$q$-deformed spin foams for Riemannian quantum gravity

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26 June 2007
LOOPS ’07
UNAM Morelia, Mexico

based on arXiv:0704.0278 [gr-qc] (with Dan Christensen)
Outline

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Barrett-Crane Model
$q$-deformation

Why?
Regularization
Cosmological Constant

How?
$q$-Barrett-Crane model
Computer Simulation

So What?
Results

Summary
Spin Foams

Start with a triangulated 4-manifold $T$ ($T^* \supset \Delta_n$ — the set of dual $n$-simplices). A spin foam is a coloring of the triangulation faces ($\Delta_2$). A spin foam model assigns an amplitude to each spin foam $F$:

$$A(F) = \prod_{f \in \Delta_2} A_F(f) \prod_{e \in \Delta_2} A_E(e) \prod_{v \in \Delta_1} A_V(v).$$

Also, to the triangulation as a whole and expectation values to observables

$$Z = \sum_F A(F), \quad \langle O \rangle = \frac{1}{Z} \sum_F O(F)A(F).$$

Sum over all histories — discrete path integral!
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Goal — compute these sums numerically.
Barrett-Crane Model

What?

A spin foam model for Riemannian General Relativity.

- Historically, obtained as a constrained version of discretized $BF$ theory.
- Can also be derived from Group Field Theory.
- Specifies vertex amplitude ($10j$ symbol):

$$A_V(v) = \begin{array}{c}
\begin{array}{ccc}
1 & \rightarrow & 0 \\
2 & \rightarrow & 0 \\
3 & \rightarrow & 0 \\
4 & \rightarrow & 0 \\
\end{array}
\end{array}$$

- The $j_{1,k}$ are balanced irreps ($j \otimes j$) of Spin(4) $\cong SU(2) \times SU(2)$.

- Several choices for amplitudes $A_F(f)$ and $A_E(e)$.
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Spin networks: graphs $\longrightarrow$ ribbon graphs.
Regularization

Application of $q$-deformation.

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- At a ROU $q$, the DFKR model is also regularized.
Cosmological Constant

Application of $q$-deformation.

In Loop Quantum Gravity, $SU(2)$ spin networks are embedded in a spatial slice.

- The spin network basis describes states of quantum spatial geometry.

$\langle |K; \Lambda \rangle$ — approximates deSitter space, a vacuum with positive Cosmological Constant, $\Lambda > 0$.

Smolin (1995) argues that invariance under large gauge transformations discretizes the CC, $\Lambda \sim 1/r$.

Expansion coefficients give topological link and graph invariants:

$\langle |K \rangle \sim \langle | \rangle^q$
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$$\langle \mathcal{K} \rangle \sim \langle \mathcal{B} \rangle_q$$

- With precisely $q = \exp(i\pi/r)$!
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Intersection structure of $10j$ symbol (only non-planar spin network) fixed from the Crane-Yetter model (1994):

Retains permutation symmetry.

Computer Simulation

How?

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tetrahedral network vs. $q$
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- Works well since $A(F) \geq 0$ when $q = 1$ or ROU, in the absence of boundaries.
Models

Perez-Rovelli (2000):

\[ A_F(f) = i \bigcirc \bullet, \quad A_E(e) = \frac{j_1 j_2 j_3 j_4}{j_1 j_2 j_3 j_4}. \]

DFKR (2000):

\[ A_F(f) = i \bigcirc \bullet, \quad A_E(e) = \begin{bmatrix} j_1 & j_2 & j_3 & j_4 \end{bmatrix}^{-1}. \]

Baez-Christensen (2002):

\[ A_F(f) = 1, \quad A_E(e) = \begin{bmatrix} j_1 & j_2 & j_3 & j_4 \end{bmatrix}^{-1}. \]
Observables

Spin foam observables depend on face spin labels:

spin avg. \( J(F) = \frac{1}{|\Delta_2|} \sum_{f \in \Delta_2} \lfloor j(f) \rfloor \),

spin var. \( (\delta J)^2(F) = \frac{1}{|\Delta_2|} \sum_{f \in \Delta_2} \left( \lfloor j(f) \rfloor - \langle J \rangle \right)^2 \),

area avg. \( A(F) = \frac{1}{|\Delta_2|} \sum_{f \in \Delta_2} \sqrt{\lfloor j(f) \rfloor \lfloor j(f) + 1 \rfloor} \),

spin corr. \( C_d(F) = \frac{1}{N_d} \sum_{\text{dist}(f,f')=d} \frac{\lfloor j(f) \rfloor \lfloor j(f') \rfloor - \langle J \rangle^2}{\langle (\delta J)^2 \rangle} \).

Quantum half integers \( \lfloor j \rfloor = j \) when \( q = 1 \), but \( \lfloor j \rfloor \sim \sin(2j\pi/r) \) when \( q = e^{i\pi/r} \).
Observables Discontinuous as $r \to \infty$

So What?
Single Spin Distribution

So What?

SSD — frequency of occurrence of $j$.

BA — $A(F)$, where $F$ contains minimal bubble.
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For PR and BCh, bubbles dominate!
Single Spin Distribution

- So What?

(a) Probability distribution for different bubble amplitudes:
- `bubble amplitude $r = 50` (dashed line)
- `bubble amplitude $q = 1$` (solid line)
- Spin distribution $r = 50$ (open circles)
- Spin distribution $q = 1$ (solid circles)

(b) Graph showing:
- $j$ vs. $(j(j+1))^{1/2}$ (solid line)
- $|j|$ (dashed line)
- $(|j|)(|j+1|)^{1/2}$ (dotted line)

- SSD — frequency of occurrence of $j$.
- BA — $A(F)$, where $F$ contains minimal bubble.
- For PR and BCh, bubbles dominate!
- Not for DFKR.
Spin Correlation

Consistent with isolated bubble hypothesis.
Summary and Outlook

- Computer simulation of $q$-Barrett-Crane models now possible and practical, for modest sized triangulations.

- Observables show a discontinuity as $q \to 1$ through roots of unity. At odds with cosmological constant interpretation.

- BC models show strong dependence on edge and face amplitudes.

Outlook

- Simulations with $|q| \sim 1$.
- Spin correlation on larger triangulations.
- Lorentzian signature.

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