \in G$ and $\alpha \in g$. When convenient and for simplicity of notation, we shall implicitly treat Lie group and Lie algebra elements as if they were faithfully represented as matrices.

The principal $G$-bundle $P \to M$ is called flat when it is endowed with a flat connection or a notion of flat parallel transport, which are compatible with the structure group action. The details of these notions are discussed in the next subsections. The arguments presented therein roughly establish the following

**Proposition 17.** Let $P \to M$ be a flat principal $G$-bundle and $\pi = \pi(M)$ be the fundamental group of $M$. We can define the following structures associated to it: (a) the sheaf $F_g$ of locally flat sections of the associated bundle $g_P \to M$, (b) the twisted de Rham complex $(\Lambda^\bullet M \otimes g_P, D)$, and (c) the monodromy representation $\rho: \pi \to G$. Then the following cohomology groups (respectively the sheaf, twisted de Rham and group cohomologies) are all isomorphic, by reason of each being isomorphic to the space of equivalence classes of infinitesimal deformations of the flat principal $G$-bundle structure of $P \to M$: \[
H^1(M, F_g) \cong H^1(\Lambda^\bullet M \otimes g_P, D) \cong H^1(\pi, Ad_\rho). \tag{149}
\]

We defer to the standard references \[50, 57, 6\] for detailed proofs.

### D.1 Flat principle bundle cocycle

There are multiple ways to construct a principal $G$-bundle over a manifold $M$. The one that will be important for us here defines also a bit more structure than principal bundle itself, it also defines a flat connection thereon. We shall refer to these structures as flat principal $G$-bundles. It is well known that this data can be specified as follows. Let $\mathcal{U} = (U_i)$ be an open cover of $M$ and $(U, V) \mapsto t_{U,V} \in G$ an assignment of a structure group element to every ordered pair of opens $U, V \in \mathcal{U}$. Each $t_{U,V}$ is called a transition map. The transition maps define a principle $G$-bundle with a flat connection if they satisfy the following
coclty identities,

\[ t_{U,V}t_{V,U} = \text{id}, \]
\[ t_{U,V}t_{V,W}t_{W,U} = \text{id}. \]

A change of trivialization is an assignment \( U \mapsto a_U \in G \) for every open \( U \in \mathcal{U} \). The modified transition functions \( t'_{U,V} = a_U t_{U,V} a_V^{-1} \) define an equivalent flat principal \( G \)-bundle.

Next, we describe infinitesimal deformations of a flat bundle cocycle \( t_{U,V} \). Namely, suppose that \( t_{U,V}(s) \) is a smooth 1-parameter family of flat bundle cocycles, with \( t_{U,V}(0) = t_{U,V} \). Let us denote the derivative at \( s = 0 \) as \( t_{U,V} = \tau_{U,V}t_{U,V} \), with \( \tau_{U,V} \in \mathfrak{g} \). Then, the defining relations (150) and (151) impose the following constraints on the infinitesimal deformation \( \tau_{U,V} \):

\[ \tau_{U,V} = -t_{V,U}^{-1}\tau_{V,U}t_{U,V}, \]
\[ \tau_{U,V} + t_{U,V}\tau_{V,W}t_{W,U}^{-1} - \tau_{W,U} = 0. \]

On the other hand, suppose that \( a_U(s) \) is a smooth 1-parameter family of trivialization changes, with \( a_U(0) = \text{id} \). Let us write the derivative at \( s = 0 \) as \( \sigma_U = -\sigma_U \). The induced infinitesimal deformation in the transition functions \( t_{U,V}(s) = a_U(s)t_{U,V}a_V^{-1} \) is

\[ \tau_{U,V} = -\sigma_U + t_{U,V}\sigma_V t_{U,V}^{-1}. \]

The point of the above calculations is to show that infinitesimal deformations of the flat principal bundle cocycle, up to infinitesimal trivialization changes, correspond precisely to the cohomology classes of a certain sheaf. To complete the argument, we need only introduce the basic definitions of Čech cohomology, which is known to compute the cohomology spaces of a corresponding sheaf [16, 45]. We will take the sheaf to be \( F_\mathfrak{g} \), where \( F_\mathfrak{g}(U) \) consists of the locally flat sections of the bundle \( \mathfrak{g}_P \to M \), associated to the flat principal \( G \)-bundle \( P \to M \). Let us now fix an open cover \( \mathcal{U} = (U_i) \) of such that each \( U_i \) is contractible and any multiple intersection of the \( U_i \) is also contractible. On a manifold, any open cover can be refined to such a good cover [15, Thm.5.1]. In particular, the flat principal bundle cocycle can always be refined to a good cover. The good cover hypothesis ensures that the Čech cohomology spaces are in fact isomorphic to the actual sheaf cohomologies.

We define a Čech \( q \)-cochain \( \sigma \) as an assignment \((U_{i_1}, \ldots, U_{i_{q+1}}) \mapsto \sigma_{i_1 \cdots i_{q+1}} \in F_\mathfrak{g}(U_{i_1} \cap \cdots \cap U_{i_{q+1}})\) to every ordered \((q+1)\)-tuple of opens from \( \mathcal{U} \). By local flatness, for any \( U \in \mathcal{U} \), \( F_\mathfrak{g}(U) \cong F_\mathfrak{g} \cong \mathfrak{g} \)(cf. Section 3.1). It is convenient to think of a Čech cocycle \( \sigma_{i_1 \cdots i_{q+1}} \) as taking values in \( \mathfrak{g} \cong F_\mathfrak{g}(U_i) \). This means that \( \sigma_{i_1 \cdots i_{q+1}} \) and \( \sigma_j \) restrict to the same element of \( F_\mathfrak{g}(U_i \cap U_j \cap \cdots) \) if \( \sigma_{i_1 \cdots i_{q+1}} = \text{Ad}(t_{U_i, V_j})\sigma_{j \cdots} = (t_{U_i, V_j})\sigma_j \cdots (t_{U_i, V_j})^{-1} \). We shall only need the Čech differential to be defined on 0- and 1-cochains:

\[ (\delta \sigma)_{ij} = \sigma_j|_{U_i \cap U_j} - \sigma_i|_{U_i \cap U_j} \]
\[ = \text{Ad}(t_{U_i, U_j})\sigma_j - \sigma_i \]
\[ = t_{U_i, U_j}\sigma_j t_{U_i, U_j}^{-1} - \sigma_i, \]  

(155)

\[ (\delta \tau)_{ijk} = \tau_{jk}|_{U_i \cap U_j \cap U_k} - \tau_{ik}|_{U_i \cap U_j \cap U_k} + \tau_{ij}|_{U_i \cap U_j \cap U_k} \]
\[ = \text{Ad}(t_{U_i, U_j})\tau_{jk} - \tau_{ik} + \tau_{ij} \]
\[ = t_{U_i, U_j}\tau_{jk} t_{U_i, U_j}^{-1} - \tau_{ik} + \tau_{ij}. \]  

(156)
The space of closed Čech $q$-cocycles modulo the Čech coboundaries then is isomorphic to the sheaf cohomology group in degree $q$, which in our case is $H^q(F_g)$.

It should now be clear, from Equations (153) and (154), that the infinitesimal deformation of the flat bundle cocycle defines a Čech 1-cocycle $\tau_{ij} = \tau_{U_i U_j}$, and an infinitesimal change in trivialization defines a Čech coboundary $\tau_{ij} = (\delta \sigma)_{ij}$, with $\sigma_i = \sigma_{U_i}$.

D.2 Flat connection on a principal bundle

Another, ultimately equivalent, way to specify a principal bundle with a flat connection is as follows.

A principal $G$-connection on a principal $G$-bundle $P$ is a $g$-valued 1-form $\omega$ on the total space $P$ (an element of $\Omega^1(P) \otimes g$) such that (i) $\omega$ is Ad-equivariant ($R^*_a \omega = \text{Ad}(a^{-1}) \omega$, where $R_a : P \to P$ is the action of $a \in G$ on $P$ by right multiplication) and (ii) $\omega(\beta) = \beta$ for any vertical vector $\beta \in TP$. Recall that vertical vectors are those annihilated by the tangent map of the projection $P \to M$ and that the vertical subspace of $TP$ at any point of $P$ may be naturally identified with $g$, which we have used in the preceding definition. The defining condition on the form $\omega$ is clearly linear inhomogeneous. Thus, the space of all principal $G$-connections forms an affine subspace of $\Omega^1(P) \otimes g$. So, the difference $A = \omega' - \omega$ between any two principal connections belongs to the subspace of $\Omega^1(P) \otimes g$ that is Ad-equivariant and horizontal (annihilates vertical vectors). This subspace is in fact isomorphic, by pullback along the projection $P \to P/G \cong M$, to the space of sections $\Gamma(\Lambda^1 M \otimes g_P)$ of the associated bundle $\Lambda^1 M \otimes g_P \to M$. In fact, we can identify the spaces of sections $\Gamma(\Lambda^p M \otimes g_P)$ with the Ad-equivariant, horizontal subspaces of $\Omega^p (P) \otimes g$. The first order differential operator $DA = dA + [\omega \wedge A]$ (see below for notation) preserves these subspaces and hence can be projected down to a first order differential operator $D : \Gamma(\Lambda^p M \otimes g_P) \to \Gamma(\Lambda^{p+1} M \otimes g_P)$, which we shall refer to as the twisted differential (cf. Section 2.4).

The curvature $\Omega$ of a principal $G$-connection $\omega$ is defined to be the following $g$-valued 2-form on $P$:

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega],$$

where the bracketed wedge product is definite to satisfy $[(\lambda \otimes \alpha) \wedge (\mu \otimes \beta)] = \lambda \wedge \mu \otimes [\alpha, \beta]$ for any $\lambda, \mu \in \Omega^1(P)$ and $\alpha, \beta \in g$. Since $\Omega$ is Ad-equivariant and horizontal, we can equally write $\Omega \in \Gamma(\Lambda^2 M \otimes g_P)$. The twisted differential $D$ is not nilpotent, $D^2 \neq 0$. However, its square is $C^\infty(M)$-linear and so is a differential operator of order 0. In fact, we can compute it to be

$$D^2 A = [\Omega \wedge A],$$

for any $A \in \Gamma(\Lambda^p M \otimes g_P)$. If $\Omega = 0$, then the connection is said to be flat. This is a sufficient condition for the twisted differential to become nilpotent, $D^2 = 0$. A necessary and sufficient condition would simply be that the curvature $\Omega$ takes values in the center of $g$, upon local trivialization of $g_P$.

Given any two flat connections $\omega$ and $\omega'$, their difference can be represented by a section $\omega' - \omega = A \in \Gamma(\Lambda^1 M \otimes g_P)$ (or rather its pullback to $P$) that
necessarily satisfies the following equation:

\[ 0 = d\omega' + \frac{1}{2} [\omega' \wedge \omega'] \] (159)

\[ = d(\omega + A) + \frac{1}{2} (\omega + A) \wedge (\omega + A) \] (160)

\[ = dA + [\omega \wedge A] + \frac{1}{2} [A \wedge A] \] (161)

\[ = DA + \frac{1}{2} [A \wedge A]. \] (162)

Where the last expression can be interpreted as computed on \( M \) rather than on \( P \). Equating this last expression to zero gives a differential equation on sections \( A \in \Gamma(\Lambda^1 M \otimes g_P) \) identifying those that parametrize the space of flat principal \( G \)-connections (relative to \( \omega \), which defines the twisted differential \( D \)).

An automorphism of a principal \( G \)-bundle \( P \to M \) is a bundle map \( f: P \to P \) that covers the identity on \( M \) and is equivariant with respect to the right action of \( G \) on \( P \). It is a standard fact that such maps can be expressed as functions \( a_f: P \to G \) that are Ad-equivariant (where Ad is the left action of \( G \) on \( G \) by conjugation) with respect to the right action of \( G \) on \( P \). In turn, the set of such maps is in bijection with the space of sections of the associated bundle \( G_P = P \times_{Ad} G \). Since the map \( f: P \to P \) is an automorphism, the pullback connection \( f^* \omega \) is considered equivalent to the original one. Given its equivariance, the map \( f \) corresponds to a section \( a_f \in \Gamma(G_P) \).

Given a section \( a \in \Gamma(G_P) \), with \( a(0) = id \) and \( \dot{a}(0) = \alpha \in \Gamma(g_P) \), defines a smooth 1-parameter family of automorphisms \( f_{a(s)}: P \to P \), then the corresponding infinitesimal deformation of the original flat connection is is equal to \( A = d\alpha - [a, \omega] = D\alpha \).

It should now be clear that infinitesimal deformations of a given flat principal \( G \)-connection, up to infinitesimal automorphisms of the underlying principal \( G \)-bundle, are in bijections with the cohomology vector space \( H^1(\Lambda^* \otimes g_P, D) \) of the twisted de Rham complex defined by the original flat connection.

### D.3 Monodromy representation

A connection, in the sense of Ehresemann, can be defined as a splitting of the tangent space of \( P \) into \( T_{x,a} P \cong T_x M \oplus g \), for \( (x, a) \in P \), with the \( g \) summand canonically identified with the subspace of vertical vectors, such that...
the splitting is smooth in \( x \) and equivariant in \( a \). The \( T_xM \) summand is called the horizontal subspace of \( T_xM \). This formulation leads naturally to the idea of parallel transport. Given a point \((x, a) \in P\) and a smooth curve \( \gamma: [0, 1] \to M \) such that \( \gamma(0) = x \) and \( \gamma(1) = 1 \), there exists a unique lift \( \tilde{\gamma} \) of \( \gamma \) to \( P \) such that \( \tilde{\gamma}(0) = (x, a) \) and the tangent vector \( \dot{\tilde{\gamma}} \) is always horizontal. With \( x \) and \( y \) fixed, the endpoint \((y, b) = \tilde{\gamma}(1) \) defines \( b \) as the image of \( a \) parallel transported along \( \gamma \). Since the splitting of \( TP \) is equivariant with respect to the right action of \( G \) on \( P \), so is parallel transport. It is easy to see that parallel transport does not depend on the parametrization of \( \gamma \), is well defined also when \( \gamma \) is piecewise smooth, and respects concatenation, \( a_{\gamma \eta} = a_\gamma a_\eta \) for \( \gamma(1) = \eta(0) \) and \( \gamma \eta \) being the concatenated curve. In particular, if \( \gamma \) is a closed curve based at \( x \in M \) \((\gamma(0) = \gamma(1) = x)\), then the effect of parallel transport is equivalent to the right action on the fiber \( P_x \) by some element \( a_\gamma \in G \).

Let \( \pi = \pi_1(M, x) \) be the fundamental group of \( M \) based at some point \( x \). The connection splitting of \( TP \) is called flat when the parallel transport along any closed contractible curve \( \gamma \) is trivial, \( a_{\gamma} = \text{id} \), and thus the group element \( a_{\gamma} \) effecting parallel transport along a closed curve \( \gamma \) based at \( x \) depends only on its homotopy type \([\gamma] \in \pi\). In other words, parallel transport defines a homomorphism \( \rho: \pi \to G, \rho([\gamma]) = a_{\gamma} \), which we call the monodromy representation of the fundamental group of \( M \) in the structure group of \( P \to M \) (cf. the introduction to Section 4).

Thus, any flat principal \( G \)-bundle gives rise to a representation \( \rho: \pi \to G \). Two isomorphic flat principal bundles give rise to equivalent monodromy representations, where two representations \( \rho' \) and \( \rho \) are equivalent if there exists an element \( a \in G \) such that \( \rho'([\gamma]) = a\rho([\gamma])a^{-1} \). Conversely, any homomorphism \( \rho: \pi \to G \) allows us to construct a flat principal \( G \)-bundle with a monodromy representation equivalent to \( \rho \). Namely, consider the universal cover \( \tilde{M} \to M \) as a principal \( \pi \)-bundle and define the total space of the corresponding principal \( G \)-bundle as \( P = \tilde{M} \times_{\rho} G \). A flat connection can be defined on the trivial principal \( G \)-bundle \( \tilde{M} \times \tilde{G} \) using the construction of Section D.1 applied to an cover by contractible open sets and transition maps defined by \( \rho \). This flat connection then projects down to \( P \).

Next, we describe infinitesimal deformations of a fixed monodromy representation \( \rho \). Let \( \rho_s: \pi \to G \) be a smooth 1-parameter family of monodromy representations, with \( \rho(s) = \rho \) and \( \dot{\rho}_s(a) = \tau(s)\rho(a) \) for some \( \tau: \pi \to \mathfrak{g} \). The representation property \( \rho_s([\gamma][\eta]) = \rho_s([\gamma])\rho_s([\eta]) \) imposes the following constraint on the infinitesimal deformation:

\[
\tau([\gamma]) + \rho([\gamma])\tau([\eta])\rho([\eta])^{-1} - \tau([\gamma][\eta]) = 0. \tag{164}
\]

A family of trivial deformations is given by \( \rho_s([\gamma]) = a_s\rho([\gamma])a_s^{-1} \) for a smooth 1-parameter family \( a_s \in G \), with \( a_0 = \text{id} \) and \( a_0 = -\sigma \in \mathfrak{g} \). The corresponding infinitesimal deformation of the representation is given by

\[
\tau([\gamma]) = -\sigma + \rho([\gamma])\sigma\rho([\gamma])^{-1}. \tag{165}
\]

The point of the above calculations is to show that these infinitesimal deformations can be identified with certain group cohomology classes. To see that, we need to introduce some basic definitions [88, Ch.6], [91]. Group cohomology is defined given a group and a representation thereof. We will give the definitions by directly taking the group to be \( \pi \) and the representation to be the composite
adjoint representation of \(\pi\) on \(\mathfrak{g}\), \(\text{Ad}_\rho = \text{Ad} \circ \rho: \pi \to \text{GL}(\mathfrak{g})\). The vector space \(C^p(\pi, \text{Ad}_\rho)\) of \(p\)-cochains consists of functions \(\sigma: \pi^p \to \mathfrak{g}\), where \(\pi^p = \pi \times \cdots \times \pi\) is the \(p\)-fold product. The cochain differentials \(\delta: C^p(\pi, \text{Ad}_\rho) \to C^{p+1}(\pi, \text{Ad}_\rho)\) are defined by the formula

\[
\delta \sigma([\gamma_1], \ldots, [\gamma_{p+1}]) = (-1)^{p+1} \sigma([\gamma_1], \ldots, [\gamma_p]) \\
+ \text{Ad}_\rho([\gamma_1]) \sigma([\gamma_2], \ldots, [\gamma_{p+1}]) \\
+ \sum_{q=1}^{p} (-1)^q \sigma(\ldots, [\gamma_q][\gamma_{q+1}], \ldots).
\]

(166)

For 0- and 1-cochains, we have the following explicit formulas:

\[
\delta \sigma([\gamma]) = -\sigma + \text{Ad}_\rho([\gamma]) \sigma \\
= -\sigma + \rho([\gamma]) \sigma \rho([\gamma])^{-1}, \quad (167)
\]

\[
\delta \tau([\gamma], [\eta]) = \tau([\gamma]) + \text{Ad}_\rho([\gamma]) \tau([\eta]) - \tau([\gamma][\eta]) \\
= \tau([\gamma]) + \rho([\gamma]) \tau([\eta]) \rho([\gamma])^{-1} - \tau([\gamma][\eta]). \quad (168)
\]

It is worth noting that the degree-0 group cohomology is isomorphic to the subspace of the representation on which the group acts trivially, \(H^0(\pi, \text{Ad}_\rho) \cong \mathfrak{g}^\pi\).

It should now be clear from Equations (164) and (165) that infinitesimal deformations of a monodromy representations \(\rho: \pi \to G\), up to deformations by conjugation, are in bijection with the group cohomology \(H^1(\pi, \text{Ad}_\rho)\) of the group \(\pi\) with coefficients in the composite adjoint representation of \(\pi\) on \(\mathfrak{g}\).

References


