harmonic analysis on amenable networks and the Bose-Einstein Condensation

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abstract

We present the new and unexpected results concerning the Bose–Einstein Condensation for the Pure Hopping Model, describing the thermodynamic behaviour of Bardeen–Cooper pairs in arrays of Josephson junctions located on non homogeneous amenable networks. For this particular situation arising from physics, the results are obtained by a detailed analysis concerning harmonic analysis of the adjacency matrix of the involved networks. The non amenable cases will also be briefly outlined.

introduction

In the early paper: Burioni R., Cassi D., Rasetti M., Sodano P., Vezzani A. J. Phys. B 34 (2001), 4697–4710, it was shown the surprising fact that the critical density describing the
condensation of Bardeen–Cooper Bosons for the pure hopping model can be finite also for low dimensional networks. Motivated by such a result, we investigated relevant spectral properties of the adjacency operator of non homogeneous networks. The graphs under investigation are obtained by adding density zero perturbations to periodic amenable networks, the case of perturbations of Cayley Trees can be also managed. Apart from the natural mathematical interest, such spectral properties are relevant for the Bose Einstein Condensation for the pure hopping model describing Bardeen–Cooper pairs of Fermions (i.e. BCS Bosons) in arrays of Josephson junctions on non homogeneous networks. The resulting topological model is described by a one particle Hamiltonian which is, up to an additive constant, the opposite of the adjacency operator on the graph. In the condensation regime, the particles condensate on the perturbed graph, even
in the configuration space due to non homogeneity. Roughly speaking, the system undergoes a sort of ”dimension transition”. We show for both amenable and non amenable situations, that it is enough to perturb in a negligible way the original graph in order to obtain a new network whose mathematical and physical properties dramatically change.

The present talk is based on the following papers:


To end we point out that, in recent experiments (M. Cirillo et al.: Phys. Lett. A 370 (2007), 499–503, arXiv:1309.2836 (2013)) it was found that the current is enhanced at low temperatures for non–homogeneous arrays of Josephson junctions in the Comb Graph and the Star Graph, see Fig 2 below. Such a phenomenon might be indeed explained by the BEC of Bardeen–Cooper pairs.
The model

The framework is a sea of Bardeen–Cooper pairs in arrays of Josephson junctions on a network $G$: particles are located on vertices $V_G$, and edges $E_G$ describe the presence of a Josephson junction. The distribution of the particles on the superconducting islands (vertices of the network) is governed by the Bose–Gibbs grand–canonical distribution.

The Hamiltonian of the system is the Bose Hubbard Hamiltonian

$$H_{BH} = m \sum_i n_i + \sum_{i,j} A_{i,j}(Vn_in_j - J_0a_i^\dagger a_j).$$ (1)

Here, $a_i^\dagger$ is the Bosonic creator, and $n_i = a_i^\dagger a_i$ the number operator on the site $i$. Finally, $A$ is the adjacency operator whose matrix element $A_{i,j}$ in the place $ij$ is the number of the edges connecting the site $i$ with the site $j$. 
When $m$ and $V$ are negligible with respect to $J_0$, it might be expected that the hopping term dominates the physics of the system, at least for low temperatures. Thus, under this approximation, (1) becomes the quadratic pure hopping Hamiltonian given by

$$H_{PH} = -J \sum_{i,j} A_{i,j} a_i^\dagger a_j,$$

where the constant $J > 0$ is a mean field coupling constant which is in general different from the $J_0$ appearing in the more realistic Hamiltonian (1).

Being the previous Hamiltonian a quasi–free (quadratic) one, it is enough to study the self-adjoint operator $-A$ on the one–particle space $\ell^2(VG)$. We put $J_0 = 1$ in (2), and normalizing such that the bottom of the spectrum of the energy is zero. The resulting Hamiltonian for the purely topological model under consideration is

$$H = \|A\| \mathbf{1} - A,$$
where $A$ is the adjacency of the fixed graph $G$, acting on the Hilbert space $\ell^2(VG)$.

The appearance of the BEC is connected with the asymptotics near zero, of the spectrum of the Hamiltonian. For vectors in the spectral subspace near zero (i.e. for small values of energies, and for $\mu \approx 0$), the Taylor expansion for the "Bose occupation function" relative to the chemical potential $\mu < 0$ leads to

$$\frac{1}{e^{\beta(H-\mu I)} - 1} \approx \left[ \beta(H - \mu I) \right]^{-1}$$

$$= \frac{1}{\beta} \left( (\|A\| - \mu) \mathbf{I} - A \right)^{-1} \equiv \frac{1}{\beta} R_A(\|A\| - \mu).$$

Then the mathematics of the BEC is reduced to the investigation of the spectral properties of the (more familiar object for mathematicians which is the) resolvent $R_A(\lambda)$, for $\lambda \approx \|A\|$.
The non homogeneous graphs we dealt with at the beginning are density zero additive perturbations of periodic lattices, Fig 1: the comb graph $\mathbb{Z} \rhd \mathbb{Z}$, see also Fig 2: the star graph (amenable situation). Very surprisingly, essentially the same situation also arises for some interesting non amenable examples such as negligible additive perturbations of homogeneous Cayley trees (Fig 3 and Fig 4).

**mathematical aspects and physical applications**

We fix a network $G$, together with an exhaustion $\{G_n\}$. In all the model under consideration (including those which are non amenable), the density of eigenvalues (called in physics integrated density of the states) of the Adjacency $A$ can be computed in the infinite volume limit. In fact, for $f \in C(\mathbb{R})$,

$$\lim_{n} \frac{1}{|VG_n|} \text{Tr}_n(f(A_n)) =: \tau_A(f(A))$$
exists and defines a state $\tau^A$ on the Abelian $C^*$–algebra generated by $A$. By the Riesz–Markov Theorem, there is a cumulative function $N_A$ (whose support as a measure is included in $\sigma(A)$) such that $\tau^A$ is given by the Stieltjes integral

$$
\tau^A(f(A)) = \int f(x) \, dN_A(x).
$$

**hidden spectrum.** It can happen that $\text{supp}(N_A) \subsetneq \sigma(A)$. Let $X$ be a graph equipped with an exhaustion $\{X_n\}_{n \in \mathbb{N}}$. The graph $Y$ with the exhaustion $\{Y_n\}_{n \in \mathbb{N}}$ is a *negligible additive perturbation* of $X$ (involving only edges) if $Y$ differs from $X$ only by additional edges (i.e. $VX_n = VY_n$ and $EX_n \subset EY_n$), and in addition,

$$
\lim_{n} \frac{EY_n \setminus EX_n}{VX_n} = 0.
$$

Suppose that Adjacency of $X$ admits IDS (w.r.t. the exhaustion $\{X_n\}_{n \in \mathbb{N}}$). Consider the one–particle pure hopping Hamiltonian $\|A_X\| \mathbf{1} - A_X$
together with the relative IDS $F_X$. Define 
\[ \delta := \|A^X\| - \|A^Y\| < 0, \]
which has the meaning of an "effective chemical potential", see below.

**Theorem** The IDS $F_Y$ of $\|A_Y\| \mathbf{I} - A_Y$ also exists (w.r.t. the exhaustion $\{Y_n\}_{n \in \mathbb{N}}$), and we have
\[ F_Y(x) = F_X(x + \delta). \]

Suppose now that $\delta < 0$ (i.e. $\|A_Y\| > \|A_X\|$). In this case, we have the so-called *Hidden Spectrum* in the low part of the spectrum of the energies: it is the combined effect of two phenomena. The first one is that the perturbation is too small to alter the density of eigenvalues in the thermodynamical limit, but a shift due to the normalisation of the Hamiltonian which is described by $\delta$, provided it is non null. The physical effect in presence of
Hidden Spectrum is that the critical density of the perturbed model is always finite:

$$\rho_c^Y(\beta) = \int \frac{dF_X(x)}{e^{\beta(x-\delta)} - 1} = \rho^X(\beta, \delta) < +\infty.$$  

The problem is now reduced to see whether $\|A_Y\| > \|A_X\|$, which is addressed by the Secular Equation

$$\left\| DR_{A_X}(\lambda) P_{\ell^2(V_Z)} \right\| = 1$$

where $Z \subset Y$ is the subgraph supporting the perturbation $D$. Notice that the Secular Equation (3) holds true for general additive perturbations, providing that $DR_{A_X}(\lambda) P_{\ell^2(V_Z)}$ is self–adjoint.*

transience/recurrence character. A general surprising result is that the appearance of the

*In the general case, one conjectures that the Secular Equation would assume the form

$$\text{sprt}(DR_{A_X}(\lambda) P_{\ell^2(V_Z)}) = 1.$$
BEC for the pure hopping model on an arbitrary network is connected with the transience/recurrence character of the Adjacency and not just with the finiteness of the critical density as we see below.

The (analytic) definition of \( T/R \) character of the adjacency is as follows. The Adjacency is said to be recurrent if

\[
\lim_{\lambda \downarrow \|A\|} \langle R_A(\lambda) \delta_x, \delta_x \rangle = +\infty
\]

for some, and then equivalently for all \( x \in G \). Otherwise it is transient.\(^*\) From a mathematical point of view, this means that \( \delta_x \in \mathcal{D}_{R_A(\|A\|)^{1/2}} \), which turns out to be equivalent that for the quadratic form,

\[
\langle (e^{\beta H} - 1)^{-1}\delta_x, \delta_x \rangle < +\infty, \quad x \in G.
\]

\(^*\)When the Hamiltonian is (the opposite of) the discrete Laplacian, it is the generator of a random walk on the network and the \( T/R \) character has the standard probabilistic meaning.
The last condition is the necessary and sufficient condition for the existence of locally normal states (which turns out to be equivalent for the models under consideration, that the local density is finite) describing BEC. Thus, it is possible to have the BEC (for the pure hopping model) if and only if the Adjacency is transient.

the algebra, the states and the existence of the dynamics. The two–point function candidates for locally normal states exhibiting BEC (i.e. at chemical potential $\mu = 0$) assumes the form

$$\omega_D(a^\dagger(u_1)a(u_2)) := \langle (e^{\beta H} - \mathbf{1})^{-1}u_1, u_2 \rangle + D\langle u_1, v \rangle\langle v, u_2 \rangle, \quad u_1, u_2 \in \mathfrak{h}. \quad (4)$$

Typically, $v$ is a Perron–Frobenius weight (i.e. non normalizable) and $\mathfrak{h} \subset \ell^2(G)$ is a dense
subspace containing \( \{ \delta_x \mid x \in G \} \) (in order to have a reasonable local description for the theory) which is invariant for the dynamics. The delicate issue is to show that the two addenda in the l.h.s. of (4) are meaningful for a dense subspace which can be chosen as

\[
\mathfrak{h} := \text{span} \left\{ e^{it H} \delta_j \mid t \in \mathbb{R}, j \in Y \right\}.
\]

The first one is meaningful because of the transience character of the Adjacency (otherwise it can be proven that the diagonal part diverges in the infinite volume limit in recurrent case) and, concerning the second one describing the amount of the condensate, we can prove that it is also meaningful for each PF weight \( \nu \) of the Adjacency. It is possible to prove in general that (4) defines the two–point function of a locally normal quasi free state exhibiting BEC (provided \( D > 0 \)) which is a KMS state on the CCR algebra \( CCR(\mathfrak{h}) \), w.r.t. the free dynamics generated by the pure hopping Hamiltonian.
the wave function of the ground state. It is possible to prove in all the models under consideration, that the sequence made of the unique Perron–Frobenius eigenvectors for the finite volume Adjacency (i.e. the finite volume ground state wave function) all normalised at 1 on a fixed root, converges point-wise to a Perron–Frobenius weight which can be considered as the ground state wave function, describing the density of possible condensate.‡

A PF (generalized) eigenvector is nothing but the (generalized) wave function of a physical ground state.§. As noticed above, it is uniquely determined for all the model under consideration (after fixing the exhaustion). Then it describes the distribution of the condensate in the configuration space. For additive negligible

‡The Perron–Frobenius weight is in general no longer unique in transient case.
§Here ”generalized” stands for non normalizable.
perturbations, it is possible to prove as a general fact, that it decays exponentially far away to the perturbation: due to inhomogeneity, particles condensate also in configuration space, and the system undergoes a "dimension transition" which can be well explained as follows. In the amenable cases, the condensate distribution is well described by the Perron–Frobenius dimension $d_{PF}$ (D. Guido, T. Isola, F.F.). Consider the ball $\Lambda_n \uparrow G$ of radius $n$ centered in any fixed root of the graph.\footnote{Such balls $\{\Lambda_n\}_{n \in \mathbb{N}}$ provides the fixed exhaustion of the network under consideration.} Consider the Perron–Frobenius eigenvector $v$, previously described. The geometrical dimension $d_G$ of $G$ is defined to be $a$ if $|\Lambda_n| \sim n^a$. The Perron–Frobenius dimension $d_{PF}(G)$ of $G$ is defined to be $b$ if $\|v[\ell^2(\Lambda_n)]\| \sim n^{b/2}$. Looking at (4), the portion of condensate is described by the last addendum of the l.h.s.. In the finite
region $\Lambda$, such a density of the condensate is roughly given by

$$C_D(\Lambda) \approx \frac{1}{|\Lambda|} \sum_{x \in \Lambda} D(\delta_x, v)\langle v, \delta_x \rangle = D\frac{||v\parallel_\Lambda^2}{|\Lambda|}.$$ 

Suppose that the graph is transient (condition under which it is possible to exhibit locally normal states describing BEC), and $\rho_c < +\infty$ (then is meaningful to ask whether it is possible to construct locally normal states exhibiting BEC such that for the mean density

$$\rho(\omega) := \limsup_{\Lambda \uparrow G} \rho_\Lambda(\omega)$$

we have $\rho(\omega) > \rho_c$). In this situation we should look at $d_{PF}$. If $d_{PF} < d_G$, $C_D(\Lambda) \to 0$ when $\Lambda \uparrow G$. it is then impossible to exhibit locally normal states exhibiting BEC for which $\rho(\omega) > \rho_c$. Conversely, if $d_{PF} > d_G$ then $\rho(\omega) = +\infty$ due to the huge amount of the condensate localised on the perturbed zone described by the "shape" of the PF eigenvector. In order to
construct such states as infinite volume limits of states satisfying the Bose–Gibbs grand canonical ensemble prescription, it is more natural for non homogeneous model, to fix the amount of the condensate. On the other hand, if one fix the mean energy (as it is usually done for homogeneous systems) \( \rho > \rho_c \), we obtain that the finite volume two point function \( \omega_{\Lambda_n}(a^\dagger(\delta_x)a(\delta_x)) \), the two alternative. If \( d_{PF} < d_G \) then

\[
\lim_{n} \omega_{\Lambda_n}(a^\dagger(\delta_x)a(\delta_x)) = +\infty,
\]

that is it still diverges. Conversely, if \( d_{PF} > d_G \) then we get

\[
\lim_{n} \omega_{\Lambda_n}(a^\dagger(\delta_x)a(\delta_x)) = \langle(e^{\beta H} - 1)^{-1}u_1, u_2 \rangle,
\]

that is we always obtain the state in (4) corresponding to \( D = 0 \), i.e. without any amount of condensate.

To summarise, such a very intriguing new, surprising and fascinating situation is summarised
in the following table describing what happens in the amenable situation.

<table>
<thead>
<tr>
<th>$\mathbb{Z}^d, d &lt; 3$</th>
<th>$\rho_c$</th>
<th>$R/T$</th>
<th>$d_G$</th>
<th>$d_{PF}$</th>
<th>0</th>
<th>$\rho$</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}^d, d \geq 3$</td>
<td>$\infty$</td>
<td>$T$</td>
<td>$d$</td>
<td>$d$</td>
<td>no</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>star graph</td>
<td>$\infty$</td>
<td>$R$</td>
<td>1</td>
<td>0</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>$\mathbb{Z}^d \sqsubset \mathbb{Z}, d &lt; 3$</td>
<td>$\infty$</td>
<td>$R$</td>
<td>$d + 1$</td>
<td>$d$</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>$\mathbb{Z}^d \sqsubset \mathbb{Z}, d \geq 3$</td>
<td>$\infty$</td>
<td>$T$</td>
<td>$d + 1$</td>
<td>$d$</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>$\mathbb{N}$</td>
<td>$\infty$</td>
<td>$T$</td>
<td>1</td>
<td>3</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>$\mathbb{N} \sqsubset \mathbb{Z}$</td>
<td>$\infty$</td>
<td>$T$</td>
<td>2</td>
<td>3</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>$\mathbb{N} \sqsubset \mathbb{Z}^2$</td>
<td>$\infty$</td>
<td>$T$</td>
<td>3</td>
<td>3</td>
<td>no</td>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>

Here, $G \sqsubset H$ is the comb shaped graph whose base graph is $G$, $\rho_c$ is the critical density, $R/T$ denotes the recurrence/transience character of the Adjacency, 0, $\rho$ and $\infty$ denote the existence of locally normal states exhibiting BEC (that is exhibiting some amount of condensate) at mean density $\rho = \rho_c$, $\rho \in (\rho_c, +\infty)$, and finally $\rho = +\infty$, respectively.
Here, $A \downarrow B$ is the comb–shaped graph whose base–point is $A$, $\rho_c$ is the critical density, $R/T$ denotes the transience/recurrence of the adjacency, BEC ($\rho$–BEC) denotes the existence of locally normal states exhibiting BEC (exhibiting BEC at mean densities $\rho > \rho_c$). The last examples involving $\mathbb{N}$ deserve of further explanation. The network $\mathbb{N}$ admits the BEC (i.e. there exist locally normal states exhibiting BEC) because it is transient. In addition, the combined effect that the critical density is infinite and $d_{PF} > d_{G}$ tells us that the states with BEC have infinite mean density. The case $\mathbb{N} \downarrow \mathbb{Z}$ is also very interesting. In fact, it admits BEC because it is transient. But there is a gap in $(\rho_c, +\infty)$ for the possible mean density of locally normal states because $\rho_c < +\infty$ and $d_{PF} > d_{G}$: the states without condensate have mean density in the interval $(0, \rho_c]$ whereas the states exhibiting BEC has infinite mean density.
A very natural question can be naively addressed in the following way: \textit{there exists any reasonable model exhibiting BEC for which } \( \rho_c < \rho(\omega) < +\infty \) \textit{such that } \( d_G < 3 \).\textsuperscript{II} \textit{Here, ”reasonable” stands for states whose local density is finite (which is equivalent in our framework to be locally normal (w.r.t. the Fock representation). After the above analysis, our request is satisfied if we find a network } X \textit{ such that}

- \( A_X \text{ is transient,} \)

- \( \rho_c < +\infty, \)

- \( d_G(X) = d_{PF}(X) < 3. \)

\textbf{examples}

\textsuperscript{II}Non integer dimensions are also allowed.
We end with the description of the properties described above for some pivotal example of the graphs under consideration.

**finite additive perturbations** (see Fig 1): finite critical density (provided the perturbation is sufficiently big to modify the norm of the adjacency), recurrent (as the PF eigenvector is normalizable), $d_{PF} = 0$.

![Fig 1: the star graph.](image)

**Comb graphs** $G^d := \mathbb{Z}^d \rightarrow \mathbb{Z}$ (see fig): finite critical density, recurrent if and only if $d \leq 2$, $d = d_{PF} < d_G = d + 1$. 
Fig 2: the comb graph $\mathbb{Z} \rhd \mathbb{Z}$.

**the graph $\mathbb{N}$:**
infinite critical density, transient, $3 = d_{PF} > 1$.

**Comb graphs $H^2 := \mathbb{N} \rhd \mathbb{Z}^2$:**
finite critical density, transient, $3 = d_{PF} = d_G$.

Fig 2 bis: the comb graph $\mathbb{N} \rhd \mathbb{Z}^2$. 
The mathematical aspects of the BEC are extended to exponentially growing graphs such as the perturbed Cayley tree.

Fig 3: finite perturbation of the Cayley Tree of order 3. It is recurrent and PF—”0” dimensional.

Fig 4: perturbation of the Cayley Tree of order 3 along $\mathbb{Z}$. 
It is recurrent and PF—”1” dimensional.

Fig 4: perturbation of the Cayley Tree of order 3 along \( \mathbb{N} \).

It is transient and PF—”3” dimensional.

Fig 5: perturbation of the Cayley Tree of order 4 along a Cayley subtree of order 3.

It is transient and has the same PF—behavior as the basepoint.