Multi-Linear Algebra, Tensors and Spinors in Mathematical Physics.

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Contents

Chapter 1

Introduction

The content of this textbook is the introductory part of the lectures I gave for graduate courses in Mathematical Physics at the University of Trento during the years 2003 - 2008. In those lectures I tried to give a quick, but rigorous, picture of the basics of tensor calculus for the applications to mathematical and theoretical physics. In particular I tried to cover the gap between the traditional presentation of these topics given by physicists, based on the practical indicial notation and the more rigorous (but not always useful in the practice) approach presented by mathematicians.
Several applications are presented as examples and exercises. Some chapters concern the geometric structure of Special Relativity theory and some theoretical issues about Lorentz group.
The reader is supposed to be familiar with standard notions of linear algebra [Lang, Sernesi], especially concerning finite dimensional vector spaces. Since the end of Chapter 8 some basic tools of Lie group theory and Lie group representation theory [KNS] are requested. The definition of Hilbertian tensor product given at the end of Chapter 2 has to be seen as complementary material and requires that the reader is familiar with elementary notions on Hilbert spaces.

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Chapter 2

Multi-linear Mappings and Tensors.

Within this section we introduce basic concepts concerning multi-linear algebra and tensors. The theory of vector spaces and linear mappings is assumed to be well known.

2.1 Dual space and conjugate space, pairing, adjoint operator.

As a first step we introduce the dual space and the conjugate space of a given vector space.

Definition 2.1. **(Dual Space, Conjugate Dual Space and Conjugate space.)** Let V be a vector space on the field either $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
(a) The **dual space** of V, V^*, is the vector space of linear functionals on V, i.e., the linear mappings $f : V \to \mathbb{K}$.
(b) If $\mathbb{K} = \mathbb{C}$, the **conjugate dual space** of V, $\overline{V^*}$, is the vector space of anti-linear functionals on V, i.e., the antilinear mappings $g : V \to \mathbb{C}$. Finally the **conjugate space** of V, $\overline{V}$ is the space $(\overline{V^*})^*$. $\diamondsuit$

Comments 2.1.
(1) If V and V' are vector spaces on $\mathbb{C}$, a mapping $f : V \to V'$ is called *anti linear* or *conjugate linear* if it satisfies

$$f(\alpha u + \beta v) = \overline{\alpha} f(u) + \overline{\beta} f(v)$$

for all $u, v \in V$ and $\alpha, \beta \in \mathbb{C}$, $\overline{\lambda}$ denoting the complex conjugate of $\lambda \in \mathbb{C}$. If $V' = \mathbb{C}$ the given definition reduces to the definition of *anti-linear functional*.
(2) V^*, $\overline{V^*}$ and $\overline{V}$ turn out to be vector spaces on the field $\mathbb{K}$ when the composition rule of vectors and the product with elements of the field are defined in the usual way. For instance, if $f, g \in V^*$ or $\overline{V^*}$, and $\alpha \in \mathbb{K}$ then $f + g$ and αf are functions such that:

$$(f + g)(u) := f(u) + g(u)$$

and

$$(\alpha f)(u) := \alpha f(u)$$

for all of $u \in V$.

We remind the reader that, as a consequence of Zorn's lemma, every vector space V admits a vector basis, that is a *possibly infinite* set of vectors such that each vector $v \in V$ can be obtained as a *finite*, and depending on v, linear combination of those vectors. In Definition 2.2, we do not explicitly assume that V is finite dimensional.

Definition 2.2. **(Dual Basis, Conjugate Dual Basis and Conjugate Basis.)** Let V be a vector space on either $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\{e_i\}_{i\in I}$ be a vector basis of V. The set $\{e^{*j}\}_{j\in I} \subset V^*$ whose elements are defined by

$$e^{*j}(e_i) := \delta_i^j$$

for all $i, j \in I$ is called the **dual basis** of $\{e_i\}_{i\in I}$.
Similarly, if $\mathbb{K} = \mathbb{C}$, the set of elements $\{\overline{e^{*j}}\}_{j\in I} \subset \overline{V^*}$ defined by:

$$\overline{e^{*j}}(e_i) := \delta_i^j$$

for all $i, j \in I$ is called the **conjugate dual basis** of $\{e_i\}_{i\in I}$.
Finally the set of elements $\{\overline{e}_j\}_{j\in I} \subset \overline{V}$ defined by:

$$\overline{e}_p(\overline{e^{*q}}) := \delta_p^q$$

for all $p, q \in I$ is called the **conjugate basis** of $\{e_i\}_{i\in I}$. $\diamondsuit$

Comments 2.2.
(1) One may wonder whether or not the *dual bases* (and the conjugate bases) are *proper* vector bases of the respective vector spaces, the following theorem gives a positive answer in the finite-dimensional case. This is not true, in general, for infinite-dimensional spaces V. We shall not try to improve the definition of dual basis in the general case, since we are interested on algebraic features only and the infinite-dimensional case should be approached by convenient topological tools which are quite far form the goals of these introductory notes.
(3) In *spinor* theory – employed, in particular, to describe the *spin* of quantum particles – there is is a complex two-dimensional space V called the *space of Weyl spinors*. It is the representation space of the group $SL(2,\mathbb{C})$ viewed as a double-valued representation of the orthochronous proper Lorentz group.
Referring to a basis in V and the associated in the spaces V^*, $\overline{V}$ and $\overline{V}^*$, the following notation is often used in physics textbooks [Wald, Streater-Wightman]. The components of the spinors, i.e. vectors in V are denoted by ξ^A. The components of dual spinors, that is vectors of V^*, are denoted by ξ_A. The components of conjugate spinors, that is vectors of $\overline{V}$, are denoted either by $\xi^{A'}$ or, using the dot index notation, $\xi^{\dot{A}}$. The components of dual conjugate spinors, that is vectors of $\overline{V}^*$, are denoted either by $\xi_{A'}$ or by $\xi_{\dot{A}}$.

Theorem 2.1. *If $dimV < \infty$ concerning definition 2.2, the dual basis, the conjugate dual basis and the conjugate basis of a base $\{e_i\}_{i\in I} \subset V$, are proper vector bases for V^*, $\overline{V^*}$ and $\overline{V}$*

respectively. As a consequence $dimV = dimV^* = dim\overline{V^*} = dim\overline{V}$.

Proof. Consider the dual basis in V^*, $\{e^{*j}\}_{j\in I}$. We have to show that the functionals $e^{*j} : V \to \mathbb{K}$ are generators of V^* and are linearly independent.
(Generators.) Let us show that, if $f : V \mapsto \mathbb{K}$ is linear, then there are numbers $c_j \in \mathbb{K}$ such that $f = \sum_{j\in I} c_j e^{*j}$.
To this end define $f_j := f(e_j)$, $j \in I$, then we argue that $f = f'$ where $f' := \sum_{j\in I} f_j e^{*j}$. Indeed, any $v \in V$ may be decomposed as $v = \sum_{i\in I} v^i e_i$ and, by linearity, we have:

$$f'(v) = \sum_{j\in I} f_j e^{*j}(\sum_{i\in I} v^i e_i) = \sum_{i,j\in I} f_j v^i e^{*j}(e_i) = \sum_{i,j\in I} f_j v^i \delta_i^j = \sum_{j\in I} v^j f_j = \sum_{j\in I} v^j f(e_j) = f(v)\,.$$

(Notice that above we have used the fact that one can extract the summation symbol from the argument of each e^{*j}, this is because the sum on the index i is *finite* by hypotheses it being $dimV < +\infty$.) Since $f'(v) = f(v)$ holds for all of $v \in V$, we conclude that $f' = f$.
(Linear independence.) We have to show that if $\sum_{k\in I} c_k e^{*k} = 0$ then $c_k = 0$ for all $k \in I$.
To achieve that goal notice that $\sum_{k\in I} c_k e^{*k} = 0$ means $\sum_{k\in I} c_k e^{*k}(v) = 0$ for all $v \in V$. Therefore, putting $v = e_i$ and using the definition of the dual basis, $\sum_{k\in I} c_k e^{*k}(e_i) = 0$ turns out to be equivalent to $c_k\delta_i^k = 0$, namely, $c_i = 0$. This result can be produced for each $i \in I$ and thus $c_i = 0$ for all $i \in I$. The proof for the conjugate dual basis is very similar and is left to the reader. The proof for the conjugate basis uses the fact that $\overline{V}$ is the dual space of $\overline{V^*}$ and thus the first part of the proof applies. The last statement of the thesis holds because, in the four considered cases, the set of the indices I is the same. $\Box$

Remark.
Let $\{e_i\}_{i\in I}$ be a vector basis of the vector space V with field $\mathbb{K}$ and consider the general case with I *infinite*. Each linear or anti-linear mapping $f : V \to \mathbb{K}$ is anyway completely defined by giving the values $f(e_i)$ for all of $i \in I$. This is because, if $v \in V$ then $v = \sum_{i\in I_v} c^i e_i$ for some numbers $c^i \in \mathbb{K}$, $I_v \subset I$ being *finite*. Then the linearity of f yields $f(v) = \sum_{i\in I_v} c^i f(e_i)$. (This fact holds true no matter if $dimV < +\infty$). So, formally one may use the notation

$$f := \sum_{i\in I} f(e_i) e^{*i}$$

to expand any element $f \in V^*$ also if $\{e^{*i}\}_{i\in I}$ is not a vector basis of V^* in the proper sense (the linear combination in the right-hand side may be infinite). This can be done provided that he/she adopts the convention that, for $v = \sum_{k\in I} v^k e_k \in V$,

$$\sum_{i\in I} f(e_i) e^{*i} \sum_{j\in I} v^j e_j := \sum_{k\in I} f(e_k) v^k\,.$$

Notice that the last sum is always finite, so it makes sense.

Notation 2.1. From now on we take advantage of the following convention. Whenever an index appears twice, once as an upper index and once as a lower index, in whichever expression, the summation over the values of that index is understood in the notation. E.g.,

$$t_{ijkl}f^{rsil}$$

means

$$\sum_{i,l} t_{ijkl}f^{rsil} .$$

The range of the various indices should be evident from the context or otherwise specified.

We are naturally lead to consider the following issue. It could seem that the definition of dual space, V^* (of a vector space V) may be implemented on V^* itself producing the double dual space $(V^*)^*$ and so on, obtaining for instance $((V^*)^*)^*$ and, by that way, an infinite sequence of dual vector spaces. The theorem below shows that, *in the finite-dimensional case*, this is not the case because $(V^*)^*$ turns out to be *naturally isomorphic* to the initial space V and thus the apparently infinite sequence of dual spaces ends on the second step. We remind the reader that a vector space isomorphism $F : V \to V'$ is a linear map which is also one-to-one, i.e., injective and surjective. An isomorphism is called *natural* when it is built up using the definition of the involved algebraic structures only and it does not depend on "arbitrary choices". A more precise definition of *natural isomorphism* may be given by introducing the *theory of mathematical categories* and using the notion of *natural transformation* (between the identity functor in the category of finite dimensional vector spaces on $\mathbb{K}$ and the functor induced by F in the same category).

Theorem 2.2. *Let V be a vector space on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The following holds.*
(a) *There is an injective linear map $F : V \to (V^*)^*$ given by*

$$(F(v))\,(u) := u(v) ,$$

for all $u \in V^*$ and $v \in V$, so that V identifies naturally with a subspace of $(V^*)^*$.
(b) *If V has finite dimension, the map F is a natural isomorphism and V identifies naturally with the whole space $(V^*)^*$*

Proof. Notice that $F(v) \in (V^*)^*$ because it is a linear functional on V^*:

$$(F(v))(\alpha u + \beta u') := (\alpha u + \beta u')(v) = \alpha u(v) + \beta u'(v) =: \alpha(F(v))(u) + \beta(F(v))(u') .$$

Let us prove that $v \mapsto F(v)$ with $(F(v))\,(u) := u(v)$ is linear and injective in the general case, and surjective when V has finite dimension.
(Linearity.) We have to show that, for all $\alpha, \beta \in \mathbb{K}$, $v, v' \in V$,

$$F(\alpha v + \beta v') = \alpha F(v) + \beta F(v') .$$

This is equivalent to, by the definition of F given above,

$$u(\alpha v + \beta v') = \alpha u(v) + \beta u(v') ,$$

for all $u \in V^*$. This is obvious because, u is a linear functional on V.
(Injectivity.) Due to linearity it is enough to show that $F(v) = 0$ implies $v = 0$. $(F(v))(u) = 0$ can be re-written as $u(v) = 0$. In our hypotheses, this holds true for all $u \in V^*$. Then, define $e_1 := v$ and notice that, if $v \neq 0$, one can complete e_1 with other vectors to get an algebraic vector basis of V, $\{e_i\}_{i\in I}$ (I being infinite in general). Since u is arbitrary, we can pick out u as the unique element of V^* such that $u(e_1) = 1$ and $u(e_i) = 0$ if $i \neq 1$. (This is possible also if V – and thus V^* – is not finite-dimensional.) It contradicts the hypothisis $u(e_1) = u(v) = 0$. By consequence $v = 0$.
(Surjectivity.) Assume that V (and thus V^*) has finite dimension. Since $dim((V^*)^*) = dim(V^*)$ (V^* having finite dimension) and so $dim((V^*)^*) = dimV < +\infty$, injectivity implies surjectivity. However it is interesting to give and explicit proof. We have to show that if $f \in (V^*)^*$, there is $v_f \in V$ such that $F(v_f) = f$.
Fix a basis $\{e_i\}_{i\in I}$ in V and the dual one in V^*. Since $\{e^{*i}\}_{i\in I}$ is a (proper) vector basis of V^* (it may be false if I is infinite!), $f \in (V^*)^*$ is completely defined by the coefficients $f(e^{*i})$. Then $v_f := f(e^{*i})e_i$ fulfills the requirement $F(v_f) = f$. □

Definition 2.3. **(Pairing.)** Let V be a vector space on $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ with dual space V^* and conjugate dual space $\overline{V^*}$ when $\mathbb{K} = \mathbb{C}$. The bi-linear map $\langle\, .,\, .\rangle : V \times V^* \to \mathbb{K}$ such that

$$\langle u, v\rangle := v(u)$$

for all $u \in V$, $v \in V^*$, is called **pairing**.
If $\mathbb{K} = \mathbb{C}$, the map, linear in the right-hand entry and anti linear in the left-hand entry $\langle\, .,\, .\rangle : V \times \overline{V} \to \mathbb{C}$ such that

$$\langle u, v\rangle := v(u)$$

for all $u \in V$, $v \in \overline{V}$, is called **(conjugate) pairing**. ♢

Comments 2.3. Because of the theorem proved above, we may indifferently think $\langle u, v\rangle$ as representing either the action of $u \in (V^*)^*$ on $v \in V^*$ or the action of $v \in V^*$ on $u \in V$. The same happens for V and $\overline{V^*}$

Notation 2.2. From now on

$$V \simeq W$$

indicates that the vector spaces V and W are isomorphic under some *natural* isomorphism. If the field of V and W is $\mathbb{C}$,

$$V \overline{\simeq} W$$

indicates that there is a *natural* anti-isomorphism, i.e. there is an injective, surjective, anti-linear mapping $G : V \to W$ built up using the abstract definition of vector space (including the abstract definitions of dual vector space and (dual) conjugate vector space).

We finally state a theorem, concerning conjugated spaces, that is analogous to theorem 2.2 and with a strongly analogous proof. The proof is an exercise left to the reader.

Theorem 2.3. *If V is a vector space with finite dimension on the field $\mathbb{C}$, one has:*
(i) $V^* \simeq \overline{V^*}$, where the anti-isomorphism $G : V^* \to \overline{V^*}$ is defined by

$$(G(v))(u) := \overline{\langle u, v\rangle} \quad \textit{for all } v \in V^* \textit{ and } u \in V;$$

(ii) $V \simeq \overline{V}$, where the anti-isomorphism $F : V \to \overline{V}$ is defined by

$$(F(v))(u) := \langle v, G^{-1}(u)\rangle \quad \textit{for all } v \in V \textit{ and } u \in V^*;$$

(iii) $\overline{V}^* \simeq \overline{V^*}$, where the involved isomorphism $H : \overline{V}^* \to \overline{V^*}$ arises by applying theorem 2.2 to the definition of $\overline{V}$.
Finally, with respect to any fixed basis $\{e_i\}_{i\in I} \subset V$, and the canonically associated bases $\{\overline{e}_i\}_{i\in I} \subset \overline{V}$, $\{e^{*i}\}_{i\in I} \subset V^*$, $\{\overline{e^{*}}^{i}\}_{i\in I} \subset \overline{V^*}$, $\{\overline{e}^{*i}\}_{i\in I} \subset \overline{V}^*$, one also has

$$F : v^i e_i \mapsto \overline{v^i}\overline{e}_i\,, \quad G : v_i e^{*i} \mapsto \overline{v_i}\overline{e^{*}}^{i}\,, \quad H : v_i\overline{e^{*}}^{i} \mapsto v_i\overline{e}^{*i}$$

where the bar over the components denotes the complex conjugation.

Exercises 2.1.

1. Show that if $v \in V$ then $v = \langle v, e^{*j}\rangle e_j$, where $\{e_j\}_{j\in I}$ is any basis of the finite dimensional vector space V.
(Hint. Decompose $v = c^i e_i$, compute $\langle v, e^{*k}\rangle$ taking the linearity of the the left entrance into account. Alternatively, show that $v - \langle v, e^{*j}\rangle e_j = 0$ proving that $f(v - \langle v, e^{*j}\rangle e_j) = 0$ for every $f \in V^*$.)

2. Show that $V^* \simeq \overline{V^*}$ if $dimV < \infty$ (and the field of V is $\mathbb{C}$). Similarly, show that $V \simeq \overline{V}$ under the same hypotheses.
(Hint. The anti-isomorphism $G : V^* \to \overline{V^*}$, is defined by $(G(v))(u) := \overline{\langle u, v\rangle}$ and the anti-isomorphism $F : V \to \overline{V}$, is defined by $(F(v))(u) := \langle v, G^{-1}(u)\rangle$.)

3. Show that if the finite-dimensional vector spaces V and V' are isomorphic or anti-isomorphic, then V^* and V'^* are isomorphic or anti-isomorphic respectively.
(Hint. If the initial (anti-) isomorphism is $F : V \to V'$ consider $G : V'^* \to V^*$ defined by $\langle F(u), v'\rangle = \langle u, G(v')\rangle$.)

4. Show that $\overline{V}^* \simeq \overline{V^*}$ if V is a finite-dimensional vector space on $\mathbb{C}$.
(Hint. Apply theorem 2.2 to the definition of $\overline{V}$.)

5. Write explicitly the (anti-)isomorphisms in exercises **2**, **3**, **4**, with respect to bases $\{e_i\}_{i\in I} \subset V$, $\{\overline{e}_i\}_{i\in I} \subset \overline{V}$ and the corresponding ones in V^*, $\overline{V^*}$, $\overline{V}$, $\overline{V}^*$.

The last notion we go to introduce is that of *adjoint operator*. We shall have very few occasions to employ this notion, however it is an important mathematical tool and it deserves mention.

Consider a (linear) operator $T : V_1 \to V_2$, where V_1, V_2 are two linear spaces with the same field $\mathbb{K}$. One can define another operator: $T^* : V_2^* \to V_1^*$ completely determined by the requirement:

$$(T^* u^*)(v) := u^*(Tv) \ , \quad \text{for all } v \in V_1 \text{ and all } u^* \in V_2^*.$$

It is obvious that T^* is linear by definition.

Definition 2.4. **(Adjoint operator.)** Let V_1, V_2 be a pair of vector spaces with the same field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and finite dimension. If $T : V_1 \to V_2$ is any linear operator, the operator $T^* : V_2^* \to V_1^*$ completely defined by the requirement

$$\langle v, T^* u^* \rangle_1 := \langle Tv, u^* \rangle_2 \ , \quad \text{for all } v \in V_1 \text{ and all } u^* \in V_2^*,$$

is called **adjoint operator** of T. $\diamondsuit$

Remark. Notice that we may drop the hypothesis of finite dimension of the involved spaces in the definition above without troubles: T^* would turn out to be well-defined also in that case. However this is not the full story, since such a definition, in the infinite-dimensional case would not be very useful for applications. In fact, a generalized definition of the adjoint operator can be given when V_1, V_2 are infinite dimensional and equipped with a suitable topology. In this case T is promoted to a continuous operator and thus one expects that T^* is continuous as well. This is not the case with the definition above as it stands: In general T^* would not be continuous also if starting with T continuous since the algebraic duals V_1^* and V_2^* are not equipped with any natural topology. However if one replaces the algebraic duals with *topological duals*, the definition abolve gives rise to a continuous operator T^* when T is continuous. It happens, in particular, whenever V_1 and V_2 are a Banach spaces.
In this scenario, there is another definition of adjoint operator, in the context of vector spaces with scalar product (including Hilbert spaces). It is worth stressing that the adjoint operator in the sense of Hilbert space theory is different form the adjoint in the sense of definition 2.4. However the two notions enjoy a nice interplay we shall discuss in the comment 7.1 later.

2.2 Multi linearity: tensor product, tensors, universality theorem.

2.2.1 Tensors as multi linear maps.

Let us consider $n \geq 1$ vector spaces $V_1, \ldots, V_n$ on the common field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and another vector space W on $\mathbb{K}$, all spaces are not necessarily finite-dimensional. In the following $\mathscr{L}(V_1, \ldots, V_n|W)$ denotes the vector space (on $\mathbb{K}$) of *multi-linear maps* from $V_1 \times \ldots \times V_n$ to W. We recall the reader that a mapping $f : V_1 \times \ldots \times V_n \to W$ is said to be *multi linear* if, arbitrarily fixing $n-1$ vectors, $v_i \in V_i$ for $i \neq k$, each mapping $v_k \mapsto f(v_1, \ldots, v_k, \ldots, v_n)$ is linear for every $k = 1, \ldots, n$. We leave to the reader the trivial proof of the fact that $\mathscr{L}(V_1, \ldots, V_n|W)$ is a *vector space* on $\mathbb{K}$, with respect to the usual sum of pair of functions and product of an element of $\mathbb{K}$ and a function. If $W = \mathbb{K}$ we use the shorter notation $\mathscr{L}(V_1, \ldots, V_n) := \mathscr{L}(V_1, \ldots, V_n|\mathbb{K})$, $\mathbb{K}$ being the common field of $V_1, \ldots, V_n$.

Exercises 2.2.

1. Suppose that $\{e_{k,i_k}\}_{i_k \in I_k}$ are bases of the finite-dimensional vector spaces V_k, $k = 1, \ldots, n$ on the same field $\mathbb{K}$. Suppose also that $\{e_i\}_{i\in I}$ is a basis of the vector space W on $\mathbb{K}$. Show that each $f \in \mathscr{L}(V_1, \ldots, V_n|W)$ satisfies

$$f(v_1, \ldots, v_n) = v_1^{i_1} \cdots v_n^{i_n} \langle f(e_{1,i_1}, \ldots, e_{n,i_n}), e^{*i} \rangle e_i \ ,$$

for all $(v_1, \ldots, v_n) \in V_1 \times \cdots \times V_n$.

2. Endow $V_1 \times \cdots \times V_n$ with the structure of vector space over the common field $\mathbb{K}$, viewed as the direct sum of $V_1, \ldots, V_n$. In other words, that vector space structure is the unique compatible with the definition:

$$\alpha(v_1, \ldots, v_n) + \beta(v'_1, \ldots, v'_n) := (\alpha v_1 + \beta v'_1, \ldots, \alpha v_n + \beta v'_n)$$

for every numbers $\alpha, \beta \in \mathbb{K}$ and vectors $v_i, v'_i \in V_i$, $i = 1, \ldots, n$.
If $f \in \mathscr{L}(V_1, \ldots, V_n|W)$, may we say that f is a linear map from $V_1 \times \cdots \times V_n$ to W? Generally speaking, is the range of f a subspace of W?
(Hint. No. Is this identity $f(\alpha(v_1, \ldots, v_n)) = \alpha f(v_1, \ldots, v_n)$ true? The answer to the second question is no anyway. For instance, $f(v_1, \ldots, v_n) + f(v'_1, \ldots, v'_n) = f(v_1 + v'_1, \ldots, v_n + v'_n)$ may not hold regardless the multi-linearity of f. This is already evident, for instance, for $n = 2$, $W = \mathbb{R}$ and $V_1 = V_2 = \mathbb{R}$ considering $f : \mathbb{R} \times \mathbb{R} \ni (a, b) \mapsto a \cdot b \in \mathbb{R}$.)

We have a first foundamental and remarkable theorem. To introduce it we employ three steps.
(1) Take $f \in \mathscr{L}(V_1, ..., V_n|W)$. This means that f associates every string $(v_1, ..., v_n) \in V_1 \times \cdots \times V_n$ with a corresponding element $f(v_1, \ldots, v_n) \in W$, and the map

$$(v_1, \ldots, v_n) \mapsto f(v_1, \ldots, v_n)$$

is multi-linear.
(2) Since, for *fixed* $(v_1, \ldots, v_n)$, the vector $f(v_1, \ldots, v_n)$ is an element of W, the action of $w^* \in W^*$ on $f(v_1, \ldots, v_n)$ makes sense, producing $\langle f(v_1, \ldots, v_n), w^* \rangle \in \mathbb{K}$.
(3) Allowing $v_1, \ldots v_n$ and w^* to range freely in the corresponding spaces, the map Ψ_f with

$$\Psi_f : (v_1, \ldots, v_n, w^*) \mapsto \langle f(v_1, \ldots, v_n), w^* \rangle \ ,$$

turns out to be multi-linear by construction. Hence, by definition $\Psi_f \in \mathscr{L}(V_1, \ldots, V_n, W^*)$.
The theorem concerns the map F which associates f with Ψ_f.

Theorem 2.4. *Let $V_1, \ldots, V_n$ be (not necessarily finite-dimensional) vector spaces on the common field $\mathbb{K}$ ($= \mathbb{C}$ or $\mathbb{R}$), and W is another finite-dimensional vector spaces on $\mathbb{K}$.*

The vector spaces $\mathscr{L}(V_1, \ldots, V_n|W)$ and $\mathscr{L}(V_1, \ldots, V_n, W^*)$ are naturally isomorphic by means of the map $F : \mathscr{L}(V_1, \ldots, V_n|W) \to \mathscr{L}(V_1, \ldots, V_n, W^*)$ with $F : f \mapsto \Psi_f$ defined by

$$\Psi_f(v_1, \ldots, v_n, w^*) := \langle f(v_1, \ldots, v_n), w^* \rangle \ , \quad \textit{for all } (v_1, \ldots, v_n) \in V_1 \times \ldots \times V_n \textit{ and } w^* \in W^* .$$

If $\dim W = +\infty$, the above-defined linear map F is injective in any cases.

Proof. Let us consider the mapping F defined above. We have only to establish that F is linear, injective and surjective. This ends the proof.
(Linearity.) We have to prove that $\Psi_{\alpha f+\beta g} = \alpha\Psi_f + \beta\Psi_g$ for all $\alpha, \beta \in \mathbb{K}$ and $f, g \in \mathscr{L}(V_1, \ldots, V_n|W)$. In fact, making use of the left-hand linearity of the pairing, one has

$$\Psi_{\alpha f+\beta g}(v_1, \ldots, v_n, w^*) = \langle (\alpha f+\beta g)(v_1, \ldots, v_n), w^* \rangle = \alpha\langle f(v_1, \ldots, v_n), w^* \rangle + \beta\langle g(v_1, \ldots, v_n), w^* \rangle$$

and this is nothing but:

$$\Psi_{\alpha f+\beta g}(v_1, \ldots, v_n, w^*) = (\alpha\Psi_f + \beta\Psi_g)(v_1, \ldots, v_n, w^*) \ .$$

Since $(v_1, \ldots, v_n, w^*)$ is arbitrary, the thesis is proved.
(Injectivity.) Due to linearity we have only to show that if $\Psi_f = 0$ then $f = 0$.
In fact, if $\Psi_f(v_1, \ldots, v_n, w^*) = 0$ for all $(v_1, \ldots, v_n, w^*)$, using the definition of Ψ_g we have $\langle f(v_1, \ldots, v_n), w^* \rangle = 0$, for all $(v_1, \ldots, v_n)$ and w^*. Then define $e_1 := f(v_1, \ldots, v_n)$, if $e_1 \neq 0$ we can complete it to a basis of W. Fixing $w^* = e^{*1}$ we should have

$$\langle f(v_1, \ldots, v_n), w^* \rangle = 1$$

which contradicts the statement above. Therefore $f(v_1, \ldots, v_n) = 0$ for all $(v_1, \ldots, v_n) \in V_1 \times \ldots \times V_n$, in other words $f = 0$.
(Surjectivity.) We have to show that for each $\Phi \in \mathscr{L}(V_1, \ldots, V_n, W^*)$ there is a $f_\Phi \in \mathscr{L}(V_1, \ldots, V_n|W)$ with $\Psi_{f_\Phi} = \Phi$. To this end fix a basis $\{e_k\}_{k\in I} \subset W$ and the associated dual one $\{e^{*k}\}_{k\in I} \subset W^*$. Then, take $\Phi \in \mathscr{L}(V_1, \ldots, V_n, W^*)$ and define the mapping $f_\Phi \in \mathscr{L}(V_1, \ldots, V_n|W)$ given by

$$f_\Phi(v_1, \ldots, v_n) := \Phi(v_1, \ldots, v_n, e^{*k})e_k \ .$$

By construction that mapping is multi-linear and, using multilinearity we find

$$\Psi_{f_\Phi}(v_1, \ldots, v_n, w^*) = \langle \Phi(v_1, \ldots, v_n, e^{*k})e_k, w^*_i e^{*i} \rangle = \Phi(v_1, \ldots, v_n, w^*_i e^{*k})\langle e_k, e^{*i} \rangle = \Phi(v_1, \ldots, v_n, w^*),$$

for all $(v_1, \ldots, v_n, w^*)$ and this is equivalent to $\Psi_{f_\Phi} = \Phi$.
If $\dim W = +\infty$ the proof of surjectivity may be false since the sum over $k \in I$ used to define f_Φ may diverge. The proofs of linearity and injectivity are independent form the cardinality of I. □

Remark.
An overall difficult point in understanding and trusting in the statement of the theorem above relies upon the fact that the function f has n entries, whereas the function Ψ_f has $n+1$ entries

but f is identified to Ψ_f by means of F. This fact may seem quite weird at first glance and the statement of the theorem may seem suspicious by consequence. The difficulty can be clarified from a practical point of view as follows.
If bases $\{e_{1,i_1}\}_{i_1\in I_1}$,...,$\{e_{n,i_n}\}_{i_n\in I_n}$, for V_1,...,V_n respectively, and $\{e_k\}_{k\in I}$ for W are fixed and $\Psi \in \mathscr{L}(V_1,\ldots,V_n,W^*)$, one has, for constant coefficients $P^k_{i_1\ldots i_n}$ depending on Ψ,

$$\Psi(v_1,\ldots,v_n,w^*) = P^k_{i_1\ldots i_n} v_1^{i_1}\ldots v_n^{i_n} w^*_k\ ,$$

where w^*_k are the components of w^* in the dual basis $\{e^{*k}\}_{k\in I}$ and $v_p^{i_p}$ the components of v_p in the basis $\{e_{p,i_p}\}_{i_p\in I_p}$. Now consider the other map $f \in \mathscr{L}(V_1,\ldots,V_n|W)$ whose argument is the string $(v_1,\ldots,v_n)$, *no further vectors being necessary*:

$$f : (v_1,\ldots,v_n) \mapsto P^k_{i_1\ldots i_n} v_1^{i_1}\ldots v_n^{i_n} e_k$$

The former mapping Ψ, which deals with $n+1$ arguments, is associated with f which deals with n arguments. The point is that we have taken advantage of the bases, $\{e_k\}_{k\in I_k}$ in particular. Within this context, theorem 2.4 just proves that *the correspondence which associates Ψ to f is linear, bijective and independent from the chosen bases.*

In the finite-dimensional case – and we are mainly interested in this case – Theorem 2.4 allow us to restrict our study to the spaces of multi-linear *functionals* $\mathscr{L}(V_1,\ldots,V_k)$, since the spaces of multi-linear *maps* are completely encompassed. Let us introduce the concept of *tensor product* of vector spaces. The following definitions can be extended to encompass the case of non-finite dimensional vector spaces by introducing suitable topological notions (e.g., Hilbert spaces).

Definition 2.5. **(Tensor product.)** Let $U_1,\ldots,U_n$ be $n \geq 1$ vector spaces (not necessarily finite-dimensional) on the common field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
(1) if $(u_1,\ldots,u_n) \in U_1\times\ldots\times U_n$, $u_1\otimes\ldots\otimes u_n$ denotes the multi linear mapping in $\mathscr{L}(U_1^*,\ldots,U_n^*)$ defined by

$$(u_1\otimes\ldots\otimes u_n)(v_1,\ldots,v_n) := \langle u_1,v_1\rangle\cdots\langle u_n,v_n\rangle\ ,$$

for all $(v_1,\ldots,v_n) \in U_1^*\times\ldots\times U_n^*$.
$u_1\otimes\ldots\otimes u_n$ is called **tensor product of vectors** $u_1,\ldots,u_n$.
(2) The mapping $\otimes : U_1\times\ldots\times U_n \to \mathscr{L}(U_1^*,\ldots,U_n^*)$ given by: $\otimes : (u_1,\ldots,u_n) \mapsto u_1\otimes\ldots\otimes u_n$, is called **tensor product map**.
(3) The vector subspace of $\mathscr{L}(U_1^*,\ldots,U_n^*)$ generated by all of $u_1\otimes\ldots\otimes u_n$ for all $(u_1,\ldots,u_n) \in U_1\times\ldots\times U_n$ is called **tensor product of spaces** $U_1,\ldots,U_n$ and is indicated by $U_1\otimes\ldots\otimes U_n$. The vectors in $U_1\otimes\ldots\otimes U_n$ are called **tensors**. $\diamondsuit$

Remarks.
(1) $U_1\otimes\ldots\otimes U_n$ is made of all the linear combinations of the form $\sum_{j=1}^N \alpha_j u_{1,j}\otimes\ldots\otimes u_{n,j}$, where $\alpha_j \in \mathbb{K}$, $u_{k,j} \in U_k$ and $N = 1,2,\ldots$.
(2) It is trivially proved that the tensor product map:

$$(u_1,\ldots,u_n) \mapsto u_1\otimes\ldots\otimes u_n\ ,$$

is *multi linear*.
That is, for any $k \in \{1, 2, \ldots, n\}$, the following identity holds for all $u, v \in U_k$ and $\alpha, \beta \in \mathbb{K}$:

$$u_1 \otimes \ldots \otimes u_{k-1} \otimes (\alpha u + \beta v) \otimes u_{k+1} \otimes \ldots \otimes u_n$$

$$= \alpha(u_1 \otimes \ldots \otimes u_{k-1} \otimes u \otimes u_{k+1} \otimes \ldots \otimes u_n)$$

$$+\beta(u_1 \otimes \ldots \otimes u_{k-1} \otimes v \otimes u_{k+1} \otimes \ldots \otimes u_n)\,.$$

As a consequence, it holds

$$(\alpha u) \otimes v = u \otimes (\alpha v) = \alpha(u \otimes v)\,,$$

and similar identities hold considering whichever number of factor spaces in a tensor product of vector spaces.
(3) From the given definition, if $U_1, \ldots, U_n$ are given vector spaces on the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, what we mean by $U_1 \otimes \ldots \otimes U_n$ or $U_1^* \otimes \ldots \otimes U_n^*$ but also, for instance, $U_1^* \otimes U_2 \otimes U_3$ or $U_1 \otimes U_2^* \otimes U_3^*$ should be obvious. One simply has to take the action of U^* on $(U^*)^*$ into account.

By definition $U_1 \otimes \ldots \otimes U_n \subset \mathscr{L}(U_1^*, \ldots, U_n^*)$, more strongly, $U_1 \otimes \ldots \otimes U_n$ is, in general, a *proper* subspace of $\mathscr{L}(U_1^*, \ldots, U_n^*)$. The natural question which arises is if there are hypotheses which entail that $U_1 \otimes \ldots \otimes U_n$ coincides with the whole space $\mathscr{L}(U_1^*, \ldots, U_n^*)$. The following theorem gives an answer to that question.

Theorem 2.5. *If all the spaces U_i have finite dimension, one has:*

$$U_1 \otimes \ldots \otimes U_n = \mathscr{L}(U_1^*, \ldots, U_n^*)\,.$$

Proof. It is sufficient to show that if $f \in \mathscr{L}(U_1^*, \ldots, U_n^*)$ then $f \in U_1 \otimes \ldots \otimes U_n$. To this end fix vector bases $\{e_{k,i}\}_{i \in I_k} \subset U_k$ for $k = 1, \ldots, n$ and consider also the associated dual bases $\{e_k^{*i}\}_{i \in I_k}$. f above is completely determined by coefficients (their number is finite since $dim U_i^* = dim U_i < +\infty$ by hypotheses!) $f_{i_1,\ldots,i_n} := f(e_1^{*i_1}, \ldots, e_n^{*i_n})$. Every $v_k \in U_k^*$ can be decomposed as $v_k = v_{k,i_k} e_k^{*i_k}$ and thus, by multi linearity:

$$f(v_1, \ldots, v_n) = v_{1,i_1} \cdots v_{n,i_n} f(e_1^{*i_1}, \ldots, e_n^{*i_n})\,.$$

Then consider the tensor $t_f \in U_1 \otimes \ldots \otimes U_n$ defined by:

$$t_f := f(e_1^{*i_1}, \ldots, e_n^{*i_n}) e_{1,i_1} \otimes \ldots \otimes e_{n,i_n}\,.$$

Then, by def.2.5, one can directly prove that, by multi linearity

$$t_f(v_1, \ldots, v_n) = v_{1,i_1} \cdots v_{n,i_n} f(e_1^{*i_1}, \ldots, e_n^{*i_n}) = f(v_1, \ldots, v_n)\,,$$

for all of $(v_1, \ldots, v_n) \in U_1^* \times \ldots \times U_n^*$. This is nothing but $t_f = f$. $\Box$

Another relevant result is stated by the theorem below.

Theorem 2.6. *Consider finite-dimensional vector spaces U_i, $i = 1, \ldots, n$ with the same field $\mathbb{K}$. The following statements hold.*
(a) *The dimension of $U_1 \otimes \ldots \otimes U_n$ is:*

$$dim(U_1 \otimes \ldots \otimes U_n) = \prod_{k=1}^{n} dim\, U_k \,.$$

(b) *If $\{e_{k,i_k}\}_{i_k \in I_k}$ is a basis of U_k, $k = 1, \ldots, n$, then $\{e_{1,i_1} \otimes \ldots \otimes e_{n,i_n}\}_{(i_1,\ldots,i_n) \in I_1 \times \ldots \times I_n}$ is a vector basis of $U_1 \otimes \ldots \otimes U_n$.*
(c) *If $t \in U_1 \otimes \ldots \otimes U_n$, the components of t with respect to a basis $\{e_{1,i_1} \otimes \ldots \otimes e_{n,i_n}\}_{(i_1,\ldots,i_n) \in I_1 \times \ldots \times I_n}$ are given by:*

$$t^{i_1 \ldots i_n} = t(e_1^{*i_1}, \ldots, e_n^{*i_n})$$

and thus it holds

$$t = t(e_1^{*i_1}, \ldots, e_n^{*i_n}) e_{1,i_1} \otimes \ldots \otimes e_{n,i_n} \,.$$

Proof. Notice that **(b)** trivially implies **(a)** because the elements $e_{1,i_1} \otimes \ldots \otimes e_{n,i_n}$ are exactly $\prod_{k=1}^{n} dim U_k$. So it is sufficient to show that the second statement holds true. To this end, since elements $e_{1,i_1} \otimes \ldots \otimes e_{n,i_n}$ are generators of $U_1 \otimes \ldots \otimes U_n$, it is sufficient to show that they are linearly independent. Consider the generic vanishing linear combination

$$C^{i_1 \ldots i_n} e_{1,i_1} \otimes \ldots \otimes e_{n,i_n} = 0 \,,$$

We want to show that all of the coefficients $C^{i_1 \ldots i_n}$ vanish. The action of that linear combination of multi-linear functionals on the generic element $(e_1^{*j_1}, \ldots, e_n^{*j_n}) \in U_1^* \times \cdots \times U_n^*$ produces the result

$$C^{j_1 \ldots j_n} = 0 \,.$$

Since we can arbitrarily fix the indices $j_1, \ldots j_n$ this proves the thesis.
Concerning **(c)**, by the uniqueness of the components of a vector with respect to a basis, it is sufficient to show that, defining (notice that the number of vectors $e_{1,i_1} \otimes \ldots \otimes e_{n,i_n}$ is finite by hypotheses since $dim U_k$ is finite in the considered case)

$$t' := t(e_1^{*i_1}, \ldots, e_n^{*i_n}) e_{1,i_1} \otimes \ldots \otimes e_{n,i_n} \,,$$

it holds

$$t'(v_1, \ldots, v_n) = t(v_1, \ldots, v_n) \,,$$

for all $(v_1, \ldots, v_n) \in U_1^* \times \ldots \times U_n^*$. By multi linearity,

$$t(v_1, \ldots, v_n) = v_{1\, i_1} \cdots v_{n\, i_n} t(e_1^{*i_1}, \ldots, e_n^{*i_n}) = t'(v_1, \ldots, v_n) \,.$$

□

We Remark that, obviously, if some of the spaces U_i has infinite dimension $U_1 \otimes \ldots \otimes U_n$ cannot have finite dimension.

In case of *finite dimension* of all involved spaces, there is an important result which, *together with the identification of* V *and* $(V^*)^*$, imply that all of the spaces which one may build up using the symbols $\otimes$, V_k, * and () coherently are naturally isomorphic to spaces which are of the form $V_{i_1}^{(*)} \otimes \ldots \otimes V_{i_n}^{(*)}$. The rule to produce spaces naturally isomorphic to a given initial space is that one has to (1) ignore parentheses – that is $\otimes$ *is associative* –, (2) assume that * is *distributive with respect to* $\otimes$ and (3) assume that * is *involutive* (i.e. $(X^*)^* \simeq X$). For instance, one has:

$$((V_1^* \otimes V_2) \otimes (V_3 \otimes V_4^*))^* \simeq V_1 \otimes V_2^* \otimes V_3^* \otimes V_4 .$$

Let us state the theorems corresponding to the rules (2) and (1) respectively (the rule (3) being nothing but theorem 2.2).
If not every V_i has finite dimension the distributivity and involutivity properties of * may fail to be fulfilled, but associativity of $\otimes$ halds true anyway.

Theorem 2.7. *If* $V_1 \ldots V_n$ *are finite-dimensional vector spaces on the same field* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$, * *is* **distributive** *with respect to* $\otimes$ *by means of* **natural isomorphisms** $F : V_1^* \otimes \ldots \otimes V_n^* \to (V_1 \otimes \ldots \otimes V_n)^*$. *In other words it holds:*

$$(V_1 \otimes \ldots \otimes V_n)^* \simeq V_1^* \otimes \ldots \otimes V_n^* .$$

under F. *The isomorphism* F *is uniquely determined by the requirement*

$$\langle u_1 \otimes \ldots \otimes u_n, F(v_1^* \otimes \ldots \otimes v_n^*)\rangle = \langle u_1, v_1^*\rangle \cdots \langle u_n, v_n^*\rangle , \tag{2.1}$$

for every choice of $v_i^* \in V_i^*$, $u_i \in V_i$ *and* $i = 1, 2, \ldots, n$.

Proof. The natural isomorphism is determined by (2.1), thus employing only the general mathematical structures of dual space and tensor product. Consider, *if it exists* (see the discussion below), the linear mapping $F : V_n^* \otimes \ldots \otimes V_n^* \to (V_1 \otimes \ldots \otimes V_n)^*$ which satisfies, for all $u_1 \otimes \ldots \otimes u_n \in V_1 \otimes \ldots \otimes V_n$,

$$\langle u_1 \otimes \ldots \otimes u_n, F(v_1^* \otimes \ldots \otimes v_n^*)\rangle := \langle u_1, v_1^*\rangle \ldots \langle u_n, v_n^*\rangle . \tag{2.2}$$

Using the linearity and the involved definitions the reader can simply prove that the mapping F is surjective. Indeed, equipping each space V_k with a basis $\{e_{k,i_k}\}_{i_k \in I_k}$ and the dual space V_k^* with the corresponding duel basis $\{e_k^{*i_k}\}_{i_k \in I_k}$, if $f \in (V_1 \otimes \cdots \otimes V_n)^*$, defining the numbers $f_{i_1 \cdots i_n} := f(e_{1,i_1} \otimes \cdots \otimes e_{n,i_n})$, by direct inspection one finds that:

$$\langle u_1 \otimes \ldots \otimes u_n, F(f_{i_1 \cdots i_n} e_1^{*i_1} \otimes \cdots \otimes e_n^{*i_n})\rangle = \langle u_1 \otimes \ldots \otimes u_n, f\rangle ,$$

for all $u_1 \otimes \ldots \otimes u_n$. By linearity this result extends to any element $u \in V_1 \otimes \ldots \otimes V_n$ in place of $u_1 \otimes \ldots \otimes u_n$ so that $f(u) = \left(F(f_{i_1 \cdots i_n} e_1^{*i_1} \otimes \cdots \otimes e_n^{*i_n})\right)(u)$ for every $u \in V_1 \otimes \cdots \otimes V_n$. Therefore the functionals $f, F(f_{i_1 \cdots i_n} e_1^{*i_1} \otimes \cdots \otimes e_n^{*i_n}) \in (V_1 \otimes \ldots \otimes V_n)^*$ concide:

$$f = F(f_{i_1 \cdots i_n} e_1^{*i_1} \otimes \cdots \otimes e_n^{*i_n}) \,.$$

Finally, by the theorems proved previously,

$$dim(V_1 \otimes \cdots \otimes V_n)^* = dim(V_1 \otimes \cdots \otimes V_n) = dim(V_1^* \otimes \cdots \otimes V_n^*)$$

As a consequence, F must be also injective and thus it is an isomorphism. □

Theorem 2.8. *The tensor product is* **associative** *by means of* **natural isomorphisms**. *In other words considering a space which is made of tensor products of vector spaces on the same field $\mathbb{R}$ or $\mathbb{C}$, one may omit parenthesis everywhere obtaining a space which is naturally isomorphic to the initial one. So, for instance*

$$V_1 \otimes V_2 \otimes V_3 \simeq (V_1 \otimes V_2) \otimes V_3 \,.$$

Where the natural isomorphism $F_1 : V_1 \otimes V_2 \otimes V_3 \to (V_1 \otimes V_2) \otimes V_3$ satisfies

$$F_1 : v_1 \otimes v_2 \otimes v_3 \to (v_1 \otimes v_2) \otimes v_3$$

for every choice of $v_i \in V_i$, $i = 1, 2, 3$.

Proof. The natural isomorphism is constructed as a linear mapping, *if it exists* (see discussion below), which linearly extends the action $F_1 : V_1 \otimes V_2 \otimes V_3 \to (V_1 \otimes V_2) \otimes V_3$ such that

$$F_1 : v_1 \otimes v_2 \otimes v_3 \mapsto (v_1 \otimes v_2) \otimes v_3 \,,$$

for all $v_1 \otimes v_2 \otimes v_3 \in V_1 \otimes V_2 \otimes V_3$.
Injectivity and surjectiviy can be proved by considering the linear map *if it exists* (see discussion below), $F_1' : (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes V_2 \otimes V_3$ such that

$$F_1' : (v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes v_2 \otimes v_3 \,,$$

for all $(v_1 \otimes v_2) \otimes v_3 \in (V_1 \otimes V_2) \otimes V_3$. By construction $F_1 \circ F_1' = id_{(V_1 \otimes V_2) \otimes V_3}$ and $F_1' \circ F_1 = id_{V_1 \otimes V_2 \otimes V_3}$ are the identity maps so that F_1 is linear and bijective. □

Remark. The central point is that the mappings F and F_1 (and F_1') above have been given by specifying their action on tensor products of elements (e.g, $F(v_1 \otimes \ldots \otimes v_n)$) and not on *linear combinations* of these tensor products of elements. Recall that, for instance $V_1^* \otimes \ldots \otimes V_n^*$ is not the set of products $u_1^* \otimes \ldots \otimes u_n^*$ but it is the set of *linear combinations* of those products. Hence, in order to completely define F and F_1, one must require that F and F_1 admit *uniquely* *determined linear extensions* on their initial domains in order to encompass the whole tensor spaces generated by linear combinations of simple tensor products. In other words one has to complete the given definition, in the former case, by adding the further requirement

$$F(\alpha\, u_1^* \otimes \ldots \otimes u_n^* + \beta\, v_1^* \otimes \ldots \otimes v_n^*) = \alpha F(u_1^* \otimes \ldots \otimes u_n^*) + \beta F(v_1^* \otimes \ldots \otimes v_n^*)\,,$$

Actually, one has to assume in addition that, for any fixed $v_1^* \otimes \cdots \otimes v_n^*$, there is a unique linear extension, of the application $F(v_1^* \otimes \cdots \otimes v_n^*)$ defined in (2.2), when the argument is a linear combination of vectors of type $u_1 \otimes \cdots \otimes u_n$.
Despite these could seem trivial requirements (and they are, in the present case) they are not trivial in general. The point is that the class of all possible products $v_1^* \otimes \ldots \otimes v_n^*$ is a system of generators of $V_1^* \otimes \ldots \otimes V_n^*$, but it is by no means a basis *because these elements are linearly dependent*. So, in general any attempt to assign a linear application with domain $V_1^* \otimes \ldots \otimes V_n^*$ by assigning the values of it on all the elements $v_1^* \otimes \ldots \otimes v_n^*$ may give rise to contraddictions. For instance, $v_1^* \otimes \ldots \otimes v_n^*$ can be re-written using linear combinations:

$$v_1^* \otimes \ldots \otimes v_n^* = [(v_1^* + u_1^*) \otimes \ldots \otimes v_n^*] \,-\, [u_1^* \otimes \ldots \otimes v_n^*]\,.$$

Now consider the identities, which has to hold as a consequence of the assumed linearity of F, such as:

$$F(v_1^* \otimes \ldots \otimes v_n^*) = F((v_1^* + u_1^*) \otimes \ldots \otimes v_n^*) - F(u_1^* \otimes \ldots \otimes v_n^*)$$

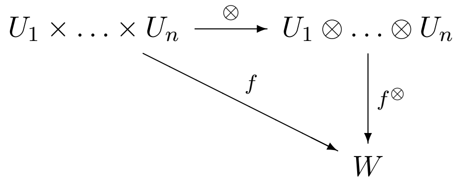
Above $F(v_1^* \otimes \ldots \otimes v_n^*)$, $F(u_1^* \otimes \ldots \otimes v_n^*)$, $F((v_1^* + u_1^*) \otimes \ldots \otimes v_n^*)$ are *independently* defined as we said at the beginning and there is no reason, in principle, for the validity of the constraint:

$$F(v_1^* \otimes \ldots \otimes v_n^*) = F((v_1^* + u_1^*) \otimes \ldots \otimes v_n^*) - F(u_1^* \otimes \ldots \otimes v_n^*)\,.$$

Similar problems may arise concerning F_1.
The general problem which arises by the two considered cases can be stated as follows. Suppose we are given a tensor product of vector spaces $V_1 \otimes \ldots \otimes V_n$ and we are interested in the possible *linear extensions* of a mapping f on $V_1 \otimes \ldots \otimes V_n$, with values in some vector space W, when f is initially defined on the whole calss of the simple products $v_1 \otimes \ldots \otimes v_n \in V_1 \otimes \ldots \otimes V_n$ only.
Is there any general prescription on the specification of values $f(v_1 \otimes \ldots \otimes v_n)$ which assures that f can be extended, uniquely*, to a linear mapping from $V_1 \otimes \ldots \otimes V_n$ to W?*
An answer is given by the following very important **universality theorem** discussed in the following subsection.

2.2.2 Universality theorem and its applications.

Theorem 2.9. **(Universality Theorem.)** *Given $n \geq 1$ vector spaces $U_1, \ldots, U_n$ (not necessarily finite-dimensional) on the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the following so-called* **universal property** *holds for the pair $(U_1 \otimes \ldots \otimes U_n, \otimes)$. For any vector space W (not necessarily finite-dimensional) and any multi-linear mapping $f : U_1 \times \ldots \times U_n \to W$, there is a unique linear* *mapping* $f^{\otimes} : U_1 \otimes \ldots \otimes U_n \to W$ *such that the diagram below* **commutes** *(in other words,* $f^{\otimes} \circ \otimes = f$*).*



Proof. Fix bases $\{e_{k,i_k}\}_{i_k \in I_k} \subset U_k$, $k = 1, \ldots, n$. By linearity, a linear mapping $g : U_1 \otimes \ldots \otimes U_n \to W$ is uniquely assigned by fixing the set of vectors

$$\{g(e_{1,i_1} \otimes \cdots \otimes e_{n,i_n})\}_{i_1 \in I_1, \ldots, i_n \in I_n} \subset W \, .$$

This is obviously true also if some of the spaces U_k, W have infinite dimension. Define the linear function $f^{\otimes} : U_1 \otimes \ldots \otimes U_n \to W$ as the unique linear map with

$$f^{\otimes}(e_{1,i_1} \otimes \cdots \otimes e_{n,i_n}) := f(e_{1,i_1}, \ldots, e_{n,i_n}) \, .$$

By construction, using linearity of $f^{\otimes}$ and multi-linearity of f, one gets immediately:

$$f^{\otimes}(v_1 \otimes \ldots \otimes v_n) = f(v_1, \ldots, v_n)$$

for all $(v_1, \ldots, v_n) \in U_1 \times \ldots \times U_n$. In other words $f^{\otimes} \circ \otimes = f$.
The uniqueness of the mapping $f^{\otimes}$ is obvious: suppose there is another mapping $g^{\otimes}$ with $g^{\otimes} \circ \otimes = f$ then $(f^{\otimes} - g^{\otimes}) \circ \otimes = 0$. This means in particular that

$$(f^{\otimes} - g^{\otimes})(e_{1,i_1} \otimes \ldots \otimes e_{n,i_n}) = 0 \, ,$$

for all $i_k \in I_k$, $k = 1, \ldots, n$. Since the coefficients above completely determine a map, the considered mapping must be the null mapping and thus: $f^{\otimes} = g^{\otimes}$. $\Box$

Let us now explain how the universality theorem gives a precise answer to the question formulated before the universality theorem. The theorem says that a *linear extension* of any function f with values in W, initially defined on simple tensor products only, $f(v_1 \otimes \ldots \otimes v_n)$ with $v_1 \otimes \ldots \otimes v_n \in U_1 \otimes \ldots \otimes U_n$, does *exist* on the whole domain space $U_1 \otimes \ldots \otimes U_n$ and it is *uniquely* determined provided $f(v_1 \otimes \ldots \otimes v_n) = g(v_1, \ldots, v_n)$ where $g : U_1 \times \ldots \times U_n \to W$ is some *multi-linear function.*
Concerning the mappings F and F_1 introduced above, we may profitably use the universality theorem to show that they are well-defined on the whole domain made of linear combinations of simple tensor products. In fact, consider F for example. We can define the *multi-linear* mapping $G : V_1^* \times \ldots \times V_n^* \to (V_1 \otimes \ldots \otimes V_n)^*$ such that $G(v_1^*, \ldots, v_n^*) : V_1 \otimes \ldots \otimes V_n \to \mathbb{C}$ is the unique linear function (assumed to exist, see below) which linearly extends the requirement

$$(G(v_1^*, \ldots, v_n^*))\,(u_1 \otimes \cdots \otimes u_n) := \langle u_1, v_1^* \rangle \cdots \langle u_n, v_n^* \rangle, \quad \text{for all } (u_1, \ldots, u_n) \in U_1 \times \cdots \times U_n \quad (2.3)$$

to the whole space $V_1 \otimes \ldots \otimes V_n$. Then the universality theorem with $U_k = V_k^*$ and $W = (V_1 \otimes \ldots \otimes V_n)^*$ assures the existence of a linear mapping $F : V_1^* \otimes \ldots \otimes V_n^* \to (V_1 \otimes \ldots \otimes V_n)^*$ with the required properties because $F := G^{\otimes}$ is such that

$$F(v_1^* \otimes \ldots \otimes v_n^*) = (G^{\otimes} \circ \otimes)(v_1^*, \ldots, v_n^*) = G(v_1^*, \ldots, v_n^*) \; ,$$

so that (2.2) is fulfilled due to (2.3). Finally, the existence of the linear function $G(v_1^*, \ldots, v_n^*) : V_1 \otimes \ldots \otimes V_n \to \mathbb{K}$ fulfilling (2.3) can be proved, once again, employing the universality theorem for $U_k := V_k$ and $W := \mathbb{K}$. Starting form the multi linear function $G'(v_1^*, \ldots, v_n^*) : V_1 \times \ldots \times V_n \to \mathbb{C}$ such that

$$\left(G'(v_1^*, \ldots, v_n^*)\right)(u_1, \cdots, u_n) := \langle u_1, v_1^* \rangle \cdots \langle u_n, v_n^* \rangle$$

it is enough to define $G(v_1^*, \ldots, v_n^*) := G'(v_1^*, \ldots, v_n^*)^{\otimes}$.
A similar multi-linear mapping G_1 can be found for F_1:

$$G_1 : (v_1, v_2, v_3) \mapsto (v_1 \otimes v_2) \otimes v_3 \; .$$

Then: $F_1 := G_1^{\otimes}$ can be used in the proof of theorem 2.8. If V_1, V_2, V_3 have finite dimension, the proof of theorem 2.8 ends because the map F_1 so obtained is trivially surjective and it is also injective since

$$dim((V_1 \otimes V_2) \otimes V_3) = dim(V_1 \otimes V_2) \times dim V_3 = dim(V_1) \times dim(V_2) \times dim(V_3) = dim(V_1 \otimes V_2 \otimes V_3)$$

by Theorem 2.6.
If not every V_1, V_2, V_3 has finite dimension the proof Theorem 2.8 is more complicated and the existence of the function F_1' used in its proof has to be explicitly checked. The construction of the function F_1' used in the proof of theorem 2.8 can be obtained with a little change in the first statement of the universality theorem. With trivial changes one gets that:

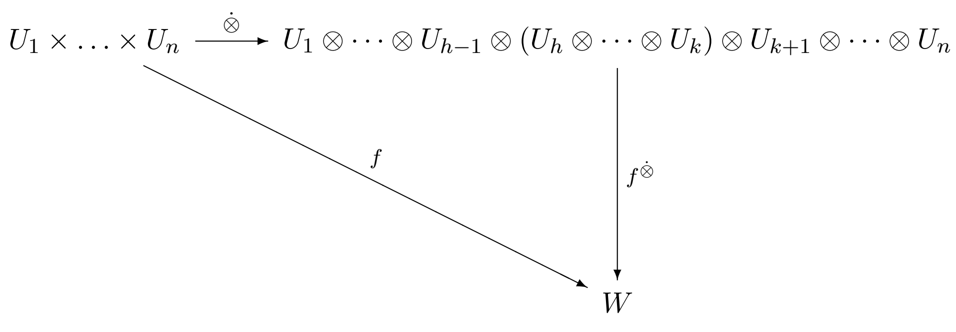
Proposition 2.1. *Given $n \geq 1$ vector spaces $U_1, \ldots, U_n$ on the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the following statements hold.*
If $h < k = 2, \ldots, n$ are fixed, for any vector space W and any multi-linear mapping

$$f : U_1 \times \ldots \times U_n \to W \; ,$$

there is a unique linear mapping

$$f^{\dot{\otimes}} : U_1 \otimes \cdots \otimes U_{h-1} \otimes (U_h \otimes \cdots \otimes U_k) \otimes U_{k+1} \otimes \cdots \otimes U_n \to W$$

such that the diagram below commute.



Above

$$\dot{\otimes} : U_1 \times \ldots \times U_n \to U_1 \otimes \cdots \otimes U_{h-1} \otimes (U_h \otimes \cdots \otimes U_k) \otimes U_{k+1} \otimes \cdots \otimes U_n \,.$$

is defined as the multi-linear function such that:

$$\dot{\otimes} : (u_1, \ldots, u_n) \mapsto u_1 \otimes \cdots \otimes u_{h-1} \otimes (u_h \otimes \cdots \otimes u_k) \otimes u_{k+1} \otimes \cdots \otimes u_n \,.$$

Now the linear mapping $F'_1 : (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes V_2 \otimes V_3$ such that linearly extends

$$(v_1 \otimes v_2) \otimes v_3 \to v_1 \otimes v_2 \otimes v_3$$

is nothing but the function $H_1^{\dot{\otimes}}$

$$H : (v_1, v_2, v_3) \to v_1 \otimes v_2 \otimes v_3$$

when $n = 3$, $h = 1$, $k = 2$ and $W := V_1 \otimes V_2 \otimes V_3$.

Exercises 2.3.

1. Consider a finite-dimensional vector space V and its dual V^*. Show by the universality theorem that there is a natural isomorphism such that

$$V \otimes V^* \simeq V^* \otimes V \,.$$

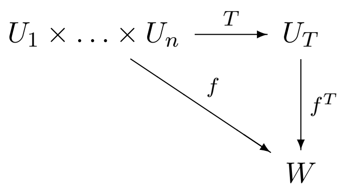
(Hint. Consider the bilinear mapping $f : V \times V^* \to V^* \otimes V$ with $f : (v_1, v_2^*) \mapsto v_2^* \otimes v_1$. Show that $f^{\otimes}$ is injective and thus surjective because $dim(V^* \otimes V) = dim(V \otimes V^*)$.)

2. Extend the result in exercise 1 to the infinite-dimensional case.

2.2.3 Abstract definition of the tensor product of vector spaces.

The property of the pair $(\otimes, U_1 \otimes \ldots \otimes U_n)$ stated in the theorem 2.9, the *universality property (of the tensor product)*, is very important from a theoretical point of view. As a matter of fact, it can be used, adopting a more advanced theoretical point of view, to *define* the tensor product of given vector spaces. This is due to the following important result.

Theorem 2.10. *Given $n \geq 1$ vector spaces $U_1, \ldots, U_n$ (not necessarily finite dimensional) on the same field $\mathbb{K}$ (not necessarily $\mathbb{R}$ or $\mathbb{C}$)*[1] *suppose there is a pair (T, U_T) where U_T is a vector space on $\mathbb{K}$ and $T : U_1 \times \ldots \times U_n \to U_T$ a multi linear map, fulfilling the universal property. That is, for any vector space W on $\mathbb{K}$ and any multi linear map $f : U_1 \times \ldots \times U_n \to W$, there is a unique linear map $f^T : U_T \to W$ such that the diagram below commute,*

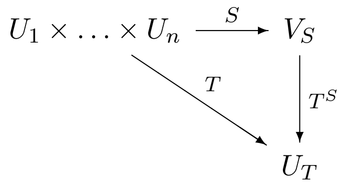


[1] The theorem can be proved assuming that $U_1, \ldots, U_n, U_T$ are *modules* on a common *ring* R.

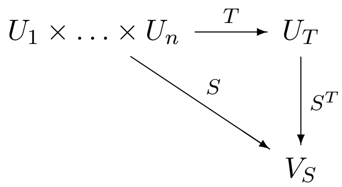
Under these hypotheses the pair (T, U_T) is determined up to vector space isomorphisms.
In other words, for any other pair (S, V_S) fulfilling the universality property with respect to $U_1, \ldots, U_n$, there is a unique isomorphism $\phi : V_S \to U_T$ such that $f^T \circ \phi = f^S$ for every multi linear map $f : U_1 \times \ldots \times U_n \to W$ and every vector space W.

Proof. Suppose that there is another pair (S, V_S) fulfilling the universality property with respect to $U_1, \ldots, U_n$. Then, using the universal property with $f = T$ and $W = U_T$ we have the diagram below, with the former diagram commutation relation:

$$T^S \circ S = T \,. \tag{2.4}$$



On the other hand, using the analogous property of the pair (T, U_T) with $f = S$ and $W = V_S$ we also have the commutative diagram



which involves a second diagram commutation relation:

$$S^T \circ T = S \,.$$

The two obtained relations imply

$$(T^S \circ S^T) \circ T = T$$

and

$$(S^T \circ T^S) \circ S = S \,,$$

In other words, if $RanT \subset U_T$ is the range of the mapping T and $RanS \subset V_S$ is the analogue for S:

$$(T^S \circ S^T)|_{RanT} = Id_{RanT}$$

and

$$(S^T \circ T^S)|_{RanS} = Id_{RanS}$$

Then consider the former. Notice that $RanT$ is not a subspace, however, the subspace spanned by $RanT$, $Span(RanT)$, coincides with the whole space U_T. This is due to the uniqueness property of f^T for a fixed f. (For, if $Span(RanT) \neq U_T$, then $U_T = Span(RanT) \oplus Z$, $Z \neq \{0\}$

being some proper subspace of U_T with $Z \cap Span(RanT) = \{0\}$. So, a mapping f^T would not be uniquely determined by the requirement $f^T \circ T = f$ because such a relation would be preserved under modifications of f^T in the subspace Z.) Hence the identity $(T^S \circ S^T)|_{RanT} = Id_{RanT}$ implies $T^S \circ S^T = Id_{U_T}$ by linearity, as $(T^S \circ S^T)$ being linear. The analog holds for $S^T \circ T^S$, i.e. $S^T \circ T^S = Id_{V_S}$. Summarizing, we have found that the linear map $\phi := T^S : V_S \to U_T$ is a vector-space isomorphism. As far as the property $f^T \circ \phi = f^S$ is concerned, we may observe that, by definition of f^S and f^T the maps f^S and $f^T \circ \phi$ satisfy the universal property for f with respect to the pair (S, V_S):

$$f^S \circ S = f \quad \text{and} \quad (f^T \circ \phi) \circ S = f \, , \tag{2.5}$$

and thus f^S and $f^T \circ \phi$ have to coincide.
Indeed, the former in (2.5) is valid by definition. Furthermore, using (2.4), one achieves the latter:

$$(f^T \circ \phi) \circ S = f^T \circ T^S \circ S = f^T \circ (T^S \circ S) = f^T \circ T = f \, .$$

To conclude we prove that $\phi : V_S \to U_T$ verifying $f^T \circ \phi = f^S$ is unique. Indeed if $\phi' : V_S \to U_T$ is another isomorphism verifying $f^T \circ \phi' = f^S$, we have $f^T \circ (\phi' - \phi) = 0$ for every $f : U_1 \times \cdots \times U_n \to W$ multilinear. Choose $W = U_T$ and $f = T$ so that $f^T = id_{U_T}$ and $f^T \circ (\phi' - \phi) = 0$ implies $\phi' - \phi = 0$, that is $\phi' = \phi$. $\Box$

Obviously, first of all this theorems applies on the pair $(\otimes, U_1 \otimes, \cdots \otimes U_n)$ on the field $\mathbb{R}$ or $\mathbb{C}$. In fact, we wish to stress that, taking advantage of such a result, given the generic vector spaces $U_1, \ldots, U_n$ on the generic field $\mathbb{K}$, one may says that a vector space U_T on $\mathbb{K}$, equipped with a multi linear map $T : U_1 \times \ldots \times U_n \to U_T$, is the **tensor product** of the spaces and T is the **tensor product map**, if the pair (T, U_T) fulfills the universality property. In this way (T, U_T) turns out to be defined *up to vector-space isomorphisms*. Obviously, if one aims to follow this very general way to define tensor products, he/she still has to show that a pair (T, U_T) exists for the considered set of vector spaces $U_1, \ldots, U_n$. This is exactly what we done within our constructive approach case when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. However the existence of a pair (T, U_T) satisfying the universality property can be given similarly also for a larger class of vector spaces than that of finite-dimensional ones on the field $\mathbb{R}$ or $\mathbb{C}$ and also for *modules on rings*. Moreover the explicit construction of a pair (T, U_T) fulfilling the universality property can be produced, within a more abstract approach, by taking the quotient of a suitable freely generated module with respect to suitable sub modules [Lang2].

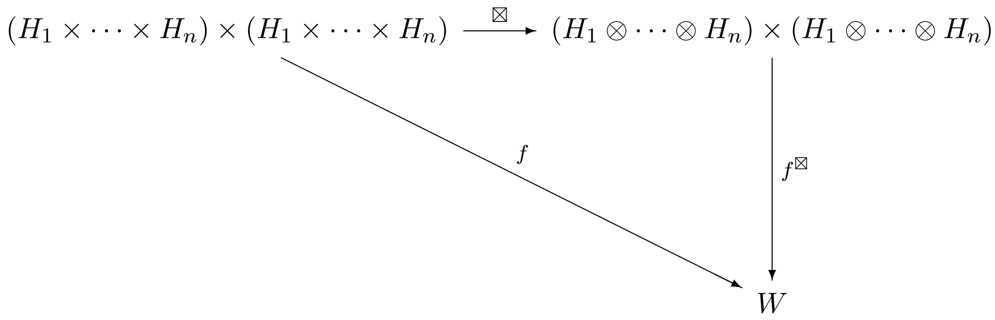
2.2.4 Hilbertian tensor product of Hilbert spaces.

This subsection is devoted to define the Hilbertian tensor product of Hilbert spaces which plays a key role in applications to Quantum Mechanics in reference to composite systems. The issue is quite far from the main stream of these lecture notes and it could be omitted by the reader who is not strictly interested, also because it involves some advanced mathematical tools [Rudin] which will not be treated in detail within these notes (however see Chapter 5 for some elementary pieces of information about Hilbert spaces). We assume here that the reader is familiar with the basic notions of complex (generally nonseparable) Hilbert space and the notion of H bases (also called orthonormal systems).

Consider a set of complex Hilbert spaces $(H_i, (\cdot|\cdot)_i)$, where $i = 1, \ldots, n < +\infty$ and the spaces are not necessarily separable. Here $(\cdot|\cdot)_i$ denotes the Hermitean scalar product on H_i. (In these lectures, concerning Hermitean scalar products, we always assume the convention that each $(\cdot|\cdot)_i$ is antilinear in the left entry.) Since the H_i are vector spaces on $\mathbb{C}$, their tensor product $H_1 \otimes \cdots \otimes H_n$ is defined as pure algebraic object, but it does not support any preferred Hermitean scalar product. We wish to specialize the definition of tensor product in order to endow $H_1 \otimes \cdots \otimes H_n$ with the structure of Hilbert space, induced by the structures of the $(H_i, (\cdot|\cdot)_i)$ naturally. Actually, this specialization requires an extension of $H_1 \otimes \cdots \otimes H_n$, to assure the completeness of the space. The key tool is the following theorem whose nature is very close to that of the universality theorem (Theorem 2.9). This result allow us to define a preferred hermitean product $(\cdot|\cdot)$ on $H_1 \otimes \cdots \otimes H_n$ induced by the $(\cdot|\cdot)_i$. The Hilbert structure generated by that scalar product on $H_1 \otimes \cdots \otimes H_n$ will be the wanted one.

Theorem 2.11. *Given $n \geq 1$ complex Hilbert spaces $(H_i, (\cdot|\cdot)_i)$, $i = 1, \ldots, n < +\infty$ the following holds.*
(a) *For any complex vector space W and a mapping $f : (H_1 \times \cdots \times H_n) \times (H_1 \times \cdots \times H_n) \to W$ that is anti linear in each one of the first n entries and linear in each one of the remaining n ones, there is a unique mapping $f^{\boxtimes} : (H_1 \otimes \ldots \otimes H_n) \times (H_1 \otimes \ldots \otimes H_n) \to W$ which is anti linear in the left-hand argument and linear in the right-hand one and such that the diagram below commutes (in other words, $f^{\boxtimes} \circ \boxtimes = f$), where we have defined:*

$$\boxtimes : (v_1, \ldots, v_n, u_1, \ldots, u_n) \mapsto (v_1 \otimes \cdots \otimes v_n, u_1 \otimes \cdots \otimes u_n) \quad \textit{for all } u_i, v_i \in H_i,\ i = 1, \ldots, n.$$



(b) *If $W := \mathbb{C}$ and it holds*

$$f((v_1, \ldots, v_n), (u_1, \ldots, u_n)) = \overline{f((u_1, \ldots, u_n), (v_1, \ldots, v_n))} \quad \textit{for all } u_i, v_i \in H_i,\ i = 1, \ldots, n,$$

then the map $f^{\boxtimes}$ as in (a) fulfills

$$f^{\boxtimes}(t, s) = \overline{f^{\boxtimes}(s, t)} \quad \text{for all } s, t \in H_1 \otimes \cdots \otimes H_n.$$

Proof. (a) Fix (algebraic) bases $\{e_{k,i_k}\}_{i_k\in I_k} \subset H_k$, $k = 1,\dots,n$. By (anti)linearity, a mapping $g : (H_1 \otimes \dots \otimes H_n) \times (H_1 \otimes \dots \otimes H_n) \to W$, which is anti linear in the left-hand entry and linear in the right-hand one, is uniquely assigned by fixing the set of vectors

$$\{g(e_{1,i_1} \otimes \cdots \otimes e_{n,i_n}, e_{1,j_1} \otimes \cdots \otimes e_{n,j_n})\}_{i_1\in I_1,\dots,i_n\in I_n, j_1\in I_1,\dots,j_n\in I_n} \subset W \,.$$

Define the function $f^{\boxtimes} : (H_1 \otimes \dots \otimes H_n) \times (H_1 \otimes \dots \otimes H_n) \to W$ as the unique, anti linear in left-hand argument and linear in the reamining one, map with

$$f^{\boxtimes}(e_{i_1} \otimes \cdots \otimes e_{i_n}, e_{j_1} \otimes \cdots \otimes e_{j_n}) := f(e_{i_1},\dots,e_{i_n}, e_{j_1},\dots,e_{j_n}) \,.$$

By construction, using (anti)linearity of $f^{\boxtimes}$ and multi-(anti)linearity of f, one gets immediately:

$$f^{\boxtimes}((v_1 \otimes \dots \otimes v_n), (u_1 \otimes \dots \otimes u_n)) = f(v_1,\dots,v_n,u_1,\dots,u_n)$$

for all $(v_1,\dots,v_n),(u_1,\dots,u_n) \in H_1 \times \dots \times H_n$. In other words $f^{\boxtimes} \circ \boxtimes = f$.
The uniqueness of the mapping $f^{\boxtimes}$ is obvious: suppose there is another mapping $g^{\boxtimes}$ with $g^{\boxtimes} \circ \boxtimes = f$ then $(f^{\boxtimes} - g^{\boxtimes}) \circ \boxtimes = 0$. This means in particular that

$$(f^{\boxtimes} - g^{\boxtimes})(e_{i_1} \otimes \dots \otimes e_{i_n}, e_{j_1} \otimes \dots \otimes e_{j_n}) = 0 \,,$$

for all $i_k \in I_k$, $j_k \in I_k$ $k = 1,\dots,n$. Since the coefficients above completely determine a map, the considered mapping must be the null mapping and thus: $f^{\boxtimes} = g^{\boxtimes}$. The proof of (b) follows straightforwardly by the construction of $f^{\boxtimes}$. □

Equipped with the established result, consider the map $f : (H_1 \times \cdots \times H_n) \times (H_1 \times \cdots \times H_n) \to \mathbb{C}$ that is anti linear in the first n entries and linear in the remaining n entries and it is defined by the requirement:

$$f((v_1,\dots,v_n),(u_1,\dots,u_n)) := (v_1|u_1)_1 (v_2|u_2)_2 \cdots (v_n|u_n)_n \,, \quad \text{for all } v_i, u_i \in H_i, i = 1,\dots,n.$$

As $(v_k|u_k)_k = \overline{(u_k|v_k)_k}$ by definition of Hermitean scalar product, the hypotheses in (b) of the theorem above is verified and thus the application

$$f^{\boxtimes} : (H_1 \otimes \cdots \otimes H_n) \times (H_1 \otimes \cdots \otimes H_n) \to \mathbb{C}$$

defined in (a) of the mentioned theorem, is linear in the right-hand argument, antilinear in the left-hand one and fulfils

$$f^{\boxtimes}(s|t) = \overline{f^{\boxtimes}(t|s)} \,, \quad \text{for all } s,t \in H_1 \otimes \cdots \otimes H_n.$$

If we were able to prove that $f^{\boxtimes}(\cdot|\cdot)$ is positive defined, i.e $f^{\boxtimes}(s|s) \geq 0$ for every $s \in H_1 \otimes \cdots \otimes H_n$ as well as $f^{\boxtimes}(s|s) = 0$ implies $s = 0$, then $f^{\boxtimes}(\cdot|\cdot)$ would define a Hermitean scalar product on $H_1 \otimes \cdots \otimes H_n$. This is just the last ingredient we need to give the wanted defintion of H tensor product. Let us prove it in form of a proposition.

Proposition 2.2. *Given $n \geq 1$ complex Hilbert spaces $(H_i, (\cdot|\cdot)_i)$, $i = 1, \ldots, n < +\infty$, the unique map $(\cdot|\cdot) : (H_1 \otimes \cdots \otimes H_n) \times (H_1 \otimes \cdots \otimes H_n) \to \mathbb{C}$ which is linear in the right-hand argument and anti linear in the left-hand one and verify*

$$(v_1 \otimes \cdots \otimes v_n | u_1 \otimes \cdots \otimes u_n) := (v_1|u_1)_1 (v_2|u_2)_2 \cdots (v_n|u_n)_n \,, \quad \textit{for all } v_i, u_i \in H_i, i = 1, \ldots, n,$$

is a Hermitean scalar product on $H_1 \otimes \cdots \otimes H_n$.

Proof. The only point we have to demonstrate is that $(\cdot|\cdot)$ is positive defined. To this end, fix a algebraic basis $\{e_{k,i_k}\}_{i_k \in I_k}$ for every H_k. If $S = S^{i_1 \cdots i_n} e_{1,i_1} \otimes \cdots \otimes e_{n,i_n} \in H_1 \otimes \cdots \otimes H_n$, the sum over repeated indices is always finite in view of the algebraic nature of the bases, also if each I_k may be infinite and uncountable. Similarly, fix a Hilbert base $\{h_{k,\alpha_k}\}_{\alpha_k \in A_k}$ for every H_k. Notice that also the set of indices A_k may be uncountable, however, as is well known, in every Hilbert decomposition $(u|v)_k = \sum_{\alpha_k \in A_k} (u|h_{k,\alpha_k})_k (h_{k,\alpha_k}|v)_k$ the set of nonvanishing terms $(u|h_{k,\alpha_k})_k$ and $(h_{k,\alpha_k}|v)_k$ is coutable and the sum reduces to a (absolutely convergent) standard series. In the following we do *not* omit the symbol of summation over the indices varying in every A_k, but we do follow that convention concerning algebraic indices $i_k, j_k \in I_k$. Notice also that the order of infinite summs cannot be interchanged in general. With our hypotheses we have for every $S \in H_1 \otimes \cdots \otimes H_n$:

$$(S|S) = \overline{S^{i_1 \cdots i_n}} S^{j_1 \cdots j_n} (e_{1,i_1}|e_{1,j_1})_1 \cdots (e_{n,i_n}|e_{n,j_n})_n$$

$$= \sum_{\alpha_1 \in A_1} \cdots \sum_{\alpha_n \in A_n} \overline{S^{i_1 \cdots i_n}} S^{j_1 \cdots j_n} (e_{1,i_1}|h_{1,\alpha_1})_1 (h_{1,\alpha_1}|e_{1,j_1})_1 \cdots (e_{n,i_n}|h_{n,\alpha_n})_n (h_{n,\alpha_n}|e_{n,j_n})_n$$

$$= \sum_{\alpha_1 \in A_1} \cdots \sum_{\alpha_n \in A_n} \overline{S^{i_1 \cdots i_n}} S^{j_1 \cdots j_n} \overline{(h_{1,\alpha_1}|e_{1,i_1})_1} (h_{1,\alpha_1}|e_{1,j_1})_1 \cdots \overline{(h_{n,\alpha_n}|e_{n,i_n})_n} \cdots (h_{n,\alpha_n}|e_{n,j_n})_n$$

$$= \sum_{\alpha_1 \in A_1} \cdots \sum_{\alpha_n \in A_n} \overline{S^{i_1 \cdots i_n} (h_{1,\alpha_1}|e_{1,i_1})_1 \cdots (h_{n,\alpha_n}|e_{n,i_n})_n} \; S^{j_1 \cdots j_n} (h_{1,\alpha_1}|e_{1,j_1})_1 \cdots (h_{n,\alpha_n}|e_{n,j_n})_n$$

$$= \sum_{\alpha_1 \in A_1} \cdots \sum_{\alpha_n \in A_n} \left| S^{j_1 \cdots j_n} (h_{1,\alpha_1}|e_{1,j_1})_1 \cdots (h_{n,\alpha_n}|e_{n,j_n})_n \right|^2 \geq 0 \,.$$

Semi positivity is thus established. We have to prove that $(S|S) = 0$ entails $S = 0$. Form the last line of the expansion of $(S|S)$ obtained above, we conclude that $(S|S) = 0$ implies $S^{i_j \cdots j_n} (h_{1,\alpha_1}|e_{1,j_1})_1 (h_{n,\alpha_n}|e_{n,j_n})_n = 0$ for every given set of elements $h_{1,\alpha_1}, \ldots, h_{n,\alpha_n}$. Now fix $\alpha_2, \ldots, \alpha_n$ to (arbitrarily) assigned values. By linearity and continuity of the scalar product, we have,

$$\sum_{\alpha_1 \in A_1} S^{j_1 j_2 \cdots j_n} c^{\alpha_1} (h_{1,\alpha_1}|e_{1,j_1})_1 (h_{2,\alpha_1}|e_{2,j_2})_2 \cdots (h_{n,\alpha_n}|e_{n,j_n})_n = 0 \,,$$

provided $\sum_{\alpha_1 \in A_1} c^{\alpha_1} h_{1,\alpha_1}$ converges to some element of H_1. Choosing $e_{1,i_1} = \sum_{\alpha_1 \in A_1} c^{\alpha_1} h_{1,\alpha_1}$, for any fixed $i_1 \in I_1$, the identity above reduces to

$$(e_{1,i_1}|e_{1,i_1})_1 S^{i_1 j_2 \cdots j_n} (h_{2,\alpha_1}|e_{2,j_2})_2 \cdots (h_{n,\alpha_n}|e_{n,j_n})_n = 0 \quad \text{so that} \quad S^{i_1 j_2 \cdots j_n} (h_{2,\alpha_1}|e_{2,j_2})_2 \cdots (h_{n,\alpha_n}|e_{n,j_n})_n = 0.$$

Iterating the procedure we achieve the final identity:

$$S^{i_1 i_2 \cdots i_n} = 0\ , \quad \text{for every } i_1 \in I_1, \ldots, i_n \in I_n,$$

that is $S = 0$. □

We have just established that the algebraic tensor product $H_1 \otimes \cdots \otimes H_n$ can be equipped with a natural Hermitean scalar product $(\cdot|\cdot)$ naturally induced by the Hermitean scalar products in the factors H_i. At this point, it is worth reminding the reader that, given a complex vector space V with a Hermitean scalar product, $(\cdot, \cdot)$, there is a unique (up to Hilbert space isomorphisms) Hilbert space which admits V as a dense subspace and whose scalar product is obtained as a continuous extension of $(\cdot, \cdot)_V$. This distinguished Hilbert space is the **Hilbert completion** of $(V, (\cdot, \cdot)_V)$ [Rudin]. We have all the ingredients to state the definition of H tensor product.

Definition 2.6. (**Hilbertian tensor product**.) Given $n \geq 1$ complex Hilbert spaces $(H_i, (\cdot|\cdot)_i)$, $i = 1, \ldots, n < +\infty$, the **Hilbertian tensor product** $H_1 \otimes_H \cdots \otimes_H H_n$ of those spaces, is the Hilbert completion of the space $H_1 \otimes \cdots \otimes H_n$ with respect to the unique scalar product $(\cdot|\cdot) : H_1 \otimes \cdots \otimes H_n \times H_1 \otimes \cdots \otimes H_n \to \mathbb{C}$ verifying

$$(v_1 \otimes \cdots \otimes v_n | u_1 \otimes \cdots \otimes u_n) := (v_1|u_1)_1 (v_2|u_2)_2 \cdots (v_n|u_n)_n\ , \quad \text{for all } v_i, u_i \in H_i, i = 1, \ldots, n\ .$$

♢

To conclude, we prove the following useful result.

Proposition 2.3. *If $\{h_{k,\alpha_k}\}_{\alpha_k \in A_k}$ is a Hilbert basis of the complex Hilbert space H_k, for every $k = 1, \ldots, n$, then $\{h_{1,\alpha_1} \otimes \cdots \otimes h_{n,\alpha_n}\}_{\alpha_k \in A_k, k=1,\ldots,n}$ is a Hilbert basis of $H_1 \otimes_H \cdots \otimes_H H_n$.*
As a consequence $H_1 \otimes_H \cdots \otimes_H H_n$ is separable whenever all the factors H_k are separable.

Proof. In the following $B_k := \{h_{k,\alpha_k}\}_{\alpha_k \in A_k}$, $< B_k >$ denotes the dense subspace of H_k generated by B_k, $B := \{h_{1,\alpha_1} \otimes \cdots \otimes h_{n,\alpha_n}\}_{\alpha_k \in A_k, k=1,\ldots,n}$, and $< B >$ is the subspace of $H_1 \otimes_H \cdots \otimes_H H_n$ generated by B. Finally, in the rest of the proof, the isometric linear map with dense range $j : H_1 \otimes \cdots \otimes H_n \to H_1 \otimes_H \cdots \otimes_H H_n$ defining the Hilbert completion of $H_1 \otimes \cdots \otimes H_n$ will be omitted in order to semplify the notation (one can always reduce to this case). Using the definition of the scalar product $(\cdot|\cdot)$ in $H_1 \otimes_H \cdots \otimes_H H_n$ given in definition 2.6 one sees that, trivially, B is an orthonormal set. Therefore, to conclude the proof, it is sufficient to prove that $\overline{< B >} = H_1 \otimes_H \cdots \otimes_H H_n$, where the bar indicates the topological closure. To this end consider also algebraic bases $D_k := \{e_{k,i_k}\}_{i_k \in I_k} \subset H_k$ made of *normalized vectors*. Since $\overline{< B_k >} = H_k$, given $m = 1, 2, \ldots$, there is $u_{k,i_k} \in < B_k >$ such that $||e_{k,i_k} - u_{k,i_k}||_k < 1/m$. Thus

$$||u_{k,i_k}||_k \leq ||e_{k,i_k}||_k + ||e_{k,i_k} - u_{k,i_k}||_k < 1 + 1/m\ .$$

Therfore, using $||v_1 \otimes \cdots \otimes v_n|| = ||v_1||_1 \cdots ||v_n||_n$ and the triangular inequality (see comments after definition 5.3), we achieve: $||e_{1,i_1} \otimes \cdots \otimes e_{n,i_n} - u_{1,i_1} \otimes \cdots \otimes u_{n,i_n}||$

$$= ||(e_{1,i_1} - u_{1,i_1}) \otimes e_{2,i_2} \otimes \cdots \otimes e_{n,i_n} + u_{1,i_1} \otimes (e_{2,i_2} - u_{2,i_2}) \otimes \cdots \otimes e_{n,i_n} + \cdots + u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes (e_{n,i_n} - u_{n,i_n})||$$

$$\leq ||(e_{1,i_1} - u_{1,i_1}) \otimes e_{2,i_2} \otimes \cdots \otimes e_{n,i_n}|| + ||u_{1,i_1} \otimes (e_{2,i_2} - u_{2,i_2}) \otimes \cdots \otimes e_{n,i_n}|| + \cdots$$
$$+||u_{1,i_1} \otimes u_{2,i_2} \otimes \cdots \otimes (e_{n,i_n} - u_{n,i_n})|| \leq n(1+1/m)^{n-1} \leq n\frac{2^{n-1}}{m} \,.$$

We conclude that, fixing $\epsilon > 0$, choosing m large enough, there is $y_{i_1 \ldots i_n} \in < B >$ such that

$$||e_{1,i_1} \otimes \cdots \otimes e_{n,i_n} - y_{i_1 \ldots i_n}|| < \epsilon \,. \tag{2.6}$$

The vectors $e_{1,i_1} \otimes \cdots \otimes e_{n,i_n}$ generates $H_1 \otimes \cdots \otimes H_n$ and linear combinations of elements $y_{i_1 \ldots i_n}$ as above belong to $< B >$. Hence, (2.6) together with homogeneity property of the norm as well as the triangular inequality (see comments after definition 5.3), imply that, for every $v \in H_1 \otimes \cdots \otimes H_n$ and every $\epsilon > 0$, there is $u \in < B >$ such that:

$$||u - v|| \leq \epsilon/2 \,. \tag{2.7}$$

Now notice that, for every $v_0 \in H_1 \otimes_H \cdots \otimes_H H_n$ and any fixed $\epsilon > 0$, there is $v \in H_1 \otimes \cdots \otimes H_n$ with $||v_0 - v|| \leq \epsilon/2$, because $\overline{H_1 \otimes \cdots \otimes H_n} = H_1 \otimes_H \cdots \otimes_H H_n$. Therefore, fixing $u \in < B >$ as in (2.7), we have that, for every $v_0 \in H_1 \otimes_H \cdots \otimes_H H_n$ and any fixed $\epsilon > 0$, there is $u \in < B >$ with $||v_0 - u|| \leq \epsilon$. In other words $\overline{< B >} = H_1 \otimes_H \cdots \otimes_H H_n$ as wanted. $\Box$

Remark. Consider the case, quite usual in quantum mechanics, where the relevant Hilbert spaces are spaces of *square-integrable functions* [Rudin], $H_i := L^2(X_i, d\mu_i)$, where every X_i is a σ-finite space with measure μ_i, $i = 1, \ldots, n$. For instance $X_i = \mathbb{R}^3$ and $\mu_i = \mu$ is the standard Lebesgue measure (in this case every H_i is the state space of a spinless quantum particle). In the considered case, on a hand one can define the tensor product $H_1 \otimes_H \cdots \otimes_H H_n = L^2(X_1, d\mu_1) \otimes_H \cdots \otimes_H L^2(X_1, d\mu_1)$ on the other hand another natural object, relying on the given mathematical structures, is the Hilbert space $H := L^2(X_1 \times \cdots \times X_n, d\mu_1 \otimes \cdots \otimes d\mu_n)$, where the product measure $\mu_1 \otimes \cdots \otimes \mu_n$ is well defined since the spaces X_i are σ-finite. It turns out that these two spaces are naturally isomorphic and the isomorphism is the unique linear and continuous extension of the map:

$$L^2(X_1, d\mu_1) \otimes_H \cdots \otimes_H L^2(X_1, d\mu_1) \ni \psi_1 \otimes \cdots \otimes \psi_n \mapsto \psi_1 \cdots \psi_n \in L(X_1 \times \cdots \times X_n, d\mu_1 \otimes \cdots d\mu_n)$$

where

$$\psi_1 \cdots \psi_n(x_1, \ldots, x_n) := \psi_1(x_1) \cdots \psi_n(x_n) \quad \text{for all } x_i \in X_i,\ i = 1, \ldots, n. \,.$$

Some textbook on quantum mechanics define the Hilbert space of a composite system of n spinless particles as $L(\mathbb{R}^3 \times \cdots \times \mathbb{R}^3, d\mu \otimes \cdots \otimes d\mu)$ rather than $L^2(\mathbb{R}^3, d\mu) \otimes_H \cdots \otimes_H L^2(\mathbb{R}^3, d\mu)$. Actually, the two definitions are mathematically equivalent.

Chapter 3

Tensor algebra, abstract index notation and some applications.

3.1 Tensor algebra generated by a vector space.

Let V be vector space on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Equipped with this algebraic structure, in principle, we may build up several different tensor spaces. Notice that we have to consider also $\mathbb{K}$, $\mathbb{K}^*$ and V^* as admissible tensor factors. (In the following we shall not interested in conjugate spaces.) Obviously, we are interested in tensor products which are not identifiable by some natural isomorphism.

First consider the dual $\mathbb{K}^*$ when we consider $\mathbb{K}$ as a vector space on the field $\mathbb{K}$ itself. $\mathbb{K}^*$ is made of linear functionals from $\mathbb{K}$ to $\mathbb{K}$. Each $c^* \in \mathbb{K}^*$ has the form $c^*(k) := c \cdot k$, for all $k \in \mathbb{K}$, where $c \in \mathbb{K}$ is a fixed field element which completely determines c^*. The mapping $c \mapsto c^*$ is a (natural) vector space isomorphism. Therefore

$$\mathbb{K} \simeq \mathbb{K}^* .$$

Then we pass to consider products $\mathbb{K} \otimes \ldots \otimes \mathbb{K} \simeq \mathbb{K}^* \otimes \ldots \otimes \mathbb{K}^* = \mathscr{L}(\mathbb{K}, \ldots, \mathbb{K})$. Each multi-linear mapping $f \in \mathscr{L}(\mathbb{K}, \ldots, \mathbb{K})$ is completely determined by the number $f(1, \ldots, 1)$, since $f(k_1, \ldots, k_n) = k_1 \cdots k_n f(1, \ldots, 1)$. One can trivially show that the mapping $f \mapsto f(1, \ldots, 1)$ is a (natural) vector space isomorphism between $\mathbb{K} \otimes \ldots \otimes \mathbb{K}$ and $\mathbb{K}$ itself. Therefore

$$\mathbb{K} \otimes \ldots \otimes \mathbb{K} \simeq \mathbb{K} .$$

Also notice that the found isomorphism trivially satisfies $c_1 \otimes \ldots \otimes c_n \mapsto c_1 \cdots c_n$, and thus the tensor product mapping reduces to the ordinary product of the field.

We pass to consider the product $\mathbb{K} \otimes V = \mathscr{L}(\mathbb{K}^*, V^*) \simeq \mathscr{L}(\mathbb{K}, V^*)$. Each multi-linear functional f in $\mathscr{L}(\mathbb{K}, V^*)$ is completely determined by the element of V, $f(1, \cdot) : V^* \to \mathbb{K}$, which maps each $v^* \in V^*$ in $f(1, v^*)$. Once again, it is a trivial task to show that $f \mapsto f(1, \cdot)$ is a (natural) vector space isomorphism between $\mathbb{K} \otimes V$ and V itself.

$$\mathbb{K} \otimes V \simeq V .$$

Notice that the found isomorphism satisfies $k \otimes v \mapsto kv$ and thus the tensor product mapping reduces to the ordinary product of a field element and a vector.
Reminding that $\otimes$ is associative and * is involutive and, in the finite-dimensional case, it is also distributive, as established in chapter 2, we conclude that only the spaces $\mathbb{K}$, V, V^* and all the tensor products of V and V^* (in whichever order and number) may be significantly different. This result leads us to the following definition which we extend to the infinite dimensional case (it is however worth stressing that, in the infinite-dimensional case, some of the natural isomorphisms encountered in the finite-dimensional case are not valid, for instance $(V \otimes V)^*$ may be larger than $V^* \otimes V^*$).

Definition 3.1. **(Tensor Algebra generated by V.)** Let V be a vector space with field $\mathbb{K}$.
(1) The **tensor algebra** $\mathscr{A}_{\mathbb{K}}(V)$ generated by V with field $\mathbb{K}$ is the class whose elements are the vector spaces: $\mathbb{K}$, V, V^* and all of tensor products of factors V and V^* in whichever order and number.
(2) The tensors of $\mathbb{K}$ are called **scalars**, the tensors of V are called **contravariant vectors**, the tensors of V^* are called **covariant vectors**, the tensors of spaces $V^{n\otimes} := V \otimes \ldots \otimes V$, where V appears $n \geq 1$ times, are called **contravariant tensors of order** n or **tensors of order** $(n, 0)$, the tensors of spaces $V^{*n\otimes} := V^* \otimes \ldots \otimes V^*$, where V^* appears $n \geq 1$ times, are called **covariant tensors of order** n or **tensors of order** $(0, n)$, the remaining tensors which belong to spaces containing n consecutive factors V and m subsequent consecutive factors V^* are called **tensors of order** (n, m). The order of the remaining type of tensors is defined analogously taking the order of occurrence of the factors V and V^* into account.
$\diamondsuit$

Remark. Obviously, pairs of tensors spaces made of the same number of factors V and V^* in different order, are naturally isomorphic (see exercises 2.3.1). However, for practical reasons it is convenient to consider these spaces as different spaces and use the identifications when and if necessary.

3.2 The abstract index notation and rules to handle tensors.

3.2.1 Canonical bases and abstract index notation.

Let us introduce the **abstract index notation**. Consider a finite-dimensional vector space V with field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. After specification of a basis $\{e_i\}_{i \in I} \subset V$ and the corresponding basis in V^*, each tensor is completely determined by giving its components with respect to the induced basis in the corresponding tensor space in $\mathscr{A}_{\mathbb{K}}(V)$. We are interested in the transformation rule of these components under change of the base in V. Suppose to fix another basis $\{e'_j\}_{j \in I} \subset V$ with $e_i = A^j{}_i e'_j$. The coefficients $A^j{}_i$ determine a matrix $A := [A^j{}_i]$ in the matrix group $GL(dimV, \mathbb{K})$, i.e., the group (see section 4.1) of $dimV \times dimV$ matrices with coefficients in $\mathbb{K}$ and non-vanishing determinant. First consider a contravariant vector $t = t^i e_i$, passing to the other basis, we have $t = t^i e_i = t'^j e'_j$ and thus $t'^j e'_j = t^i A^j{}_i e'_j$. This is equivalent to $(t'^j - A^j{}_i t^i) e'_j = 0$ which implies

$$t'^j = A^j{}_i t^i \ ,$$

because of the linear independence of vectors e'_j. Similarly, if we specify a set of components in $\mathbb{K}$, $\{t^i\}_{i\in I}$ for each basis $\{e_i\}_{i\in I} \subset V$ and these components, changing the basis to $\{e'_j\}_{j\in I}$, transform as

$$t'^j = A^j{}_i t^i \ ,$$

where the coefficients $A^j{}_i$ are defined by

$$e_i = A^j{}_i e'_j \ ,$$

then a contravariant tensor t is defined. It is determined by $t := t^i e_i$ in each basis. The proof is self evident.

Concerning covariant vectors, a similar result holds. Indeed a covariant vector $u \in V^*$ is completely determined by the specification of a set of components $\{u_i\}_{i\in I}$ for each basis $\{e^{*i}\}_{i\in I} \subset V^*$ (dual basis of $\{e_i\}_{i\in I}$ above) when these components, changing the basis to $\{e'^{*j}\}_{j\in I}$ (dual base of $\{e'_j\}_{j\in I}$ above), transform as

$$u'_j = B_j{}^i u_i \ ,$$

where the coefficients $B_l{}^r$ are defined by

$$e^{*i} = B_j{}^i e'^{*j} \ .$$

What is the relation between the matrix $A = [A^j{}_i]$ and the matrix $B := [B_k{}^h]$? The answer is obvious: it must be

$$\delta^i_j = \langle e_j, e^{*i} \rangle = A^l{}_j B_k{}^i \langle e'_l, e'^{*k} \rangle = A^l{}_j B_k{}^i \delta^k_l = A^k{}_j B_k{}^i \ .$$

In other words it has to hold $I = A^t B$, which is equivalent to

$$B = A^{-1t} \ .$$

(Notice that ^{t} and $^{-1}$ commute.)

Proper tensors have components which transform similarly to the vector case. For instance, consider $t \in V \otimes V^*$, fix a basis $\{e_i\}_{i\in I} \subset V$, the dual one $\{e^{*i}\}_{i\in I} \subset V^*$ and consider that induced in $V \otimes V^*$, $\{e_j \otimes e^{*j}\}_{(i,j)\in I\times I}$. Then $t = t^i{}_j e_i \otimes e^{*j}$. By bi linearity of the tensor product map, if we pass to consider another basis $\{e'_i\}_{i\in I} \subset V$ and those associated in the relevant spaces as above, concerning the components $t'^k{}_l$ of t in the new tensor space basis, one trivially gets

$$t'^k{}_l = A^k{}_i B_l{}^j t^i{}_j \ ,$$

where the matrices $A = [A^j{}_i]$ and $B := [B_k{}^h]$ are those considered above. It is obvious that the specification of a tensor of $V \otimes V^*$ is completely equivalent to the specification of a set of components for each basis of $V \otimes V^*$, $\{e_j \otimes e^{*j}\}_{(i,j)\in I\times I}$, provided these components transform as specified above under change of basis.
We can generalize the obtained results after a definition.

Definition 3.2. **(Canonical bases.)** Let $\mathscr{A}_{\mathbb{K}}(V)$ be the tensor algebra generated by the finite-dimensional vector space V on the field $\mathbb{K}$ ($= \mathbb{C}$ or $\mathbb{R}$). If $B = \{e_i\}_{i\in I}$ is a basis in V with dual basis $B^* = \{e^{*i}\}_{i\in I} \subset V^*$, the **canonical bases** associated to the former are the bases in the tensor spaces of $\mathscr{A}_{\mathbb{K}}(V)$ obtained by tensor products of elements of B and B^*. $\diamondsuit$

Remark. Notice that also $\{e_i \otimes e'_j\}_{i,j\in I}$ is a basis of $V \otimes V$ if $\{e_i\}_{i\in I}$ and $\{e'_j\}_{j\in I}$ are bases of V. However, $\{e_i \otimes e'_j\}_{i,j\in I}$ is *not* canonical unless $e_i = e'_i$ for all $i \in I$.

Theorem 3.1. *Consider the tensor algebra $\mathscr{A}_{\mathbb{K}}(V)$ generated by a finite-dimension vector space V with field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and take a tensor space $V^{n\otimes} \otimes V^{*m\otimes} \in \mathscr{A}_{\mathbb{K}}(V)$. The specification of a tensor $t \in V^{n\otimes} \otimes V^{*m\otimes}$ is completely equivalent to the specification of a set of components*

$$\{t^{i_1\ldots i_n}{}_{j_1\ldots j_m}\}_{i_1,\ldots,i_n,j_1,\ldots,j_m\in I}$$

with respect to each canonical basis of $V^{n\otimes} \otimes V^{*m\otimes}$,

$$\{e_{i_1} \otimes \ldots \otimes e_{i_n} \otimes e^{*j_1} \otimes \ldots \otimes e^{*j_n}\}_{i_1,\ldots i_n,j_1,\ldots j_m\in I}$$

which, under change of basis:

$$\{e'_{i_1} \otimes \ldots \otimes e'_{i_n} \otimes e'^{*j_1} \otimes \ldots \otimes e'^{*j_n}\}_{i_1,\ldots i_n,j_1,\ldots j_m\in I}$$

transform as:

$$t'^{i_1\ldots i_n}{}_{j_1\ldots j_m} = A^{i_1}{}_{k_1} \ldots A^{i_n}{}_{k_n} B_{j_1}{}^{l_1} \ldots B_{j_m}{}^{l_m} t^{k_1\ldots k_n}{}_{l_1\ldots l_m}\ ,$$

where

$$e_i = A^j{}_i e'_j\ ,$$

and the coefficients $B_j{}^l$ are those of the matrix:

$$B = A^{-1t}\ ,$$

with $A := [A^j{}_i]$. The associated tensor t is represented by

$$t = t^{i_1\ldots i_n}{}_{j_1\ldots j_m} e_{i_1} \otimes \ldots \otimes e_{i_n} \otimes e^{*j_1} \otimes \ldots \otimes e^{*j_m}$$

for each considered canonical basis. Analogous results hold for tensor spaces whose factors V and V^* take different positions.

Notation 3.1. In the **abstract index notation** a tensor is indicated by writing its generic component in a non-specified basis. E.g. $t \in V^* \otimes V$ is indicated by $t_i{}^j$.

Somewhere in the following we adopt a cumulative index notation, i.e., letters $A, B, C, \ldots$ denote set of covariant, contravariant or mixed indices. For instance $t^{ijk}{}_{lm}$ can be written as t^A with ${}^A = {}^{ijk}{}_{lm}$. Similarly e_A denotes the element of a canonical basis $e_i \otimes e_j \otimes e_k \otimes e^{*l} \otimes e^{*m}$. Moreover, if ^{A} and ^{B} are cumulative indices, the indices of the cumulative index ^{AB} are those of ^{A} immediately followed by those of ^{B}, e.g, if ${}^A = {}^{ijk}{}_{lm}$, and ${}^B = {}^{pq}{}_u{}^n$, ${}^{AB} = {}^{ijk}{}_{lm}{}^{pq}{}_u{}^n$.

3.2.2 Rules to compose, decompose, produce tensors form tensors.

Let us specify the allowed mathematical rules to produces tensors from given tensors. To this end we shall use both the synthetic and the index notation.

Linear combinations of tensors of a fixed tensor space. Take a tensor space $S \in \mathscr{A}_{\mathbb{K}}(V)$. This is a vector space by definition, and thus picking out $s, t \in S$, and $\alpha, \beta \in \mathbb{K}$, linear combinations can be formed which still belong to S. In other words we may define the tensor of S

$$u := \alpha s + \beta t \quad \text{or, in the abstract index notation} \quad u^A = \alpha s^A + \beta t^A \,.$$

The definition of u above given by the abstract index notation means that the components of u are related to the components of s and t by a linear combination which has *the same form in every canonical basis of the space* S and the coefficients α,β *do not depend on the basis.*

Products of tensors of generally different tensor spaces. Take two tensor spaces $S, S' \in \mathscr{A}_{\mathbb{K}}(V)$ and pick out $t \in S$, $t' \in S'$. Then the tensor $t \otimes t' \in S \otimes S'$ is well defined. Using the associativity of the tensor product by natural isomorphisms, we find a unique tensor space $S'' \in \mathscr{A}_{\mathbb{K}}(V)$ which is isomorphic to $S \otimes S'$ and thus we can identify $t \otimes t'$ with a tensor in S'' which we shall indicate by $t \otimes t'$ once again with a little misuse of notation. $t \otimes t'$ is called the **product** of tensors t and t'. Therefore, the product of tensors coincides with the usual tensor product of tensors up to a natural isomorphism. What about the abstract index notation? By theorem 2.8 one sees that, fixing a basis in V, the natural isomorphism which identify $S \otimes S$ with S'' transforms the products of elements of canonical bases in S and S' in the corresponding elements of the canonical basis of S'' obtained by cancelling every parantesis; e.g. for $S = V \otimes V^*$ and $S' = V^*$, the natural isomorphism from $(V \otimes V^*) \otimes V^*$ to $V \otimes V^* \otimes V^*$ transforms each $(e_k \otimes e^{*h}) \otimes e^{*r}$ to $e_k \otimes e^{*h} \otimes e^{*r}$. As a consequence

$$(t \otimes t')^{AB} = t^A t'^B \,.$$

Therefore, for instance, if $S = V \otimes V^*$ and $S' = V^*$ the tensors t and t' are respectively indicated by $t^i{}_j$ and s_k and thus $(t \otimes s)^i{}_{jk} = t^i{}_j s_k$.

Contractions. Consider a tensor space of $\mathscr{A}_{\mathbb{K}}(V)$ of the form

$$U_1 \otimes \ldots \otimes U_k \otimes V \otimes U_{k+1} \otimes \ldots \otimes U_l \otimes V^* \otimes U_{l+1} \otimes \ldots \otimes U_n$$

where U_i denotes either V or V^*. Everything we are going to say can be re-stated for the analogous space

$$U_1 \otimes \ldots \otimes U_k \otimes V^* \otimes U_{k+1} \otimes \ldots \otimes U_l \otimes V \otimes U_{l+1} \otimes \ldots \otimes U_n \, .$$

Then consider the *multi-linear* mapping C with domain

$$U_1 \times \ldots \times U_k \times V \times U_{k+1} \times \ldots \times U_l \times V^* \times U_{l+1} \times \ldots \times U_n \, ,$$

and values in

$$U_1 \otimes \ldots \otimes U_k \otimes U_{k+1} \otimes \ldots \otimes U_l \otimes U_{l+1} \otimes \ldots \otimes U_n$$

defined by:

$$(u_1, \ldots, u_k, v, u_{k+1}, \ldots, u_l, v^*, u_{l+1}, \ldots, u_n) \mapsto \langle v, v^* \rangle u_1 \otimes \ldots \otimes u_k \otimes u_{k+1} \otimes \ldots \otimes u_l \otimes u_{l+1} \otimes \ldots \otimes u_n \, .$$

By the universality theorem there is a *linear* mapping $C^{\otimes}$, called **contraction** of V and V^*, defined on the whole tensor space

$$U_1 \otimes \ldots \otimes U_k \otimes V \otimes U_{k+1} \otimes \ldots \otimes U_l \otimes V^* \otimes U_{l+1} \otimes \ldots \otimes U_n$$

taking values in

$$U_1 \otimes \ldots \otimes U_k \otimes U_{k+1} \otimes \ldots \otimes U_l \otimes U_{l+1} \otimes \ldots \otimes U_n$$

such that, on simple products of vectors reduces to

$$u_1 \otimes \ldots \otimes u_k \otimes v \otimes u_{k+1} \otimes \ldots \otimes u_l \otimes v^* \otimes u_{l+1} \otimes \ldots \otimes u_n \mapsto \langle v, v^* \rangle u_1 \otimes \ldots \otimes u_k \otimes u_{k+1} \otimes \ldots \otimes u_l \otimes u_{l+1} \otimes \ldots \otimes u_n .$$

This linear mapping takes tensors in a tensor product space with $n+2$ factors and produces tensors in a space with n factors. The simplest case arises for $n = 0$. In that case $C : V \times V^* \to \mathbb{K}$ is nothing but the bilinear pairing $C : (v, v^*) \mapsto \langle v, v^* \rangle$ and $C^{\otimes}$ is the linear associated mapping by the universality theorem.
Finally, let us represent the contraction mapping within the abstract index picture. It is quite simple to show that, $C^{\otimes}$ takes a tensor $t^{AiB}{}_j{}^C$ where ^{A}, ^{B} and ^{C} are arbitrary cumulative indices, and produces the tensor $(C^{\otimes} t)^{ABC} := t^{AkB}{}_k{}^C$ where we remark the convention of *summation of the twice repeated index k*. To show that the abstract-index representation of contractions is that above notice that the contractions are linear and thus

$$C^{\otimes}(t^{AiB}{}_j{}^C e_A \otimes e_i \otimes e_B \otimes e^{*j} \otimes e_C) = t^{AiB}{}_j{}^C C^{\otimes}(e_A \otimes e_i \otimes e_B \otimes e^{*j} \otimes e_C) = t^{AiB}{}_j{}^C \delta^j_i e_A \otimes e_B \otimes e_C \, ,$$

and thus

$$C^{\otimes}(t^{AiB}{}_j{}^C e_A \otimes e_i \otimes e_B \otimes e^{*j} \otimes e_C) = t^{AkB}{}_k{}^C e_A \otimes e_B \otimes e_C \, .$$

This is nothing but:

$$(C^{\otimes} t)^{ABC} := t^{AkB}{}_k{}^C \, .$$

3.2.3 Linear transformations of tensors are tensors too.

To conclude we pass to consider a final theorem which shows that there is a one-to-one correspondence between linear mappings on tensors and tensors them-selves.

Theorem 3.2. **(Linear mappings and tensors.)** *Let S, S' be a pair of tensor spaces of a tensor algebra $\mathscr{A}_{\mathbb{K}}(V)$ where V is finite dimensional. The vector space of linear mappings from S to S' is naturally isomorphic to $S^* \otimes S'$ (which it is naturally isomorphic to the corresponding tensor space of $\mathscr{A}_{\mathbb{K}}(V)$). The isomorphism $F : S^* \otimes S' \to \mathscr{L}(S|S')$ such that $F : t \mapsto f_t$, where the linear function $f_t : S \to S'$ is defined by:*

$$f_t(s) := C^{\otimes}(s \otimes t)\ , \quad \textit{for all } s \in S,$$

where $C^{\otimes}$ is the contraction of all the indices of s and the corresponding indices in S^* of t. Moreover, fixing a basis $\{e_i\}_{i\in I}$ in V, let $\{e_A\}$ denote the canonical basis induced in S, $\{e'_B\}$ that induced in S' and $\{e'^{*C}\}$ that induced in S'^*. With those definitions if $s = s^A e_A \in S$,

$$f_t(s)^C = s^A t_A{}^C$$

and

$$t_A{}^C = \langle f_t(e_A), e'^{*C}\rangle\ .$$

The isomorphism F^{-1} is the composition of the isomorphisms in theorems 2.4 and 2.2: $\mathscr{L}(S|S') \simeq \mathscr{L}(S, S'^*) = S^* \otimes (S'^*)^* \simeq S^* \otimes S'$.

Proof. Defining f_t as said above it is evident that the map F is linear. The given definition implies that, in components $f_t(s)^C = s^A t_A{}^C$. Since $S^* \otimes S'$ and $\mathscr{L}(S|S') \simeq \mathscr{L}(S, S'^*) \simeq S^* \otimes S'$ have the same dimension, injectivity of F implies surjectivity. Injectivity has a straightforward proof: $f_t = 0$ is equivalent to, fixing canonical bases as indicated in the hypoteses, $s^A t_A{}^C = 0$ for every choice of coefficients s^A. This is equivalent, in turn, to $t_A{}^C = 0$ for every component of t and thus $t = 0$. The formula $t_A{}^C = \langle f_t(e_A), e'^{*C}\rangle$ can be proved as follows: if $s = e_A$, the identity $f_t(s)^C = s^A t_A{}^C$ reduces to $f_t(e_A)^C = t_A{}^C$, therefore $f_t(e_A) = f_t(e_A)^C e'_C = t_A{}^C e'_C$ and so $\langle f_t(e_A), e'^{*C}\rangle = t_A{}^C$. The last statement in the thesis can be obtained by direct inspection. □

Let us illustrate how one may use that theorem. For instance consider a linear mapping $f : V \to V$ where V is finite-dimensional with field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Then f individuates a tensor $t \in V^* \otimes V$. In fact, fixing a basis $\{e_i\}_{i\in I}$in V and considering those canonically associated, by the linearity of f and the pairing:

$$f(v) = \langle f(v), e^{*k}\rangle e_k = v^i \langle f(e_i), e^{*k}\rangle e_k\ .$$

The tensor $t \in V^* \otimes V$ associated with f in view of the proved theorem has components

$$t_i{}^k := \langle f(e_i), e^{*k}\rangle\ .$$

Moreover, using the abstract index notation, one has

$$(f(v))^k = v^i t_i{}^k \, .$$

In other words the action of f on v reduces to (1) a product of the involved tensors:

$$v^i t_j{}^k \, ,$$

(2) followed by a convenient contraction:

$$(f(v))^k = v^i t_i{}^k \, .$$

More complicate cases can be treated similarly. For example, linear mappings f from $V^* \otimes V$ to $V^* \otimes V \otimes V$ are one-to-one with tensors $t^i{}_{jk}{}^{lm}$ of $V \otimes V^* \otimes V^* \otimes V \otimes V$ and their action on tensors $u_p{}^q$ of $V^* \otimes V$ is

$$f(u)_k{}^{lm} = u_i{}^j t^i{}_{jk}{}^{lm}$$

i.e., a product of tensors and two contractions. Obviously

$$t^i{}_{jk}{}^{lm} = \langle f(e^{*i} \otimes e_j), e_k \otimes e^{*l} \otimes e^{*m} \rangle \, .$$

Remark. In the applications it is often convenient defining the tensor t in $S' \otimes S^*$ instead of $S^* \otimes S'$. In this case the isomorfism between $S' \otimes S^*$ and $\mathscr{L}(S|S')$ associates $t \in S' \otimes S^*$ with $g_t \in \mathscr{L}(S|S')$ where

$$g_t(s) := C^{\otimes}(t \otimes s) \, , \quad \text{for all } s \in S,$$

where $C^{\otimes}$ is the contraction of all the indices of s and the corresponding indices in S^* of t.

3.3 Physical invariance of the form of laws and tensors.

A physically relevant result is that the rules given above to produce a new tensor from given tensors have the same form independently from the basis one use to handle tensors. For that reason the abstract index notation makes sense. In physics the choice of a basis is associated with the choice of a reference frame. As is well known, various relativity principles (*Galileian Principle*, *Special Relativity Principle* and *General Relativity Principle*) assume that
"*the law of Physics can be written in the same form in every reference frame*".
The allowed reference frames range in a class singled out by the considered relativity principle, for instance in Special Relativity the relevant class is that of inertial reference frames.
It is obvious that the use of tensors and rules to compose and decompose tensors to represent physical laws is very helpful in implementing relativity principles. In fact the theories of Relativity can be profitably formulated in terms of tensors and operations of tensors just to assure the invariance of the physical laws under change of the reference frame. When physical laws are given in terms of tensorial relations one says that those laws are *covariant*. It is worthwhile stressing that covariance is *not* the only way to state physical laws which preserve their form under changes of reference frames. For instance the Hamiltonian formulation of mechanics, in relativistic contexts, is invariant under change of reference frame but it is not formulated in terms of tensor relations (in spacetime).

3.4 Tensors on Affine and Euclidean spaces.

We recall the reader the definitions of *affine space* and **Euclidean space**. We assume that the reader is familiar with these notions and in particular with the notion of scalar product and orthonormal basis. [Sernesi]. A general theory of scalar product from the point of view of tensor theory will be developed in chapter 5.

Definition 3.3. **(Affine space).** A **real n-dimensional affine space** [Sernesi] is a triple $(\mathbb{A}^n, V, \vec{\cdot\cdot})$ where $\mathbb{A}^n$ is a set whose elements are called **points**, V is a real n-dimensional vector space called **space of translations** and $\vec{\cdot\cdot} : \mathbb{A}^n \times \mathbb{A}^n \to V$ is a mapping such that the two following requirements are fulfilled.
(i) For each pair $p \in \mathbb{A}^n$, $v \in V$ there is a *unique* point $q \in \mathbb{A}^n$ such that $\overrightarrow{pq} = v$.
(ii) $\overrightarrow{pq} + \overrightarrow{qr} = \overrightarrow{pr}$ for all $p, q, r \in \mathbb{A}^n$.
$\overrightarrow{pq}$ is called vector with **initial point** p and **final point** q. $\diamondsuit$

Definition 3.4. **(Euclidean space).** A (real) **Euclidean** space is a pair $(\mathbb{E}^n, (\cdot|\cdot))$, where $\mathbb{E}^n$ is an affine space with space of translations V and $(\cdot|\cdot) : V \times V \to \mathbb{R}$ is a scalar product on V. $\diamondsuit$

A **Cartesian coordinate systems** on the affine space $\mathbb{A}^n$ is a bijective map $f : \mathbb{A}^n \to \mathbb{R}^n$ defined as follows. Fix a point $O \in \mathbb{A}^n$ called **origin** of the coordinates and fix a vector basis $\{e_i\}_{i=1,\ldots,n} \subset V$ whose elements are called **axes** of the coordinates. Varying $p \in \mathbb{A}^n$, the components $(x^1(p), \ldots, x^n(p)) \in \mathbb{R}^n$ of any vector $\overrightarrow{OP}$ with respect the chosen basis, define a mapping $f : \mathbb{A}^n \to \mathbb{R}^n$. Property (i) above implies that f is bijective.
It is simply proved that, if f' is another Cartesian coordinate system with origin O' and axes $\{e'_j\}_{j=1,\ldots,n}$, the following coordinate transformation law holds:

$$x'^j = \sum_{i=1}^{n} A^j{}_i x^i + b^j \,, \tag{3.1}$$

where $e_i = \sum_j A^j{}_i e'_j$ and $\overrightarrow{O'O} = \sum_j b^j e'_j$.
An **orthonormal Cartesian coordinate systems** on the Euclidean space $(\mathbb{E}^n, (\cdot|\cdot))$ is a Cartesian coordinate systems whose axes form an orthonormal basis of the vector space V associated with $\mathbb{E}^n$ with respect to $(\cdot|\cdot)$, namely $(e_i|e_j) = \delta_{ij}$.
In that case the matrices A of elements $A^j{}_i$ employed to connect different orthonormal Cartesian coordinate system are all of the elements of the **orthogonal group** of order n:

$$O(n) = \{R \in M(n, \mathbb{R}) \mid R^t R = I \,.\}$$

This is evident from the requirement that both the bases appearing in the identities $e_i = \sum_j A^j{}_i e'_j$ are orthonormal.

Comments 3.1.
(1) The simplest example of affine space is $\mathbb{R}^n$ itself with $\mathbb{A}^n = \mathbb{R}^n$ and $V = \mathbb{R}^n$. The map $\vec{\cdot} : \mathbb{A}^n \times \mathbb{A}^n \to V$ in this case is trivial: $\overrightarrow{(x^1, \ldots, x^n)(y^1, \ldots, y^n)} = (y^1 - x^1, \ldots, y^n - x^n)$.
From a physical point of view affine and Euclidean spaces are more interesting than $\mathbb{R}^n$ to describe physical space (of inertial reference frames) because the affine structure can naturally be used to represent homogeneity and isotropy of space. Conversely $\mathbb{R}^n$ has a unphysical structure breaking those symmetries: it admits a preferred coordinate system!
(2) Three-dimensional Euclidean spaces are the spaces of usual Euclidean geometry.
(3) The whole class of Cartesian coordinate systems, allow the introduction of a Hausdorff second-countable topology on $\mathbb{A}^n$. This is done by fixing a Cartesian coordinate system $f : \mathbb{A}^n \to \mathbb{R}^n$ and defining the open sets on $\mathbb{A}^n$ as the set of the form $f^{-1}(B)$, $B \subset \mathbb{R}^n$ being any open set of $\mathbb{R}^n$. Since affine transformation (3.1) from $\mathbb{R}^n$ to $\mathbb{R}^n$ are homomorphisms, the given definition does not depend on the used Cartesian coordinate system. In this way, every Cartesian coordinate system becomes a homeomorphism and thus Hausdorff and second countability properties are inherited from the topology on $\mathbb{R}^n$.
(4) it is possible to give the notion of a *differentiable function* $h : A \to B$, where $A \subset \mathbb{A}^r$, $B \subset \mathbb{A}^s$ are open sets. In particular it could be either $\mathbb{A}^r = \mathbb{R}^r$ or $\mathbb{A}^s = \mathbb{R}^s$.

Definition 3.5. If $A \subset \mathbb{A}^r$, $B \subset \mathbb{A}^s$ are open sets in the corresponding affine spaces, we say that $h : A \to B$ is of **class** $C^k(A)$ if $f' \circ h \circ f^{-1} : f(A) \to \mathbb{R}^s$ is of class $C^k(f(A))$ for a pair of Cartesian coordinate systems $f : \mathbb{A}^r \to \mathbb{R}^r$ and $f' : \mathbb{A}^s \to \mathbb{R}^s$. $\diamondsuit$

The definition is well-posed, since affine transformations (3.1) are C^∞, the given definition does not matter the choice of f and f'.

Remark. As a final comment we notice that, from a more abstract point of view, the previous comments proves that every n-dimensional affine (or euclidean) space has a natural structure of *n-dimensional C^∞ differentiable manifold* induced by the class of all Cartesian coordinate systems.

3.4.1 Tensor spaces, Cartesian tensors.

Consider an $\mathbb{A}^n$ is an affine space and a Cartesian coordinate system $f : \mathbb{A}^n \ni p \mapsto (x^1(p), \ldots, x^n(p)) \in \mathbb{R}^n$ with origin O and axis $e_1, \ldots, e_n$. A standard notation to denote the elements of that basis and those of the associated dual basis is:

$$\frac{\partial}{\partial x^i} := e_i \,, \quad \text{and} \quad dx^i := e^{*i} \,, \quad \text{for } i = 1, \ldots, n. \tag{3.2}$$

This notation is standard notation in differential geometry, however we want not to address here the general definition of vector tangent and cotangent to a manifold. The simplest explanation of such a notation is that, under changes of coordinates (3.1), formal computations based on the fact that:

$$\frac{\partial x'^j}{\partial x^i} = A^j{}_i \,,$$

show that (writing the summation over repeated indices explicitly)

$$\frac{\partial}{\partial x^i} = \sum_{j=1}^{n} A^j{}_i \frac{\partial}{\partial x'^j} \,, \quad \text{and} \quad dx'^j = \sum_{i=1}^{n} A^j{}_i dx^i \,. \tag{3.3}$$

And, under the employed notation, those correspond to the *correct* relations

$$e_i = \sum_{j=1}^{n} A^j{}_i e'_j \,, \quad \text{and} \quad e'^{*j} = \sum_{i=1}^{n} A^j{}_i e^{*i} \,.$$

Very often, the notation for $\frac{\partial}{\partial x^i}$ is shortened to ∂_{x^i}. We make use of the shortened notation from now on.

Let us pass to consider tensors in the algebra $\mathscr{A}_{\mathbb{R}}(V)$. A Change of Cartesian coordinates

$$x'^j = \sum_{i=1}^{n} A^j{}_i x^i + b^j \,,$$

involves a corresponding change in the associated bases of V and V^*. Under these changes of bases, the transformation of components of contravariant tensors uses the same coefficients $A^j{}_i$ appearing in the relation connecting different Cartesian coordinate systems: if $t \in V \otimes V$ one has

$$t = t^{ij} \partial_{x^i} \otimes \partial_{x^j} = t'^{kl} \partial_{x'^i} \otimes \partial_{x'^l} \,, \quad \text{where } t'^{kl} = A^k{}_i A^l{}_j t^{ij}.$$

Covariant tensors uses the coefficients of the matrix A^{-1t} where A is the matrix of coefficients $A^j{}_i$. Therefore, if we restrict ourselves to employ only bases in V and V^* associated with Cartesian coordinate systems, the transformation law of components of tensors use the same coefficients as those appearing in the transformation law of Cartesian coordinates. For this reason tensors in affine spaces are said *Cartesian tensors*. A straightforward application of the given notation is the tangent vector to a differentiable curve in $\mathbb{A}^n$.

Definition 3.6. **(Tangent vector.)** Let $\mathbb{A}^n$ be an affine space with space of translations V. If $\gamma : (a,b) \to \mathbb{A}^n$ is $C^1((a,b))$, one defines the **tangent vector** $\dot{\gamma}(t)$ at γ in $\gamma(t)$ as

$$\dot{\gamma}(t) := \frac{dx^i}{dt} \partial_{x^i} \,,$$

where $f : \mathbb{A}^n \ni p \mapsto (x^1(p), \ldots, x^n(p)) \in \mathbb{R}^n$ is any Cartesian coordinate system on $\mathbb{A}^n$. $\diamondsuit$

The definition is well-posed because $\dot{\gamma}(t)$ does not depend on the chosen Cartesian coordinate system. Indeed, under a changes of coordinates (3.1) one finds

$$\frac{dx'^j}{dt} = A^j{}_i \frac{dx^i}{dt} ,$$

and thus

$$\frac{dx'^j}{dt} \partial_{x'^j} = A^j{}_i \frac{dx^i}{dt} \partial_{x'^j} ,$$

so that, by (3.3),

$$\frac{dx'^j}{dt} \partial_{x'^j} = \frac{dx^i}{dt} \partial_{x^i} .$$

3.4.2 Applied tensors.

In several physical applications it is convenient to view a tensor as applied in a point $p \in \mathbb{A}^n$. This notion is based on the following straightforward definition which is equivalent to that given using the structure of differentiable manifold of an affine space.

Definition 3.7. **(Tangent and cotangent space.)** If $\mathbb{A}^n$ is an affine space with space of translations V and $p \in \mathbb{A}^n$, the **tangent space** at $\mathbb{A}^n$ in p is the vector space

$$T_p\mathbb{A}^n := \{(p, v) \mid v \in V\}$$

with vector space structure naturally induced by that of V (i.e. by the definition of linear composition of vectors

$$a(p, v) + b(p, u) := (p, au + bv) , \quad \text{for all } a, b \in \mathbb{R} \text{ and } u, v \in V.)$$

The **cotangent space** at $\mathbb{A}^n$ in p is the vector space

$$T_P^*\mathbb{A}^n := \{(p, v^*) \mid v^* \in V^*\}$$

with vector space structure naturally induced by that of V^*.
The bases of $T_p\mathbb{A}^n$ and $T_p^*\mathbb{A}^n$ associated with a Cartesian coordinate system $f : \mathbb{A}^n \ni p \mapsto (x^1(p), \ldots, x^n(p)) \in \mathbb{R}^n$ are respectively denoted by $\{\partial_{x^i}|_p\}_{i=1,\ldots,n}$ and $\{dx^i|_p\}_{i=1,\ldots,n}$ (but the indication of the application point p is very often omitted).
$\diamondsuit$

It is obvious that $T_p^*\mathbb{A}^n$ is canonically isomorphic to the dual space $(T_p\mathbb{A}^n)^*$ and every space of tensor $S_p \in \mathscr{A}_{\mathbb{R}}(T_p\mathbb{A}^n)$ is canonically isomorphic to a vector space

$$\{(p, t) \mid t \in S\} ,$$

where $S \in \mathscr{A}_{\mathbb{R}}(V)$.

Definition 3.8. **(Tensor fields.)** Referring to definition 3.7, the **tensor algebra of applied tensors** at $\mathbb{A}^n$ in p is the algebra $\mathscr{A}(T_p\mathbb{A}^n)$.
An assignment $\mathbb{A}^n \ni p \mapsto t_p \in S_p$ such that, every space $S_p \in \mathscr{A}(T_p\mathbb{A}^n)$ is of the same order, and the components of t_p define C^k functions with respect to the bases of tensors applied at every $p \in \mathbb{A}^n$ associated with a fixed Cartesian coordinate system, is a C^k **tensor field** on $\mathbb{A}^n$. The order of the tensor field is defined as the order of each tensor t_p.
$\diamondsuit$

The definition is well-posed in the sense that it does not depend on the choice of the used Cartesian coordinate system. This is due to the fact that the components of a fixed tensor field transforms, under changes of Cartesian coordinates, by means of a linear transformation with constant coefficients.

3.4.3 The natural isomorphism between V and V^* for Cartesian vectors in Euclidean spaces

Let us consider an Euclidean space $\mathbb{E}^n$ and, from now on, *we restrict ourselves to only use orthonormal Cartesian coordinate frames.* We are going to discuss the existence of a natural isomorphism between V and V^* that we shall consider much more extensively later from a wider viewpoint.
If $\{e_i\}_{i=1,...,n}$ and $\{e^{*j}\}_{j=1,...,n}$ are an orthonormal basis in V and its associated basis in V^* respectively, we can consider the isomorphism:

$$V \ni v = v^i e_i \mapsto v_j e^{*j} \in V^* \quad \text{where } v_j := v^j \text{ for } j = 1, 2, \ldots, n.$$

In principle that isomorphism may seem to depend on the chosen basis $\{e_i\}_{i=1,...,n}$, actually it does not! Indeed, if we chose another *orthonormal* basis and its dual one $\{e'_k\}_{k=1,...,n}$ and $\{e'^{*h}\}_{h=1,...,n}$, so that $e_i = A^j{}_i e'_j$ and $e^{*j} = B_h{}^j e'^{*h}$, the isomorphism above turns out to be written:

$$H : V \ni v = v'^j e'_j \mapsto v'_h e'^{*h} \in V^* \quad \text{where } v'_h := B_h{}^k (A^{-1})^k{}_j v'^j \text{ for } j = 1, 2, \ldots, n.$$

We know that $B = A^{-1t}$, however, in our case $A \in O(n)$, so that $A^t = A^{-1}$ and thus $B = A$. Consequently:

$$B_h{}^k (A^{-1})^k{}_j = (BA^{-1})_{hj} = \delta_{hj} \, .$$

We conclude that the isomorphism

$$H : V \ni v = v^i e_i \mapsto v_j e^{*j} \in V^* \quad \text{where } v_j := v^j \text{ for } j = 1, 2, \ldots, n.$$

is actually independent from the chosen orthonormal basis and, in that sense, is natural. In other words, in the presence of a scalar product there is a natural identification of V and V^* that can be written down as, in the abstract index notation:

$$v^i \mapsto v_j \quad \text{where } v_j := v^j \text{ for } j = 1, 2, \ldots, n,$$

provided we confine ourselves to handle vectors by employing orthonormal basis only. Exploiting the universality property, this isomorphism easily propagates in the whole tensor algebra generated by V. So, for instance there exists natural isomorphisms between $V \otimes V^*$ and $V \otimes V$ or $V \otimes V^*$ and $V^* \otimes V$ acting as follows in the abstract index notation and referring to canonical bases associated with orthogonal bases in V:

$$t_i{}^j \mapsto t_{ij} := t_i{}^j \,,$$

or, respectively

$$t_i{}^j \mapsto t^i{}_j := t_i{}^j \,.$$

In practice the height of indices does not matter in vector spaces equipped with a scalar product, provided one uses orthonormal bases only.
All that said immediately applies to Cartesian tensors in Euclidean spaces when V is the space of translations or is the tangent space $T_p\mathbb{E}^n$ at some point $p \in \mathbb{E}^n$

Examples 3.1. In classical continuum mechanics, consider an internal portion $\mathscr{C}$ of a continuous body $\mathscr{B}$. Suppose that $\mathscr{C}$ is represented by a regular set in the Euclidean space $\mathbb{E}^3$: the physical space where the body is supposed to be at rest. Let $\partial\mathscr{C}$ be the boundary of $\mathscr{C}$ assumed to be a regular surface. If $p \in \partial\mathscr{C}$, the part of $\mathscr{B}$ external to $\mathscr{C}$ acts on $\mathscr{C}$ through the element of surface $dS \subset \partial\mathscr{C}$ about p, by a density of force called *stress vector* $\mathbf{s}(p, \mathbf{n})$. $\mathbf{n}$ denotes the versor orthogonal to dS at p. In this context, it is convenient to assume that: $\mathbf{s}(p, \mathbf{n}), \mathbf{n}_p \in T_p\mathbb{E}^3$.
In general one would expect a complicated dependence of the stress vector from $\mathbf{n}$. Actually a celebrated theorem due to Cauchy proves that (under standard hypotheses concerning the dynamics of the continuous body) this dependence must be linear. More precisely, if one defines: $\mathbf{s}(p, \mathbf{0}) := \mathbf{0}$ and $\mathbf{s}(p, \mathbf{v}) := |\mathbf{v}|\mathbf{s}(p, \mathbf{v}/|\mathbf{v}|)$, it turns out that $T_p\mathbb{E}^3 \ni \mathbf{v} \mapsto \mathbf{s}(p, \mathbf{v})$ is linear. As a consequence, there is a tensor $\sigma \in T_p\mathbb{E}^3 \otimes T_p^*\mathbb{E}^3$ (depending on the nature of $\mathscr{B}$ and on p) called *Cauchy stress tensor*, such that

$$\mathbf{s}(p, \mathbf{n}) = \sigma(\mathbf{n}) \,. \tag{3.4}$$

In order to go on with continuum mechanics it is assumed that, varying $p \in \mathbb{E}^3$, the map $p \mapsto \sigma$ gives rise to a sufficiently regular (usually C^2) tensor field of order $(1, 1)$.
Exploiting the abstract index notation:

$$s^i(p, \mathbf{n}) = \sigma(p)^i{}_j n^j \,.$$

It is possible to prove, taking the equation for the angular momentum into account that the stress tensor is symmetric. In other words, passing to the completely covariant expression for it (taking advantage) of the existence of the natural isomorphism between $V \otimes V^*$ and $V^* \otimes V^*$ – for $V = T_p\mathbb{E}^3$ – discussed above, one has:

$$\sigma(p)_{ij} = \sigma(p)_{ji} \,.$$

Chapter 4

Some application to group theory.

In this chapter we present a few applications of the theory developed previously in relation to group theory.

4.1 Some notions about Groups.

4.1.1 Basic notions on groups and matrix groups.

As is known a **group** is an algebraic structure, $(G, \circ)$, where G is a set and $\circ : G \times G \to G$ is a mapping called the **composition rule** of the group or **group product**. Moreover the following three conditions have to be satisfied.
(1) $\circ$ is *associative*, i.e.,

$$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3 \,, \quad \text{for all} \quad g_1, g_2, g_3 \in G \,.$$

(2) There is a **group unit** or **unit element**, i.e., there is $e \in G$ such that

$$e \circ g = g \circ e = g \,, \quad \text{for all} \quad g \in G \,.$$

(3) Each element $g \in G$ admits an **inverse element**, i.e.,

$$\text{for each} \quad g \in G \quad \text{there is} \quad g^{-1} \in G \quad \text{with} \quad g \circ g^{-1} = g^{-1} \circ g = e \,.$$

We remind the reader that the unit element turns out to be unique and so does the inverse element for each element of the group (the reader might show those uniqueness properties as an exercise). A group $(G, \circ)$ is said to be **commutative** or **Abelian** if $g \circ g' = g' \circ g$ for each pair of elements, $g, g' \in G$; otherwise it is said to be **non-commutative** or **non-Abelian**. A subset $G' \subset G$ of a group is called **subgroup** if it is a group with respect to the restriction to $G' \times G'$ of the composition rule of G. A subgroup N of a group G is called a **normal subgroup** if it is invariant under **conjugation**, that is, for each element $n \in N$ and each $g \in G$, the element $g \circ n \circ g^{-1}$ is still in N.

If $(G_1, \circ_1)$ and $(G_2, \circ_2)$ are groups, a group **homomorphism** from G_1 to G_2 is a mapping $h : G_1 \to G_2$ which *preserves the group structure*, i.e., the following requirement has to be fulfilled:

$$h(g \circ_1 g') = h(g) \circ_2 h(g') \quad \text{for all} \quad g, g' \in G_1 \, ,$$

As a consequence, they also hold with obvious notations:

$$h(e_1) = e_2 \, ,$$

and

$$h(g^{-1_1}) = (h(g))^{-1_2} \quad \text{for each} \quad g \in G_1 \, .$$

Indeed, if $g \in G_1$ one has $h(g) \circ e_2 = h(g) = h(g \circ e_1) = h(g) \circ h(e_1)$. Applying $h(g)^{-1}$ on the left, one finds $e_2 = h(e_1)$. On the other hand $h(g)^{-1} \circ h(g) = e_2 = h(e_1) = h(g^{-1} \circ g) = h(g^{-1}) \circ h(g)$ implies $h(g)^{-1} = h(g^{-1})$.
The **kernel** of a group homomorphism, $h : G \to G'$, is the subgroup $K \subset G$ whose elements g satisfy $h(g) = e'$, e' being the unit element of G'. Obviously, h is injective if and only if its kernel contains the unit element only. A group **isomorphism** is a *bijective* group homomorphism. A group isomorphism $h : G \to G$, so that the domain and the co-domain are the same group, is called group **automorphism** on G. The set of group automorphism on a given group G is denoted by $Aut(G)$ and it is a group in its own right when the group product is the standard composition of maps.

Examples 4.1. The algebra $M(n, \mathbb{K})$, that is the vector space of the matrices $n \times n$ with coefficients in $\mathbb{K}$ equipped with the matrix product, contains several interesting *matrix groups.* If $n > 0$, most interesting groups in $M(n, \mathbb{K})$ are non-commutative.
(1) The first example is $GL(n, \mathbb{K})$ which is the set of the $n \times n$ matrices A with components in the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and $\det A \neq 0$. It is a group with group composition rule given by the usual product of matrices.
(2) An important subgroup of $GL(n, \mathbb{K})$ is the *special group* $SL(n, \mathbb{K})$, i.e., the set of matrices A in $GL(n, \mathbb{K})$ with $\det A = 1$. (The reader might show that $SL(n, \mathbb{K})$ is a subgroup of $GL(n, \mathbb{K})$.)
(3) If $\mathbb{K} = \mathbb{R}$ another interesting subgroup is the *orthogonal group* $O(n)$ containing all of the matrices satisfying $R^t R = I$.
(4) If $\mathbb{K} = \mathbb{C}$ an interesting subgroup of $GL(n, \mathbb{C})$ is the *unitary group* $U(n)$ (and the associated subgroup $SU(n) := SL(n, \mathbb{C}) \cap U(n)$), containing the matrices A satisfying $\overline{A^t} A = I$ (the bar denoting the complex conjugation) is of great relevance in quantum physics.
(5) In section 5.2.1 we shall encounter the pseudo orthogonal groups $O(m, p)$ and, one of them, the Lorentz group $O(1, 3)$ will be studied into details in the last two chapters.

Notice that there are groups which *are not defined* as group of matrices, e.g., $(\mathbb{Z}, +)$. Some of them can be represented in terms of matrices anyway (for example $(\mathbb{Z}, +)$ can be represented as a matrix subgroup of $GL(2, \mathbb{R})$, we leave the proof as an exercise). There are however groups which cannot be represented in terms of matrices as the so-called *universal covering of* $SL(2, \mathbb{R})$.

An example of a group which is not defined as a group of matrices (but it admits such representations) is given by the *group of permutations of n elements* which we shall consider in the next subsection.

Exercises 4.1.

1. Prove the uniqueness of the unit element and the inverse element in any group.

2. Show that in any group G the unique element e such that $e^2 = e$ is the unit element.

3. Show that if G' is a subgroup of G the unit element of G' must coincide with the unit element of G, and, if $g \in G'$, the inverse element g^{-1} in G' coincides with the inverse element in G.

4. Show that if $h : G_1 \to G_2$ is a group homomorphism, then $h(G_1)$ is a subgroup of G_2.

4.1.2 Direct product and semi-direct product of groups.

We remind here two important elementary notions of group theory, the direct product and the semi-direct product of groups.

If G_1 and G_2 are groups, their **direct product**, $G_1 \times G_2$, is another group defined as follows. The elements of $G_1 \times G_2$ are, as the notation suggests, the elements (g_1, g_2) of the *Cartesian product* of the *sets* G_1 and G_2. Moreover the composition rule is $(g_1, g_2) \circ (f_1, f_2) := (g_1 \circ_1 f_1, g_2 \circ_2 f_2)$ for all $(g_1, g_2), (f_1, f_2) \in G_1 \times G_2$. Obviously the unit of $G_1 \times G_2$ is (e_1, e_2), where e_1 and e_2 are the unit elements of G_1 and G_2 respectively. The proof that these definitions determine a correct structure of group on the set $G_1 \times G_2$ is trivial and it is left to the reader. Notice that, with the given definition of direct product, G_1 and G_2 turn out to be normal subgroups of the direct product $G_1 \times G_2$.

Sometimes there are gropus with a structure close to that of direct product but a bit more complicated. These structure are relevant, especially in physics and are captured within the following definition.
Suppose that $(G_1, \circ_1)$ and $(G_2, \circ_2)$ are groups and for every $g_1 \in G_1$ there is a group isomorphism $\psi_{g_1} : G_2 \to G_2$ satisfying the following property: (i) $\psi_{g_1} \circ \psi_{g'_1} = \psi_{g_1 \circ_1 g'_1}$ and (ii) $\psi_{e_1} = id_{G_2}$, where $\circ$ is the usual composition of functions and e_1 the unit element of G_1. (In other words, $\psi_g \in Aut(G_2)$ for every $g \in G_1$ and the map $G_1 \ni g \mapsto \psi_g$ is a group homomorphism from G_1 to $Aut(G_2)$.) In this case a natural structure of group can be assigned on the Cartesian product $G_1 \times G_2$. This is done by defining the composition rule between a pair of elements, $(g_1, g_2), (f_1, f_2) \in G_1 \times G_2$, as

$$(g_1, g_2) \circ_\psi (f_1, f_2) := (g_1 \circ_1 f_1,\ g_2 \circ_2 \psi_{g_1}(f_2)) \ .$$

We leave to the reader the strightforward proof of the well posedness of the given group composition rule. Again, the unit element turns out to be (e_1, e_2). The obtained group structure $(G_1 \times_\psi G_2, \circ_\psi)$, is called the **semi-direct product** of G_1 and G_2.

Examples 4.2.
(1) Consider the group $O(3)$ of orthogonal matrices in $\mathbb{R}^3$, and $\mathbb{R}^3$ itself viewed as additive Abelian group. Define, for every $R \in O(3)$, the automorphism $\psi_R \in Aut(\mathbb{R}^3)$ as $\psi_R : \mathbb{R}^3 \ni \mathbf{v} \mapsto R\mathbf{v} \in \mathbb{R}^3$. We can endow $O(3) \times \mathbb{R}^3$ with the structure of semi-direct product group $O(3) \times_\psi \mathbb{R}^3$ when definig the composition rule

$$(R, \mathbf{v}) \circ_\psi (R', \mathbf{v}') := (RR', \mathbf{v} + R\mathbf{v}') .$$

The meaning of the structure so defined should be evident: A pair $(R, \mathbf{v})$ is a roto-translation, i.e. a generic isometry of the space $\mathbb{R}^3$ (equipped with the standard metric structure). In other words $(R, \mathbf{v})$ acts of a point $\mathbf{u} \in \mathbb{R}^3$ with a rotation followed by a translation, $(R, \mathbf{v}) : \mathbf{u} \mapsto \mathbf{v} + R\mathbf{u}$. The composition rule $\circ_\psi$ is nothing but a composition of two such transformations. It is possible to prove that every isometry of $\mathbb{R}^3$ equipped with the standard structure of Euclidean space, is necessarily an element of $O(3) \times_\psi \mathbb{R}^3$.
(2) As further examples, we may mention the Poincaré group as will be presented in definition 8.12, the group of isometries of an Eucliedean (affine) space, the Galileo group in classical mechanics, the BMS group in general relativity, the inhomogeneous $SL(2, \mathbb{C})$ group used in relativistic spinor theory. Similar structures, in physics, appear very frequently.

Exercises 4.2.

1. Consider a semi-dirct product of groups $(G \times_\psi N, \circ_\psi)$ as defined beforehand (notice that the role of G and N is not interchangeable). Prove that N is a normal subgroup of $G \times_\psi N$ and that

$$\psi_g(n) = g \circ_\psi n \circ_\psi g^{-1} \quad \text{for every } g \in G \text{ and } n \in N.$$

2. Consider a group $(H, \circ)$, let N be a normal subgroup of H and let G a subgroup of H. Suppose that $N \cap G = \{e\}$, where e is the unit element of H, and that $H = GN$, in the sense that, for every $h \in H$ there are $g \in G$ and $n \in N$ such that $h = gn$. Prove that (g, n) is uniquely determined by h and that H is isomorphic to $G \times_\psi N$, where

$$\psi_g(n) := g \circ h \circ g^{-1} \quad \text{for every } g \in G \text{ and } n \in N.$$

4.2 Tensor products of group representations.

4.2.1 Linear representation of groups and tensor representations.

We are interested in the concept of (linear) *representation of a group on a vector space.* In order to state the corresponding definition, notice that, if V is a (not necessarily finite-dimensional) vector space, $\mathscr{L}(V|V)$ contains an important group. This is $GL(V)$ which is the set of both injective and surjective elements of $\mathscr{L}(V|V)$ equipped with the usual composition rule of maps. We can give the following definition.

Definition 4.1. **(Linear group on a vector space.)** If V is a vector space, $GL(V)$ denotes the group of linear mappings $f : V \to V$ such that f is injective and surjective, with group composition rule given by the usual mappings composition. $GL(V)$ is called the **linear group on** V. $\diamondsuit$

Remarks.
(a) If $V := \mathbb{K}^n$ then $GL(V) = GL(n, \mathbb{K})$.
(b) If $V \neq V'$ it is not possible to define the analogue of $GL(V)$ considering some subset of $\mathscr{L}(V|V')$. (The reader should explain the reason.)
(c) Due to theorem 3.2, if $dimV < +\infty$, we have that $GL(V)$ coincides to a subset of $V \otimes V^*$.

Definition 4.2. **(Linear group representation on a vector space.)** Let $(G, \circ)$ be a group and V a vector space. A (linear group) **representation of** G **on** V is a homomorphism $\rho : G \to GL(V)$. Moreover a representation $\rho : G \to GL(V)$ is called:
(1) faithful if it is injective,
(2) free if, for any $v \in V \setminus \{0\}$, the subgroup of G made of the elements h_v such that $\rho(h_v)v = v$ contains only the unit element of G,
(3) transitive if for each pair $v, v' \in V \setminus \{0\}$ there is $g \in G$ with $v' = \rho(g)v$.
(4) irreducible if there is no *proper* vector subspace $S \subset V$ which is **invariant** under the action of $\rho(G)$, i.e., which satisfies $\rho(g)S \subset S$ for all $g \in G$.
Finally, the representation $\rho^d : G \to GL(V^*)$ defined by

$$\rho^d(g) := \rho(g^{-1})^* \quad \text{for all } g \in G,$$

where $\rho(g^{-1})^* : V^* \to V^*$ is the adjoint operator of $\rho(g^{-1}) : V \to V$, as definied in definition 2.4, is called the **dual representation** associated with ρ. $\diamondsuit$

Remark. **(1)** The definitions given above representations can be extended to the case of general (nonlinear) representations of a group G and replacing $GL(V)$ for a group of bijective transformations $f : V \to V$. Assuming that V is a differential manifold $GL(V)$ can be replaced for the group $Diff(V)$ of d*iffeomorphisms* of V. In that case the homomorphism $\rho : G \to Diff(V)$ is a representation of G in terms of diffeomorphisms. When V is a Riemannian manifold, representations in terms of *isometries* can be similarly defined. In these extended contexts the definitions of faithful, free (dropping the constraint $v \neq 0$) and transitive (dropping the constraint $v, v' \neq 0$)representations can be equally stated.
(2) The presence of the inverse element in the definition of $\rho^d(g) := \rho(g^{-1})^*$ accounts for the fact that adjoint operators compose in the reverse order, i.e. $(T_1T_2)^* = T_2^*T_1^*$. Thanks to the inverse element g^{-1} in the definition of $\rho^d(g)$, one gets $\rho^d(g_1)\rho^d(g_2) = \rho^d(g_1g_2)$ as requested for representations, instead of $\rho^d(g_1)\rho^d(g_2) = \rho^d(g_2g_1)$ as it would be if simply defining $\rho^d(g) := \rho(g)^*$.

Equipped with the above-given definitions we are able to study the simplest interplay of tensors and group representations. We want to show that the notion of tensor product allows the definitions of *tensor products of representations.* That mathematical object is of fundamental importance in applications to Quantum Mechanics, in particular as far as systems with many

components are concerned.
Consider a group G (from now on we omit to specify the symbol of the composition rule whenever it does not produces misunderstandings) and several representations of the group $\rho_i : G \to GL(V_i)$, where $V_1, \ldots, V_n$ are finite-dimensional vector spaces on the same field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. For each $g \in G$, we may define a multi-linear mapping $[\rho_1(g), \ldots, \rho_n(g)] \in \mathscr{L}(V_1, \ldots, V_n | V_1 \otimes \ldots \otimes V_n)$ given by, for all $(\rho_1(g), \ldots, \rho_n(g)) \in V_1 \times \ldots \times V_n$,

$$[\rho_1(g), \ldots, \rho_n(g)] : (v_1, \ldots, v_n) \mapsto (\rho_1(g)v_1) \otimes \ldots \otimes (\rho_n(g)v_n) .$$

That mapping is multi linear because of the multi linearity of the tensor-product mapping and the linearity of the operators $\rho_k(g)$. Using the universality theorem, we *uniquely* find a linear mapping which we indicate by $\rho_1(g) \otimes \ldots \otimes \rho_n(g) : V_1 \otimes \ldots \otimes V_n \to V_1 \otimes \ldots \otimes V_n$ such that:

$$\rho_1(g) \otimes \ldots \otimes \rho_n(g)(v_1 \otimes \ldots \otimes v_n) = (\rho_1(g)v_1) \otimes \ldots \otimes (\rho_n(g)v_n) .$$

Definition 4.3. **(Tensor product of representations.)** Let $V_1, \ldots V_n$ be finite-dimensional vector spaces on the same field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and suppose there are n representations $\rho_i : G \to GL(V_k)$ of the same group G on the given vector spaces. The set of linear maps

$$\{\rho_1(g) \otimes \ldots \otimes \rho_n(g) : V_1 \otimes \ldots \otimes V_n \to V_1 \otimes \ldots \otimes V_n \mid g \in G\} ,$$

defined above is called **tensor product of representations** $\rho_1, \ldots, \rho_n$. $\diamondsuit$

The relevance of the definition above is evident because of the following theorem,

Theorem 4.1. *Referring to the definition above, the map $G \ni g \mapsto \rho_1(g) \otimes \ldots \otimes \rho_n(g)$ is a* **linear group representation of G on the tensor-product space $V_1 \otimes \ldots \otimes V_n$**.

Proof. We have to show that the mapping

$$g \mapsto \rho_1(g) \otimes \ldots \otimes \rho_n(g) ,$$

is a group homomorphism from G to $GL(V_1 \otimes \ldots \otimes V_n)$. Taking account of the fact that each ρ_i is a group homomorphism, if $g, g' \in G$, one has

$$\rho_1(g')\otimes\ldots\otimes\rho_n(g')(\rho_1(g)\otimes\ldots\otimes\rho_n(g)(v_1\otimes\ldots\otimes v_n)) = \rho_1(g')\otimes\ldots\otimes\rho_n(g')((\rho_1(g)v_1)\otimes\ldots\otimes(\rho_n(g)v_n))$$

and this is

$$(\rho_1(g' \circ g)v_1) \otimes \ldots \otimes (\rho_n(g' \circ g)v_n) .$$

The obtained result holds true also using a canonical basis for $V_1 \otimes \ldots \otimes V_n$ made of usual elements $e_{1,i_1} \otimes \ldots \otimes e_{n,i_n}$ in place of $v_1 \otimes \ldots \otimes v_n$. By linearity, it means that the found identity is valid if replacing $v_1 \otimes \cdots \otimes v_n$ for every tensor in $V_1 \otimes \cdots \otimes V_n$. Therefore,

$$(\rho_1(g') \otimes \ldots \otimes \rho_n(g'))(\rho_1(g) \otimes \ldots \otimes \rho_n(g)) = \rho_1(g' \circ g) \otimes \ldots \otimes \rho_n(g' \circ g) .$$

To conclude, notice that $\rho_1(g) \otimes \ldots \otimes \rho_n(g) \in GL(V_1 \otimes \ldots \otimes V_n)$ because $\rho_1(g) \otimes \ldots \otimes \rho_n(g)$ is (1) linear and furthermore it is (2) bijective. The latter can be proved as follows:

$$\rho_1(g^{-1}) \otimes \ldots \otimes \rho_n(g^{-1}) \circ (\rho_1(g) \otimes \ldots \otimes \rho_n(g)) = (\rho_1(g) \otimes \ldots \otimes \rho_n(g)) \circ \rho_1(g^{-1}) \otimes \ldots \otimes \rho_n(g^{-1})$$

$$= \rho_1(e) \otimes \ldots \otimes \rho_n(e) = I \,.$$

The last identity follows by linearity form $(\rho_1(e) \otimes \ldots \otimes \rho_n(e))(v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n$. $\Box$

More generally, if $A_k : V_k \to U_k$ are n linear mappings (operators), and all involved vector spaces are finite dimensional and with the same field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, it is defined the *tensor product of operators*.

Definition 4.4. **(Tensor Product of Operators.)** If $A_k : V_k \to U_k$, $k = 1, \ldots, n$ are n linear mappings (operators), and all the vector spaces U_i, V_j are finite dimensional with the same field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, the **tensor product of** $A_1, \ldots, A_n$ is the linear mapping

$$A_1 \otimes \ldots \otimes A_n : V_1 \otimes \ldots \otimes V_n \to U_1 \otimes \ldots \otimes U_n$$

uniquely determined by the universality theorem and the requirement:

$$(A_1 \otimes \ldots \otimes A_n) \circ \otimes = A_1 \times \ldots \times A_n \,,$$

where

$$A_1 \times \ldots \times A_n : V_1 \times \ldots \times V_n \to U_1 \otimes \ldots \otimes U_n \,,$$

is the multi-linear mapping such that, for all $(v_1, \ldots, v_n) \in V_1 \times \ldots \times V_n$:

$$A_1 \times \ldots \times A_n : (v_1, \ldots, v_n) \mapsto (A_1 v_1) \otimes \ldots \otimes (A_n v_n) \,.$$

$\diamondsuit$

Remark. **(1)** Employing the given definition and the same proof used for the relevant part of the proof of theorem 4.1, it is simply shown that if $A_i : V_i \to U_i$ and $B_i : U_i \to W_i$, $i = 1, \ldots, n$ are $2n$ linear maps and all involved spaces V_i, U_j, W_k are finite dimensional with the same field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, then

$$B_1 \otimes \ldots \otimes B_n \;\; A_1 \otimes \ldots \otimes A_n = (B_1 A_1) \otimes \ldots \otimes (B_n A_n) \,.$$

(2) Suppose that $A : V \to V$ and $B : U \to U$ are linear operators. How does $A \otimes B$ work in components? Or, that is the same, what is the explicit expression of $A \otimes B$ when employing the abstract index notation?
To answer we will make use of Theorem 3.2. Fix a basis $\{e_i\}_{i \in I} \subset V$ and another $\{f_j\}_{j \in J} \subset U$ and consider the associated canonical bases in tensor spaces constructed out of V, U and their dual spaces. In view of the mentioned theorem we know that if $v \in V$ and $u \in U$ then indicating

again by $A \in V \otimes V^*$ and $B \in U \otimes U^*$ the tensors associated with the linear maps A and B, respectively:

$$(Av)^i = A^i{}_j v^j \quad \text{and} \quad (Bu)^r = B^r{}_s u^s \, .$$

Passing to the tensor products and making use of the definition of tensor product of linear operators:

$$((A \otimes B)(v \otimes u))^{ir} = ((Av) \otimes (Bu))^{ir} = A^i{}_j v^j B^r{}_s u^s = A^i{}_j B^r{}_s v^j u^s \, .$$

Now, since $A \otimes B : V \otimes U \to V \otimes U$ is linear, we can write, in general for $t = t^{rs} e_r \otimes f_s$:

$$((A \otimes B)t)^{ir} = A^i{}_j B^r{}_s t^{js} \, . \tag{4.1}$$

We conclude that the tensor in $V \otimes V^* \otimes U \otimes U^*$ associated with $A \otimes B$ has components

$$A^i{}_j B^r{}_s \, ,$$

and, as usually, the action of $A \otimes B : V \otimes U \to V \otimes U$ on $t \in V \otimes U$ is obtained by the tensor product of t and the tensor associated to $A \otimes B$, followed by a suitable contraction, as written in (4.1).

4.2.2 An example from Continuum Mechanics.

Referring to the example 3.1, consider once more an internal portion $\mathscr{C}$ of a continuous body $\mathscr{B}$. Suppose that $\mathscr{C}$ is represented by a regular set in the Euclidean space $\mathbb{E}^3$. As we said there is a tensor $\sigma \in T_p\mathbb{E}^3 \otimes T_p^*\mathbb{E}^3$ (depending on the nature of $\mathscr{B}$ and on p) called Cauchy stress tensor, such that

$$\mathbf{s}(p, \mathbf{n}) = \sigma(\mathbf{n}) \, . \tag{4.2}$$

Suppose now to rotate $\mathscr{B}$ about p with a rotation $R \in SO(3)$. If the body is isolated from any other external system, and the rest frame is inertial, assuming *isotropy of the space*, we expect that the relationship between $\mathbf{s}, \mathbf{n}$ and σ "remains fixed" under the rotation R, provided one replaces the vectors $\mathbf{s}$, $\mathbf{n}$ and the tensor σ with the corresponding objects transformed under R. How does the rotation R act on the tensor σ?
We start from the action of a rotation on vectors. We stress that we are assuming here an *active* point of view, a vector is transformed into another different vector under the rotation. It is not a passive change of basis for a fixed vector. Fix an orthonormal basis $e_1, e_2, e_3 \in T_p\mathbb{E}^3$. Referring to that basis, the action of $SO(3)$ is, if $\mathbf{v} = v^i e_i$ and R is the matrix of coefficients $R^i{}_j$:

$$\mathbf{v} \mapsto \rho_R \mathbf{v}$$

where

$$(\rho_R \mathbf{v})^i := R^i{}_j v^j \, .$$

It is a trivial task to show that

$$\rho_R \rho_{R'} = \rho_{RR'} \, , \quad \text{for all } R, R' \in SO(3) \, ,$$

therefore $SO(3) \ni R \mapsto \rho_R$ defines, in fact, a (quite trivial) representation of $SO(3)$ on $T_p\mathbb{E}^3$. This representation acts both on $\mathbf{n}$ and $\mathbf{s}$ and enjoys a straightforward physical meaning.
Let us pass to the tensor $\sigma \in T_p\mathbb{E}^3 \otimes T_p^*\mathbb{E}^3$. In the bases $\{e_i\}_{i=1,2,3}$ and the associated dual one $\{e^{*j}\}_{j=1,2,3}$, the relation (3.4) reads:

$$s^i = \sigma^i{}_j n^j \ ,$$

and thus, by action of a rotation R, one gets

$$(R\mathbf{s})^i = R^i{}_k \sigma^k{}_j n^j \ ,$$

or also,

$$(R\mathbf{s})^i = R^i{}_k \sigma^k{}_h (R^{-1})^h{}_j (R\mathbf{n})^j \ .$$

We conclude that, if we require that the relation (4.2) is preserved under rotations, we must define the action of $R \in SO(3)$ on the tensors $\sigma \in T_p\mathbb{E}^3 \otimes T_p^*\mathbb{E}^3$ given by:

$$\sigma \mapsto \gamma_R \sigma$$

where, *in the said bases*:

$$(\gamma_R \sigma)^i{}_j = R^i{}_k \, (R^{-1})^h{}_j \, \sigma^k{}_h \ .$$

It is a trivial task to show that

$$\gamma_R \gamma_{R'} = \gamma_{RR'} \ , \quad \text{for all } R, R' \in SO(3) \ ,$$

therefore $SO(3) \ni R \mapsto \gamma_R$ defines a representation of $SO(3)$ on $T_p\mathbb{E}^3 \otimes T_p^*\mathbb{E}^3$. This representation is the tensor product of ρ acting on $T_p\mathbb{E}^3$ and another representation of $SO(3)$, ρ' acting on $T_p^*\mathbb{E}^3$ and defined as follows. If $\mathbf{w} = w_j e^{*j} \in T_p^*\mathbb{E}^3$ and $R \in SO(3)$ is the matrix of coefficients $R^i{}_j$:

$$\mathbf{w} \mapsto \rho'_R \mathbf{w}$$

where, referring to the dual basis $e^{*1}, e^{*2}, e^{*3} \in T_p^*\mathbb{E}^3$,

$$(\rho'_R \mathbf{w})_j := w_i (R^t)^i{}_j \ .$$

Indeed one has:

$$\gamma_R = \rho_R \otimes \rho'_R \ , \quad \text{for all } R, R' \in SO(3) \ .$$

It is simply proved that ρ' is nothing but the dual representation associated with ρ as defined in definition 4.2. We leave the immediate proof to the reader.

4.2.3 An example from Quantum Mechanics.

Physicists are involved with Hilbert spaces whenever they handle quantum mechanics. A Hilbert space is noting but a complex vector space equipped with a Hermitean scalar product (see the next chapter) such that it is complete with respect to the norm topology induced by that scalar product. However, here we analyze the structure of vector space only. Physically speaking, the vectors of the Hilbert space represent the states of the considered physical system (actually things are more complicated but we do not matter). To consider the simplest case we assume that the vector space which describes the states of the system is finite-dimensional (that is the case for the spin part of a quantum particle). Moreover, physics implies that the space $\mathcal{H}$ of the states of a composite system S made of two systems S_1 and S_2 associated with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, is the Hilbertian tensor product (see section 2.2.4 for more details) $\mathcal{H} = \mathcal{H}_1 \otimes_H \mathcal{H}_2$. As far as this example is concerned, the reader may omit the adjective "Hilbertian" and think of $\mathcal{H}$ as the standard algebraic tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, it being correct when $\mathcal{H}_1, \mathcal{H}_2$ are finite-dimensional. Let the system S_1 be described by a state $\psi \in \mathcal{H}_1$ and suppose to transform the system by the action of an element R of some physical group of transformations $\mathcal{G}$ (e.g. $SO(3)$). The transformed state ψ' is given by $U^{(1)}_R \psi$ where $\mathcal{G} \ni R \mapsto U^{(1)}_R$ is a representation of $\mathcal{G}$ in terms of linear transformations $U^{(1)}_R : \mathcal{H}_1 \to \mathcal{H}_1$. Actually, physics and the celebrated *Wigner's theorem* in particular, requires that every $U^{(1)}_R$ be a *unitary* (or *anti unitary*) transformation but this is not relevant for our case. The natural question concerning the representation of the action of $\mathcal{G}$ on the compositor system S is:
"If we know the representations $\mathcal{G} \ni R \mapsto U^{(1)}_R$ and $\mathcal{G} \ni R \mapsto U^{(2)}_R$, what about the representation of the action of $\mathcal{G}$ on S in terms of linear transformations in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$?"
The answer given by physics, at least when the systems S_1 and S_2 do not interact, is that $U_R := U^{(2)}_R \otimes U^{(1)}_R$.

4.3 Permutation group and symmetry of tensors.

We remind the definition of the *group of permutations of n objects* and give some know results of basic group theory whose proofs may be found in any group-theory textbook.

Definition 4.5. **(Group of permutations.)** Consider the set $I_n := \{1, \ldots, n\}$, the **group of permutations of n objects**, $\mathscr{P}_n$ is the set of the bijective mappings $\sigma : I_n \to I_n$ equipped with the composition rule given by the usual composition rule of functions. Moreover,
(a) the elements of $\mathscr{P}_n$ are called **permutations** (of n objects);
(b) a permutation of $\mathscr{P}_n$ with $n \geq 2$ is said to be a **transposition** if differs from the identity mapping and reduces to the identity mapping when restricted to some subset of I_n containing $n - 2$ elements. $\diamondsuit$

Comments 4.1.
(1) $\mathscr{P}_n$ contains $n!$ elements.

(2) Each permutation $\sigma \in \mathscr{P}_n$ can be represented by a corresponding string $(\sigma(1), \ldots, \sigma(n))$.
(3) If, for instance $n = 5$, with the notation above $(1,2,3,5,4)$, $(5,2,3,4,1)$, $(1,2,4,3,5)$ are transpositions, $(2,3,4,5,1)$, $(5,4,3,2,1)$ are not.
(4) It is possible to show that each permutation $\sigma \in \mathscr{P}_n$ can be decomposed as a product of transpositions $\sigma = \tau_1 \circ \ldots \circ \tau_k$. In general there are several different transposition-product decompositions for each permutation, however it is possible to show that if $\sigma = \tau_1 \circ \ldots \circ \tau_k = \tau'_1 \circ \ldots \circ \tau'_r$, where τ_i and τ'_j are transpositions, then $r + k$ is even. Equivalently, r is even or odd if and only if k is such. This defines the **parity**, $\epsilon_\sigma \in \{-1, +1\}$, of a permutation σ, where, $\epsilon_\sigma = +1$ if σ can be decomposed as a product of an *even* number of transpositions and $\epsilon_\sigma = -1$ if σ can be decomposed as a product of an *odd* number of transpositions.
(5) If $A = [A_{ij}]$ is a real or complex $n \times n$ matrix, it is possible to show (by induction) that:

$$det A = \sum_{\sigma \in \mathscr{P}_n} \epsilon_\sigma A_{1\sigma(1)} \cdots A_{n\sigma(n)} \, .$$

Alternatively, the identity above may be used to define the determinant of a matrix.

We pass to consider the action of $\mathscr{P}_n$ on tensors. Fix $n > 1$, consider the tensor algebra $\mathscr{A}_{\mathbb{K}}(V)$ and single out the tensor space $V^{n\otimes} := V \otimes \ldots \otimes V$ where the factor V appears n times. Then consider the following action of $\mathscr{P}_n$ on $V^{n\times} := V \times \ldots \times V$ where the factor V appears n times. For each $\sigma \in \mathscr{P}_n$ consider the mapping

$$\hat{\sigma} : V^{n\times} \to V^{n\otimes} \quad \text{such that} \quad \hat{\sigma}(v_1, \ldots, v_n) \mapsto v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)} \, .$$

It is quite straightforward to show that $\hat{\sigma}$ is multi linear. Therefore, let

$$\sigma^{\otimes} : V^{n\otimes} \to V^{n\otimes} \, ,$$

be the linear mapping uniquely determined by $\hat{\sigma}$ by means of the universality theorem. By definition, it is completely determined by linearity and the requirement

$$\sigma^{\otimes} : v_1 \otimes \ldots \otimes v_n \mapsto v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)} \, .$$

Theorem 4.2. *The above-defined linear mapping*

$$\sigma \mapsto \sigma^{\otimes}$$

with $\sigma \in \mathscr{P}_n$, is a group representation of $\mathscr{P}_n$ on $V^{n\otimes}$.

Proof. First we show that if $\sigma, \sigma' \in \mathscr{P}_n$ then

$$\sigma^{\otimes}(\sigma'^{\otimes}(v_1 \otimes \ldots \otimes v_n)) = (\sigma \circ \sigma')^{\otimes}(v_1 \otimes \ldots \otimes v_n) \, ,$$

This follows from the definition: $\sigma^{\otimes}(\sigma'^{\otimes}(v_1 \otimes \ldots \otimes v_n)) = \sigma^{\otimes}(v_{\sigma'^{-1}(1)} \otimes \ldots \otimes v_{\sigma'^{-1}(n)})$. Re-defining $u_i := v_{\sigma'^{-1}(i)}$ so that $u_{\sigma^{-1}(j)} := v_{\sigma'^{-1}(\sigma^{-1}(j))}$, one finds the identity $\sigma^{\otimes}(\sigma'^{\otimes}(v_1 \otimes \ldots \otimes v_n)) =$

$u_{\sigma^{-1}(1)} \otimes \ldots \otimes u_{\sigma^{-1}(n)} = v_{\sigma'^{-1}\circ\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma'^{-1}\circ\sigma^{-1}(n)} = v_{(\sigma\circ\sigma')^{-1}(1)} \otimes \ldots \otimes v_{(\sigma\circ\sigma')^{-1}(n)} = (\sigma \circ \sigma')^{\otimes}(v_1 \otimes \ldots \otimes v_n)$. In other words

$$\sigma^{\otimes}(\sigma'^{\otimes}(v_1 \otimes \ldots \otimes v_n)) = (\sigma \circ \sigma')^{\otimes}(v_1 \otimes \ldots \otimes v_n) .$$

In particular, that identity holds also for a canonical basis of elements $e_{i_1} \otimes \ldots \otimes e_{i_n}$

$$\sigma^{\otimes}(\sigma'^{\otimes}(e_{i_1} \otimes \ldots \otimes e_{i_n})) = (\sigma \circ \sigma')^{\otimes}(e_{i_1} \otimes \ldots \otimes e_{i_n}) .$$

By linearity such an identity will hold true for all arguments in $V^{n\otimes}$ and thus we have

$$\sigma^{\otimes}\sigma'^{\otimes} = (\sigma \circ \sigma')^{\otimes} .$$

The mappings $\sigma^{\otimes}$ are linear by constructions and are bijective because

$$\sigma^{\otimes}\sigma'^{\otimes} = (\sigma \circ \sigma')^{\otimes}$$

implies

$$\sigma^{\otimes}\sigma^{-1\otimes} = \sigma^{-1\otimes}\sigma^{\otimes} = e^{\otimes} = I .$$

The identity $e^{\otimes} = I$ can be proved by noticing that $e^{\otimes} - I$ is linear and vanishes when evaluated on any canonical base of $V^{n\otimes}$. We have shown that $\sigma^{\otimes} \in GL(V^{n\otimes})$ for all $\sigma \in \mathscr{P}_n$ and the mapping $\sigma \mapsto \sigma^{\otimes}$ is a homomorphism. This concludes the proof. $\Box$

Let us pass to consider the abstract index notation and give a representation of the action of $\sigma^{\otimes}$ within that picture.

Theorem 4.3. *If t is a tensor in $V^{n\otimes} \in \mathscr{A}_{\mathbb{K}}(V)$ with $n \geq 2$ and $\sigma \in \mathscr{P}_n$, then the components of t with respect to any canonical basis of $V^{n\otimes}$ satisfy*

$$(\sigma^{\otimes}t)^{i_1 \ldots i_n} = t^{i_{\sigma(1)} \ldots i_{\sigma(n)}} .$$

An analogous statement holds for the tensors $s \in V^{*n\otimes}$:

$$(\sigma^{\otimes}s)_{i_1 \ldots i_n} = s_{i_{\sigma(1)} \ldots i_{\sigma(n)}} .$$

Proof.

$$\sigma^{\otimes}t = \sigma^{\otimes}(t^{j_1 \ldots j_n} e_{j_1} \otimes \ldots \otimes e_{j_n}) = t^{j_1 \ldots j_n} e_{j_{\sigma^{-1}(1)}} \otimes \ldots \otimes e_{j_{\sigma^{-1}(n)}} .$$

Since $\sigma : I_n \to I_n$ is bijective, if we define $i_k := j_{\sigma^{-1}(k)}$, it holds $j_k = i_{\sigma(k)}$. Using this identity above we find

$$\sigma^{\otimes}t = t^{i_{\sigma(1)} \ldots i_{\sigma(n)}} e_{i_1} \otimes \ldots \otimes e_{i_n} .$$

That is nothing but the thesis. The final statement is an obvious consequence of the initial one just replacing the vector space V with the vector space V^* (the fact that V^* is a dual space is immaterial, the statement relies upon the only structure of vector space.)□

To conclude we introduce the concept of symmetric or anti-symmetric tensor.

Definition 4.6. **(Symmetric and antisymmetric tensors.)** Let V be a finite-dimensional vector space with field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Consider the space $V^{n\otimes} \in \mathscr{A}_{\mathbb{K}}(V)$ of tensors of order $(n, 0)$ with $n \geq 2$.
(a) $t \in V^{n\otimes}$ is said to be **symmetric** if

$$\sigma^{\otimes} t = t \,,$$

for all of $\sigma \in \mathscr{P}_n$, or equivalently, using the abstract index notation,

$$t^{j_1 \cdots j_n} = t^{j_{\sigma(1)} \cdots j_{\sigma(n)}} \,,$$

for all of $\sigma \in \mathscr{P}_n$.
(b) $t \in V^{n\otimes}$ is said to be **anti symmetric** if

$$\sigma^{\otimes} t = \epsilon_\sigma \, t \,,$$

for all of $\sigma \in \mathscr{P}_n$, or equivalently using the abstract index notation:

$$t^{j_1 \cdots j_n} = \epsilon_\sigma \, t^{j_{\sigma(1)} \cdots j_{\sigma(n)}} \,,$$

for all of $\sigma \in \mathscr{P}_n$. ♢

Comments 4.2.
(1) Concerning the definition in **(b)** notice that $\epsilon_\sigma = \epsilon_{\sigma^{-1}}$.
(2) Notice that the sets of symmetric and antisymmetric tensors in $V^{n\otimes}$ are, separately, closed by linear combinations. In other words each of them is a subspaces of $V^{n\otimes}$.

Examples 4.3.
1. Suppose $n = 2$, then a symmetric tensor $s \in V \otimes V$ satisfies $s^{ij} = s^{ji}$ and an antisymmetric tensor $a \in V \otimes V$ satisfies $a^{ij} = -a^{ji}$.
2. Suppose $n = 3$, then it is trivially shown that $\sigma \in \mathscr{P}_3$ has parity 1 if and only if σ is a **cyclic permutation**, i.e., $(\sigma(1), \sigma(2), \sigma(3)) = (1, 2, 3)$ or $(\sigma(1), \sigma(2), \sigma(3)) = (2, 3, 1)$ or $(\sigma(1), \sigma(2), \sigma(3)) = (3, 1, 2)$.
Now consider the vector space V with $dimV = 3$. It turns out that a tensor $e \in V \otimes V \otimes V$ is anti symmetric if and only if

$$e^{ijk} = 0 \,,$$

if (i, j, k) is *not* a permutation of $(1, 2, 3)$ and, otherwise,

$$e^{ijk} = \pm e^{123} \,,$$

where the sign $+$ takes place if the permutation $(\sigma(1), \sigma(2), \sigma(3)) = (i, j, k)$ is cyclic and the sign $-$ takes place otherwise. That relation between parity of a permutation and cyclicity does *not* hold true for $n > 3$.

Remarks.
(1) Consider a generic tensor space $S \in \mathscr{A}_{\mathbb{K}}(V)$ which contains $n \geq 2$ spaces V as factors. We may suppose for sake of simplicity $S = S_1 \otimes V^{n\otimes} \otimes S_2$ where $S_1 = U_1 \otimes \ldots \otimes U_k$, $S_2 = U_{k+1} \otimes \ldots \otimes U_m$ and $U_i = V$ or $U_i = V^*$. Anyway all what we are going to say holds true also if the considered n spaces V do not define a unique block $V^{n\otimes}$. We may define the action of $\sigma \in \mathscr{P}_n$ on the whole space S starting by a multi linear mapping

$$\hat{\sigma} : U_1 \times \ldots \times U_k \times V^{n\times} \times U_{k+1} \times \ldots \times U_m \to U_1 \otimes \ldots \otimes U_k \otimes V^{n\otimes} \otimes U_{k+1} \otimes \ldots \otimes U_m \ ,$$

such that reduces to the tensor-product mapping on $U_1 \times \ldots \times U_k$ and $U_{k+1} \times \ldots \times U_m$:

$$\sigma : (u_1, \ldots, u_k, v_1, \ldots v_n, u_{k+1}, \ldots, u_m) \mapsto u_1 \otimes \ldots \otimes u_k \otimes v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)} \otimes u_{k+1} \otimes \ldots \otimes u_m \, .$$

Using the universality theorem as above, we build up a representation of $\mathscr{P}_n$ on S, $\sigma \mapsto \sigma^{\otimes}$ which "acts on $V^{n\otimes}$ only". Using the abstract index notation the action of $\sigma^{\otimes}$ is well represented:

$$\sigma^{\otimes} : t^{A \ i_1 \ldots i_n \ B} \mapsto t^{A \ i_{\sigma(1)} \ldots i_{\sigma(n)} \ B} \, .$$

This allows one to define and study the symmetry of a tensor referring to a few indices singled out among the complete set of indices of the tensors. E.g., a tensor $t^{ij}{}_k{}^r$ may be symmetric or anti symmetric, for instance, with respect to the indices i and j or j and r or ijr.
(2) Notice that no discussion on the symmetry of indices of different kind (one covariant and the other contravariant) is possible.

Exercises 4.3.
1. Let t be a tensor in $V^{n\otimes}$ (or $V^{*n\otimes}$). Show that t is symmetric or anti symmetric if there is a canonical basis where the components have symmetric or anti symmetric indices, i.e., $t^{i_1 \ldots i_n} = t^{i_{\sigma(1)} \ldots i_{\sigma(n)}}$ or respectively $t^{i_1 \ldots i_n} = \epsilon_\sigma \, t^{i_{\sigma(1)} \ldots i_{\sigma(n)}}$ for all $\sigma \in \mathscr{P}_n$.
Note. The result implies that, to show that a tensor is symmetric or anti symmetric, it is sufficient to verify the symmetry or anti symmetry of its components within a *single* canonical basis.
2. Show that the sets of symmetric tensors of order $(n, 0)$ and $(0, n)$ are vector subspaces of $V^{n\otimes}$ and $V^{*n\otimes}$ respectively.
3. Show that the subspace of anti-symmetric tensors of order $(0, n)$ in $V^{*n\otimes}$ has dimension $\binom{dimV}{n}$ if $n \leq dimV$. What about $n > dimV$?
4. Consider a tensor $t^{i_1 \ldots i_n}$, show that the tensor is symmetric if and only if it is symmetric with respect to each arbitrary chosen pair of indices, i.e.

$$t^{\ldots i_k \ldots i_p \ldots} = t^{\ldots i_p \ldots i_k \ldots} \, ,$$

for all $p, k \in \{1, \dots n\}$, $p \neq k$.

5. Consider a tensor $t^{i_1 \cdots i_n}$, show that the tensor is anti symmetric if and only if it is anti symmetric with respect to each arbitrarily chosen pair of indices, i.e.

$$t^{\cdots i_k \cdots i_p \cdots} = -t^{\cdots i_p \cdots i_k \cdots} \, ,$$

for all $p, k \in \{1, \dots n\}$, $p \neq k$.

6. Show that $V \otimes V = A \oplus S$ where $\oplus$ denotes the direct sum and A and S are respectively the space of anti-symmetric and symmetric tensors in $V \otimes V$. Does such a direct decomposition hold if considering $V^{n\otimes}$ with $n > 2$?

4.4 Grassmann algebra, also known as Exterior algebra (and something about the space of symmetric tensors).

n-forms play a crucial role in several physical applications of tensor calculus. Let us review, very quickly, the most important features of n-forms.

Definition 4.7. **(Spaces $\Lambda^p(V)$ and $\Lambda^p(V^*)$.)** If V denotes a linear space with field $\mathbb{K}$ and finite dimension n, $\Lambda^p(V)$ and $\Lambda^p(V^*)$ indicates respectively the linear space of the *antisymmetric* $(p, 0)$-order tensors and *antisymmetric* $(0, p)$-order tensors. We assume the convention that $\Lambda^0(V) := \mathbb{K}$ and $\Lambda^0(V^*) := \mathbb{K}$ and furthermore, if $p > n$ is integer, $\Lambda^p(V) := \{0\}$, $\Lambda^p(V^*) := \{0\}$, $\{0\}$ being the trivial vector space made of the zero vector only. The elements of $\Lambda^p(V)$ are called **p-vectors**. The elements of $\Lambda^p(V^*)$ are called **p-forms**. $\diamondsuit$

In the following we will be mainly concerned with the spaces $\Lambda^p(V)$ and $\Lambda^p(V^*)$, however, for some applications to physics (especially quantum mechanics), it is useful to define the spaces $S^p(V)$ and $S^p(V^*)$, respectively, the linear space of the *symmetric* $(p, 0)$-order tensors and *symmetric* $(0, p)$-order tensors. As before, we assume $S^0(V) := \mathbb{K}$, $S^0(V^*) := \mathbb{K}$.

Remark. $\Lambda^p(V) = \{0\}$ if $p > n$ is a natural definition. Indeed in the considered case, if $s \in \Lambda^p(V)$, in every component $s^{i_1 \cdots i_n}$ there must be at least two indices, say i_1 and i_2, with the same value so that $s^{i_1 i_2 \cdots i_n} = s^{i_2 i_1 \cdots i_n}$. Theorem 4.3 implies: $s^{i_1 i_2 \cdots i_n} = -s^{i_2 i_1 \cdots i_n}$, and thus $s^{i_1 i_2 \cdots i_n} = s^{i_2 i_1 \cdots i_n} = 0$.

Before going on we remind some basic notions about *projectors*.
If U is a vector space with field $\mathbb{K} = \mathbb{R}$ of $\mathbb{C}$, a **projector** P **onto** the subspace $S \subset U$ is a linear operator $P : U \to U$ such that it is **idempotent**, i.e. $PP = P$ and $P(U) = S$.
We have the following well-known results concerning projectors.
(1) If P is a projector onto S, $Q := I - P$ is a projector onto another subspace $S' := Q(U)$. Moreover it holds: $U = S \oplus S'$, where $\oplus$ denotes the direct sum of S and S' (i.e. $S \oplus S'$ is the space generated by linear combianations of elements in S and elements in S' and it holds $S \cap S' = \{0\}$).

(2) Conversely, if S and S' are subspaces of U such that $S \oplus S' = U$ – and thus, for every $v \in U$ there are a pair of elements $v_S \in S$ and $v_{S'} \in S'$, uniquely determined by the decomposition $S \oplus S' = U$, such that $v = v_S + v_{S'}$ – the applications $P : U \ni v \mapsto v_S$ and $Q : U \ni v \mapsto v_{S'}$ are projectors onto S and S' respectively and $Q = I - P$.
(3) If (a) P is a projector onto S, (b) P' is a projector onto S' and (c) $PP' = P'P$, then PP' is a projector as well and it is a projector onto $S \cap S'$.
These results are valid also in the infinite dimensional case. However in that case, if U is a Banach space it is more convenient specializing the definition of projectors requiring their continuity too. With this restricted definition the statements above are valid anyway provided that the involved subspaces are closed (i.e. they are sub Banach spaces).

Proposition 4.1. *Consider the linear operators $\mathcal{A} : V^{\otimes p} \to V^{\otimes p}$ and $\mathcal{S} : V^{\otimes p} \to V^{\otimes p}$ where, respectively,*

$$\mathcal{A} := \frac{1}{p!} \sum_{\sigma \in \mathscr{P}_p} \epsilon_\sigma \sigma^\otimes \,, \tag{4.3}$$

and

$$\mathcal{S} := \frac{1}{p!} \sum_{\sigma \in \mathscr{P}_p} \sigma^\otimes \,. \tag{4.4}$$

$\mathcal{A}$ is a projector onto $\Lambda^p(V)$, i.e. $\mathcal{A}\mathcal{A} = \mathcal{A}$ and $\mathcal{A}\left(V^{\otimes p}\right) = \Lambda^p(V)$ and, similarly, $\mathcal{S}$ is a projector onto $S^p(V)$, i.e. $\mathcal{S}\mathcal{S} = \mathcal{S}$ and $\mathcal{S}\left(V^{\otimes p}\right) = S^p(V)$. Finally one has

$$\mathcal{A}\mathcal{S} = \mathcal{S}\mathcal{A} = 0 \tag{4.5}$$

where 0 is the zero-operator, which implies $\Lambda^p(V) \cap S^p(V) = \{0\}$.

Proof. Let us prove that $\mathcal{A}\mathcal{A} = \mathcal{A}$. (Actually it would be a trivial consequence of the two facts we prove immediately after this point, but this proof is interesting in its onw right because similar procedures will be used later in more complicated proofs.) Using theorem 4.2 and the fact that $\epsilon_\sigma \epsilon_\tau = \epsilon_{\sigma\circ\tau}$ for $\sigma, \tau \in \mathscr{P}_p$, one has

$$\mathcal{A}\mathcal{A} = \frac{1}{p!} \sum_{\sigma \in \mathscr{P}_p} \epsilon_\sigma \sigma^\otimes \frac{1}{p!} \sum_{\tau \in \mathscr{P}_p} \epsilon_\tau \tau^\otimes = \frac{1}{p!} \sum_{\sigma \in \mathscr{P}_p} \frac{1}{p!} \sum_{\tau \in \mathscr{P}_p} \epsilon_{\sigma\circ\tau} (\sigma\circ\tau)^\otimes = \frac{1}{p!} \sum_{\sigma \in \mathscr{P}_p} \frac{1}{p!} \sum_{\sigma\circ\tau \in \mathscr{P}_p} \epsilon_{\sigma\circ\tau} (\sigma\circ\tau)^\otimes .$$

In the last step, we have taken advantage of the fact that, for fixed σ, $\sigma\circ\tau$ ranges in the whole $\mathscr{P}_p$ if τ ranges in the whole $\mathscr{P}_p$. Therefore summing over τ is equivalent to summing over $\sigma\circ\tau$. In other words, if $\sigma' := \sigma\circ\tau$,

$$\mathcal{A}\mathcal{A} = \frac{1}{p!} \sum_{\sigma \in \mathscr{P}_p} \frac{1}{p!} \sum_{\sigma' \in \mathscr{P}_p} \epsilon_{\sigma'} \, \sigma'^\otimes = \frac{1}{p!} p! \, \mathcal{A} = \mathcal{A} \,.$$

To prove $\mathcal{A}\left(V^{\otimes p}\right) = \Lambda^p(V)$ it is sufficient to verify that, if $s \in V^{\otimes p}$, then $\sigma^\otimes \mathcal{A}s = \epsilon_\sigma \mathcal{A}s$ so that $\mathcal{A}(V^{\otimes p}) \subset \Lambda^p(V)$ and furthermore that $t \in \Lambda^p(V)$ entails $\mathcal{A}t = t$, so that $\mathcal{A}(V^{\otimes p}) \supset \Lambda^p(V)$.

Using, in particular, the fact that $\epsilon_\sigma \epsilon_{\tau\circ\sigma} = \epsilon_\sigma \epsilon_\tau \epsilon_\sigma = (\epsilon_\sigma)^2 \epsilon_\tau$, we have:

$$\sigma^{\otimes} \mathcal{A} s = \frac{1}{p!} \sum_{\tau \in \mathscr{P}_p} \epsilon_\tau \, \sigma^{\otimes}(\tau^{\otimes} s) = \frac{1}{p!} \sum_{\tau \in \mathscr{P}_p} \epsilon_\sigma \epsilon_{\tau\circ\sigma} (\sigma \circ \tau)^{\otimes} s = \epsilon_\sigma \frac{1}{p!} \sum_{\sigma\circ\tau \in \mathscr{P}_p} \epsilon_{\tau\circ\sigma} (\sigma \circ \tau)^{\otimes} s = \epsilon_\sigma \mathcal{A} s \,.$$

To conclude we prove that $t \in \Lambda^p(V)$ entails $\mathcal{A}t = t$.

$$\mathcal{A}t = \frac{1}{p!} \sum_{\sigma \in \mathscr{P}_p} \epsilon_\sigma \sigma^{\otimes} t = \frac{1}{p!} \sum_{\sigma \in \mathscr{P}_p} (\epsilon_\sigma)^2 t = \frac{1}{p!} \sum_{\sigma \in \mathscr{P}_p} t = \frac{p!}{p!} t = t \,.$$

We have used the fact that $\sigma^{\otimes} t = \epsilon_\sigma t$ since $t \in \Lambda^p(V)$.
The proof for $\mathcal{S}$ and the spaces $S^p(V)$ is essentially identical with the only difference that the coefficients ϵ_τ, ϵ_σ etc. do not take place anywhere in the proof. The last statement can be proved as follows. First notice that, if τ is a fixed transposition

$$-\sum_{\sigma \in \mathscr{P}_p} \epsilon_\sigma = \sum_{\sigma \in \mathscr{P}_p} \epsilon_\tau \epsilon_\sigma = \sum_{\sigma \in \mathscr{P}_p} \epsilon_{\tau\circ\sigma} = \sum_{\tau\circ\sigma \in \mathscr{P}_p} \epsilon_{\tau\circ\sigma} = \sum_{\sigma' \in \mathscr{P}_p} \epsilon_{\sigma'} = \sum_{\sigma \in \mathscr{P}_p} \epsilon_\sigma \,,$$

so that $\sum_{\sigma \in \mathscr{P}_p} \epsilon_\sigma = 0$. Next, notice that if $t \in V^{\otimes p}$, $\mathcal{S}t = s \in S^p(V)$ is symmetric and thus:

$$p! \mathcal{A}\mathcal{S} t = p! \mathcal{A} s = \sum_{\sigma \in \mathscr{P}_p} \epsilon_\sigma \sigma^{\otimes} s = \left(\sum_{\sigma \in \mathscr{P}_p} \epsilon_\sigma \right) s = 0$$

since $\sum_{\sigma \in \mathscr{P}_p} \epsilon_\sigma = 0$. If $t \in V^{\otimes p}$, $\mathscr{A} t = a \in \Lambda^p(V)$ is antisymmetric and thus:

$$p! \mathcal{S}\mathcal{A} t = p! \mathcal{S} a = \sum_{\sigma \in \mathscr{P}_p} \sigma^{\otimes} a = \left(\sum_{\sigma \in \mathscr{P}_p} \epsilon_\sigma \right) a = 0 \,,$$

again, since $\sum_{\sigma \in \mathscr{P}_p} \epsilon_\sigma = 0$. Using the property (3) of projector, mentioned above, it results that $\Lambda^p(V) \cap S^p(V) = \{0\}$. $\square$

Remark. In view of the last statement, one may wonder if the space $V^{\otimes}$ is the direct sum of $\Lambda^p(V)$ and $S^p(V)$. This is not the case if $p > 2$ and $\Lambda^p(V) \oplus S^p(V)$ is a proper subspace of $V^{\otimes p}$ for $p > 2$. The proof is left to the reader as an execise.

Examples 4.4. This example is very important in Quantum Mechanics. Consider a set of p identical quantum systems S, each represented on the Hilbert space $\mathcal{H}$. As we know, in principle, the states of the overall system are represented by unitary-norm vectors in the Hilbertian tensor product of p copies of the space $\mathcal{H}$, $\mathcal{H} \otimes_H \cdots \otimes_H \mathcal{H}$ (see definition 2.6). This is not the whole story, because the axioms of Quantum Mechanics (but the following statement turns out to be a theorem in relativistic quantum field theory called the *spin-statistics theorem*) assume that only some vectors are permitted to describe states in the case the system S is a quantum particle. There are only two categories of quantum particles: *bosons* and *fermions*. The first ones are those with integer spin (or helicity) and the others are those with semi-integer spin (or helicity). The states of a system of p identical bonsons are allowed to be represented by unitary-norm vectors in $S^p(\mathcal{H})$ only, whereas the states of a system of p identical bonsons are allowed to be represented by unitary-norm vectors in $\Lambda^p(\mathcal{H})$. All the possible transformations of states by means of operators, representing some physical action on the system, must respect that requirement.

4.4.1 The exterior product and Grassmann algebra.

We are in place to give the central definition, that of *exterior product* of k-vectors.

Definition 4.8. **(Exterior product.)** *If $s \in \Lambda^p(V)$ and $t \in \Lambda^q(V)$ the* **exterior product** *of s and t is the $(p+q)$-vector (the 0-vector when $p+q > dimV$):*

$$s \wedge t := \mathcal{A}(s \otimes t) .$$

Theorem 4.4. *Consider the exterior product $\wedge$ in the spaces $\Lambda^r(V)$ where V is a linear space with field $\mathbb{K}$ and finite dimension n. It fulfills the following properties for $s \in \Lambda^p(V)$, $t \in \Lambda^q(V)$, $w \in \Lambda^r(V)$ and $a, b \in \mathbb{K}$:*
(a) *$s \wedge (t \wedge w) = (s \wedge t) \wedge w = \mathcal{A}(s \otimes t \otimes w)$;*
(b) *$s \wedge t = (-1)^{pq} t \wedge s$, in particular $u \wedge v = -v \wedge u$ if $u, v \in V$;*
(c) *$s \wedge (at + bw) = a(s \wedge t) + b(s \wedge w)$.*
Furthermore, if $u_1, \ldots, u_m \in V$, these vectors are linearly independent if and only if

$$u_1 \wedge \cdots \wedge u_m \neq 0 ,$$

where the exterior product of several vectors is well-defined (without parentheses) in accordance with the associativity property (a).

Proof. (a) Let us prove that $s \wedge (t \wedge w) = \mathcal{A}(s \otimes t \otimes w)$. Exploiting a similar procedure one proves that $(s \wedge t) \wedge w = \mathcal{A}(s \otimes t \otimes w)$ and it concludes the proof. To prove that $s \wedge (t \wedge w) = \mathcal{A}(s \otimes t \otimes w)$, we notice that, if $s \in \Lambda^p(V)$, $t \in \Lambda^q(V)$, $w \in \Lambda^r(V)$ then:

$$s \wedge (t \wedge w) = \frac{1}{(p+q+r)!} \sum_{\sigma \in \mathscr{P}_{p+q+r}} \epsilon_\sigma \sigma^{\otimes} \left(\frac{1}{(q+r)!} \sum_{\tau \in \mathscr{P}_{q+r}} \epsilon_\tau \, s \otimes \tau^{\otimes}(t \otimes w) \right) .$$

That is

$$s \wedge (t \wedge w) = \frac{1}{(q+r)!} \sum_{\tau \in \mathscr{P}_{q+r}} \left(\frac{1}{(p+q+r)!} \sum_{\sigma \in \mathscr{P}_{p+q+r}} \epsilon_\tau \epsilon_\sigma \, \sigma^{\otimes}(s \otimes \tau^{\otimes}(t \otimes w)) \right) .$$

Now we can view $\mathscr{P}_{q+r}$ as the subgroup of $\mathscr{P}_{p+q+r}$ which leaves unchanged the first p objects. With this interpretation one has

$$s \wedge (t \wedge w) = \frac{1}{(q+r)!} \sum_{\tau \in \mathscr{P}_{q+r}} \left(\frac{1}{(p+q+r)!} \sum_{\sigma \in \mathscr{P}_{p+q+r}} \epsilon_\tau \epsilon_\sigma \sigma^{\otimes}(\tau^{\otimes}(s \otimes t \otimes w)) \right) .$$

Equivalently,

$$\begin{aligned} s \wedge (t \wedge w) &= \frac{1}{(q+r)!} \sum_{\tau \in \mathscr{P}_{q+r}} \left(\frac{1}{(p+q+r)!} \sum_{\sigma \in \mathscr{P}_{p+q+r}} \epsilon_{\sigma \circ \tau} \, (\sigma \circ \tau)^{\otimes}(s \otimes t \otimes w) \right) \\ &= \frac{1}{(q+r)!} \sum_{\tau \in \mathscr{P}_{q+r}} \left(\frac{1}{(p+q+r)!} \sum_{\sigma \circ \tau \in \mathscr{P}_{p+q+r}} \epsilon_{\sigma \circ \tau} \, (\sigma \circ \tau)^{\otimes}(s \otimes t \otimes w) \right) \\ &= \frac{1}{(q+r)!} \sum_{\tau \in \mathscr{P}_{q+r}} \left(\frac{1}{(p+q+r)!} \sum_{\sigma' \in \mathscr{P}_{p+q+r}} \epsilon_{\sigma'} \, \sigma'^{\otimes}(s \otimes t \otimes w) \right) \\ &= \frac{(q+r)!}{(q+r)!} \mathcal{A}(s \otimes t \otimes w) = \mathcal{A}(s \otimes t \otimes w) . \end{aligned}$$

(b) One may pass from $u_1 \otimes \cdots \otimes u_q \otimes v_1 \otimes \cdots \otimes v_p$ to $v_1 \otimes \cdots \otimes v_p \otimes u_1 \otimes \cdots \otimes u_q$ by means of the following procedure. First, using $p+q-1$ transpositions one achieves $u_2 \otimes \cdots \otimes u_q \otimes v_1 \otimes \cdots \otimes v_p \otimes u_1$. In this case, due to the definition of $\mathcal{A}$ one has:

$$\mathcal{A}(u_1 \otimes u_2 \otimes \cdots \otimes u_q \otimes v_1 \otimes \cdots \otimes v_p) = (-1)\mathcal{A}(u_2 \otimes u_1 \otimes \cdots \otimes u_q \otimes v_1 \otimes \cdots \otimes v_p) ,$$

so that, using $p+q-1$ iterated transpositions:

$$\mathcal{A}(u_1 \otimes \cdots \otimes u_q \otimes v_1 \otimes \cdots \otimes v_p) = (-1)^{p+q-1} \mathcal{A}(u_2 \otimes \cdots \otimes u_q \otimes v_1 \otimes \cdots \otimes v_p \otimes u_1) .$$

Using again the same procedure one gets also

$$\mathcal{A}(u_1 \otimes \cdots \otimes u_q \otimes v_1 \otimes \cdots \otimes v_p) = (-1)^{p+q-1}(-1)^{p+q-1} \mathcal{A}(u_3 \otimes \cdots \otimes u_q \otimes v_1 \otimes \cdots \otimes v_p \otimes u_1 \otimes u_2) .$$

By iteration of this prescription one finally gets:

$$\mathcal{A}(u_1 \otimes \cdots \otimes u_q \otimes v_1 \otimes \cdots \otimes v_p) = (-1)^{q(p+q-1)} \mathcal{A}(v_1 \otimes \cdots \otimes v_p \otimes u_1 \otimes \cdots \otimes u_q) .$$

That is

$$u_1 \wedge \cdots \wedge u_q \wedge v_1 \wedge \cdots \wedge v_p = (-1)^{q(q-1)}(-1)^{pq} v_1 \wedge \cdots \wedge v_p \wedge u_1 \wedge \cdots \wedge u_q .$$

Since $(-1)^{q(q-1)} = 1$, by linearity the obtained identity gives rise to the thesis.
(c) is a straightforward consequence of the definition of $\mathcal{A}$.

Let us come to the last statement. If $u_1, \dots u_m$ are linearly dependent, there must be one of them, say u_1, which can be represented as a linear combination of the others. In this case $u_1 \wedge \cdots \wedge u_m$ can be re-written as: $\sum_{i=2}^m c^i u_i \wedge \cdots \wedge u_m$. Since, whenever $c_i \neq 0$, u_i appears twice in $u_i \wedge \cdots \wedge u_m = \mathcal{A}(u_i \otimes \cdots \otimes u_m)$, this term vanishes (interchanging the two copies of u^i on a hand amounts to do nothing on the other hand it amounts to a transposition which produce a sign minus) and thus $u_1 \wedge \cdots \wedge u_m = 0$. If $u_1, \dots, u_m$ are linearly independent, one can add further vectors $u_{m+1}, \dots u_n$ in order to obtain a basis for V, which we denote by $\{e_i\}_{i=1,\dots m}$. In this case $u_1 \wedge \cdots \wedge u_n = \mathcal{A}(e_1 \otimes \cdots \otimes e_n) \neq 0$. Indeed, by direct inspection one finds that if $\{e^{*i}\}_{i=1,\dots,n}$ is the dual base

$$\langle e^{*1} \otimes \cdots \otimes e^{*n}, \mathcal{A}(e_1 \otimes \cdots \otimes e_n)\rangle = \frac{1}{n!}\langle e^{*1} \otimes \cdots \otimes e^{*n}, e_1 \otimes \cdots \otimes e_n\rangle + \text{vanishing terms} = \frac{1}{n!} \neq 0$$

and so $u_1 \wedge \cdots \wedge u_n \neq 0$. As a consequence the factor $u_1 \wedge \cdots \wedge u_m$ cannot vanish as well: If not it would imply, making use of (c) in the last passage:

$$u_1 \wedge \cdots \wedge u_n = (u_1 \wedge \cdots \wedge u_m) \wedge (u_{m+1} \wedge \cdots \wedge u_n) = \mathbf{0} \wedge (u_{m+1} \wedge \cdots \wedge u_n) = 0(\mathbf{0} \wedge (u_{m+1} \wedge \cdots \wedge u_n)) = \mathbf{0}.$$

□

Theorem 4.4 has an immediate and important corollary:

Corollary. *Each space* $\Lambda^p(V)$, *with* $0 \leq p \leq n = dim(V)$ *has dimension* $\binom{n}{p}$. *Moreover, if* $\{e_i\}_{i=1,\dots,n}$ *is a basis of* V, *the elements*

$$e_{i_1} \wedge \cdots \wedge e_{i_p}\,, \quad \textit{with } 1 \leq i_1 < i_2 < \cdots < i_p \leq n$$

form a basis of $\Lambda^p(V)$.

Proof. Define $I := \{1, 2, \dots, n\}$. Notice that, with the given definitions, $e_{i_1} \wedge \cdots \wedge e_{i_p} = \mathcal{A}(e_{i_1} \otimes \cdots \otimes e_{i_p})$.
Since $e_{i_1} \otimes \cdots \otimes e_{i_p}$ with $i_1, \dots, i_p \in I$ define a basis of $V^{n\otimes}$ and $\mathcal{A} : V^{n\otimes} \to \Lambda^p(V)$ is a projector, the elements $e_{i_1} \wedge \cdots \wedge e_{i_p} = \mathcal{A}(e_{i_1} \otimes \cdots \otimes e_{i_p})$ are generators of $\Lambda^p(V)$. Moreover $e_{i_1} \wedge \cdots \wedge e_{i_p} = \pm e_{j_1} \wedge \cdots \wedge e_{j_p}$ if $(i_1, \dots, i_p)$ and $(j_1, \dots, j_p)$ differ to each other for a permutation and $e_{i_1} \wedge \cdots \wedge e_{i_p} = 0$ if there are repeated indices. All that implies that

$$S := \{e_{i_1} \wedge \cdots \wedge e_{i_p}\,, \quad \text{with } 1 \leq i_1 < i_2 < \cdots < i_p \leq n\}$$

is still a set of generators of $\Lambda^p(V)$. Finally, the elements of S are linearly independent. Indeed, suppose that

$$c^{i_1 \cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p} = 0\,, \quad \text{with } 1 \leq i_1 < i_2 < \cdots < i_p \leq n\,.$$

If we apply the functional $e_{i_1} \wedge \cdots \wedge e_{i_p}$ to $(e^{*j_1}, \dots, e^{*j_n})$ with $1 \leq j_1 < j_2 < \cdots < j_p \leq n$ fixed, only the term $e_{j_1} \wedge \cdots \wedge e_{j_p}$ gives a non-vanishing contribution:

$$0 = c^{i_1 \cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}(e^{*j_1}, \dots, e^{*j_p}) = c^{i_1 \cdots i_p} \left\langle \mathcal{A}(e_{i_1} \otimes \cdots \otimes e_{i_p}), e^{*j_1} \otimes \cdots \otimes e^{*j_p} \right\rangle$$

$$= c^{i_1\cdots i_p}\frac{1}{p!}\delta^{i_1}_{j_1}\cdots\delta^{i_p}_{j_p} = c^{j_1\cdots j_p}\frac{1}{p!}\,.$$

Thus we conclude that $c^{j_1\cdots j_p} = 0$ for all $j_1, \ldots, j_p$ with $1 \leq j_1 < j_2 < \cdots < j_p \leq n$. Therefore the elements of S are linearly independent. As a result of combinatorial calculus those elements are exactly $n!/(p!(n-p)!) = \binom{n}{p}$. $\Box$

Comments 4.3.
(1) Let $dimV = n$ and fix a basis $\{e_i\}_{i=1,\ldots,n}$ in V. If $w \in \Lambda^p(V)$, it can be represented in two different ways:

$$w = w^{i_1\cdots i_p} e_{i_1} \otimes \cdots \otimes e_{i_p}\,,$$

or

$$w = \sum_{1\leq i_1<\cdots<i_p\leq n} \tilde{w}^{i_1\cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}\,.$$

The relation between the components with $1 \leq i_1 < \cdots < i_p \leq n$ is (the reader should prove it):

$$\tilde{w}^{i_1\cdots i_p} = p!w^{i_1\cdots i_p}\,. \tag{4.6}$$

(2) Every $u^{*1} \wedge \cdots \wedge u^{*p} \in \Lambda^p(V^*) \subset V^{*n\otimes}$ acts on $V^{n\otimes}$ and thus, in particular, it acts on every tensor $v_1 \wedge \cdots \wedge v_p \in \Lambda^p(V) \subset V^{n\otimes}$. Let us focus on that action.

$$\langle v_1 \wedge \cdots \wedge v_p, u^{*1} \wedge \cdots \wedge u^{*p}\rangle = \langle \mathcal{A}(v_1 \otimes \cdots \otimes v_p), \mathcal{A}(u^{*1} \otimes \cdots \otimes u^{*p})\rangle$$

$$= \sum_{\tau\in\mathscr{P}_p} \frac{\epsilon_\tau}{p!}\langle \mathcal{A}(v_1 \otimes \cdots \otimes v_p), u^{*\tau^{-1}(1)} \otimes \cdots \otimes u^{*\tau^{-1}(p)}\rangle$$

$$= \sum_{\tau\in\mathscr{P}_p} \frac{(\epsilon_\tau)^2}{p!}\langle \mathcal{A}(v_{\tau^{-1}(1)}\otimes\cdots\otimes v_{\tau^{-1}(p)}), u^{*\tau^{-1}(1)}\otimes\cdots\otimes u^{*\tau^{-1}(p)}\rangle = \frac{p!}{p!}\langle \mathcal{A}(v_1\otimes\cdots\otimes v_p), u^{*1}\otimes\cdots\otimes u^{*p}\rangle,$$

since, as one can prove by direct inspection:

$$\langle \mathcal{A}(v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(p)}), u^{*\tau^{-1}(1)} \otimes \cdots \otimes u^{*\tau^{-1}(p)}\rangle = \langle \mathcal{A}(v_1 \otimes \cdots \otimes v_p), u^{*1} \otimes \cdots \otimes u^{*p}\rangle\,.$$

To conclude, we have found that:

$$\langle v_1 \wedge \cdots \wedge v_p, u^{*1} \wedge \cdots \wedge u^{*p}\rangle = \frac{1}{p!}\sum_{\sigma\in\mathscr{P}_p} \epsilon_\sigma \langle v_{\sigma^{-1}(1)}, u^{*1}\rangle \cdots \langle v_{\sigma^{-1}(p)}, u^{*p}\rangle\,.$$

Since summing over $\sigma \in \mathscr{P}_p$ is completely equivalent to summing over $\sigma^{-1} \in \mathscr{P}_p$ and the right-hand side is nothing but $(1/p!)\det\left([\langle v_i, u^{*j}\rangle]_{i,j=1,\ldots,p}\right)$, we have eventually proved the following formula:

$$\langle v_1 \wedge \cdots \wedge v_p, u^{*1} \wedge \cdots \wedge u^{*p}\rangle = \frac{1}{p!}\det\left([\langle v_i, u^{*j}\rangle]_{i,j=1,\ldots,p}\right)\,. \tag{4.7}$$

To go on, we recall the reader that, if $V_1, \cdots, V_k$ are vector space on the field $\mathbb{K}$, $\bigoplus_{p=0}^k V_p$ denotes the **external direct sum** i.e. the vector space defined on $V_1 \times \cdots \times V_k$ with composition rule

$$\alpha(w_1, \ldots, w_k) + \beta(u_1, \ldots, u_k) := (\alpha w_1 + \beta u_1, \ldots, \alpha w_k + \beta u_k)$$

for all $\alpha, \beta \in \mathbb{K}$ and $(w_1, \ldots, w_k), (u_1, \ldots, u_k) \in V_1 \times \cdots \times V_k$.

Definition 4.9. **(Grassmann algebra)** Let V be a linear space on the field $\mathbb{K}$ with finite dimension n and let $\wedge$ be the exterior product defined above. The **Grassmann algebra** on V (also called **exterior algebra** on V) is the pair $(\Lambda(V), \wedge)$ where the former is the external direct sum

$$\Lambda(V) = \bigoplus_{p=0}^{n} \Lambda^p(V) .$$

$\diamondsuit$

Let us finally focus on the space of the forms $\Lambda^p(V^*)$ where, as usual, V has finite dimension $n \geq p$. We show that it is nothing but $(\Lambda^p(V))^*$. This is stated in the following proposition.

Proposition 4.2. *If V is a linear space with field $\mathbb{K}$ and finite dimension n, and $0 \leq p \leq n$ is integer, there is a natural isomorphism F_p between $\Lambda^p(V^*)$ and $(\Lambda^p(V))^*$ obtained by the restriction of any functional in $\Lambda^p(V^*)$ to the space $\Lambda^p(V)$. In other words,*

$$F_p : \Lambda^p(V^*) \ni f \mapsto f\restriction_{\Lambda^p(V)} \in (\Lambda^p(V))^* .$$

Therefore $\Lambda(V^*)$ is naturally isomorphic to $(\Lambda(V))^*$.

Proof. By construction the map F is linear. Let us prove that it is injective and surjective. Concerning the first issue, it is sufficient that $f\restriction_{\Lambda^p(V)} = 0$ entails $f = 0$. Let us prove it. Fix a basis in V and generate the associated canonical bases in every relevant tensor space. Therefore, using the corollary of theorem 4.4

$$f = c_{i_1 \ldots i_p} e^{*i_1} \wedge \cdots \wedge e^{*i_p} ,$$

where the sum is extended over the set $1 \leq i_1 < \cdots < i_p \leq n$. If $f\restriction_{\Lambda^p(V)} = 0$, one has, in particular:

$$f(e_{j_1} \wedge \cdots \wedge e_{j_p}) = 0 ,$$

so that

$$c_{i_1 \ldots i_p} \langle e_{j_1} \wedge \cdots \wedge e_{j_p}, e^{*i_1} \wedge \cdots \wedge e^{*i_p} \rangle = 0 .$$

Remembering that $1 \leq i_1 < \cdots < i_p \leq n$ and $1 \leq j_1 < \cdots < j_p \leq n$ and using the definition of $\wedge$ one concludes that $\langle e_{j_1} \wedge \cdots \wedge e_{j_p}, e^{*i_1} \wedge \cdots \wedge e^{*i_p} \rangle = 0$ unless $i_1 = j_1$, $i_2 = j_2$,..., $i_p = j_p$. In this case the formula (4.7) produces $\langle e_{i_1} \wedge \cdots \wedge e_{i_p}, e^{*i_1} \wedge \cdots \wedge e^{*i_p} \rangle = (p!)^{-1} \det I = 1/p!$. We conclude that

$$\langle e_{j_1} \wedge \cdots \wedge e_{j_p}, e^{*i_1} \wedge \cdots \wedge e^{*i_p} \rangle = \frac{1}{p!} \delta^{i_1}_{j_1} \cdots \delta^{i_p}_{j_p} , \tag{4.8}$$

and thus

$$c_{j_1 \ldots j_p} = 0 \, .$$

Since $j_1, \ldots, j_n$ are arbitrary, this implies that $f = c_{i_1 \ldots i_p} e^{*i_1} \wedge \cdots \wedge e^{*i_p} = 0$.
Let us pass to the proof of surjectivity. Actually surjectivity follows from injectivity since $\Lambda^p(V^*)$ and $(\Lambda^p(V))^*$ have the same dimension, however we give also a direct proof. By direct inspection one verifies straightforwardly that, if $h \in (\Lambda^p(V))^*$, the functional of $\Lambda^p(V^*)$ (where, as usual, the sum is extended over the set $1 \leq j_1 < \cdots < j_p \leq n$)

$$f_h := p! \, h(e_{j_1} \wedge \cdots \wedge e_{j_p}) e^{*j_1} \wedge \cdots \wedge e^{*j_p}$$

satisfies:

$$f_h \restriction_{\Lambda^p(V)} = h \, .$$

The natural isomorphism from $\Lambda(V^*)$ to $(\Lambda(V))^*$ it is boviously defined as

$$F : \Lambda(V^*) \ni (u_0^*, u_1^*, \ldots, u_n^*) \mapsto (F_0(u_0^*), F_1(u_1^*), \ldots, F_n(u_n^*)) \in (\Lambda(V))^* \, .$$

This concludes the proof. $\Box$

4.4.2 Interior product.

Consider the exterior algebras $\Lambda(V)$ and $\Lambda(V^*)$ with $dimV = n$ and field $\mathbb{K}$ and the pairing: $\langle \cdot | \cdot \rangle : \Lambda(V) \times \Lambda(V^*) \to \mathbb{K}$, defined by:

$$\langle w | z^* \rangle := \sum_{p=0}^{n} p! \langle w_p, z^{*p} \rangle \, , \quad \text{for all } w = (w_0, \ldots, w_n) \in \Lambda(V) \text{ and } z^* = (z^{*0}, \ldots, z^{*n}) \in \Lambda(V^*). \tag{4.9}$$

By direct inspection one proves that the pairing $\langle \cdot | \cdot \rangle$ is bilinear. Moreover one has the following proposition.

Proposition 4.3. *Consider the exterior algebras $\Lambda(V)$ and $\Lambda(V^*)$ with $dimV = n$ and field $\mathbb{K}$ and the pairing: $\langle \cdot | \cdot \rangle : \Lambda(V) \times \Lambda(V^*) \to \mathbb{K}$ defined in (4.9).*
If $T : \Lambda(V) \to \Lambda(V)$ is a linear operator, there is a unique linear operator $T^* : \Lambda(V^*) \to \Lambda(V^*)$, called the **dual operator** *of T with respect to $\langle \cdot | \cdot \rangle$, such that:*

$$\langle Tw | z^* \rangle = \langle w | T^* z^* \rangle \, , \quad \textit{for all } w \in \Lambda(V) \textit{ and } z^* \in \Lambda(V^*) \, .$$

Proof. Fix a basis $\{e_i\}_{i=1,\ldots,n}$ in V and consider its dual one $\{e^{*i}\}_{i=1,\ldots,n}$ in V^*. These bases induce bases $\{E_{p,i_p}\}_{i_p \in I_p}$ in each space $\Lambda^p(V)$ and, taking advantage from proposition 4.2, induce associated dual bases $\{E_p^{*i_p}\}_{i_p \in I_p}$ in $\Lambda^p(V^*)$. Each element E_{p,i_p} has the form $e_{j_1} \wedge \cdots \wedge e_{j_p}$ whereas each element $E_p^{*i_p}$ has the form $e^{*j_1} \wedge \cdots \wedge e^{*j_p}$. By (4.8):

$$\langle E_{p,i_p} | E_q^{*j_q} \rangle = \delta_{pq} \delta_{i_p}^{j_q} \, .$$

Obviously $\{E_{i_p,p}\}_{i_p\in I_p, p=0,\ldots,n}$ is a basis of $\Lambda(V)$ and $\{E_p^{*i_p}\}_{i_p\in I_p, p=0,\ldots,n}$ is a basis in $\Lambda(V^*)$. Consider the operator $T^* : \Lambda(V^*) \to \Lambda(V^*)$ defined by (where we write explicitly the symbols of sum)

$$T^* z^* := \sum_{p=0}^{n} \sum_{i_p \in I_p} \langle T E_{p,i_p} | z^* \rangle E_p^{*i_p} .$$

With the given definitions one finds that $\langle Tw|z^*\rangle = \langle w|T^* z^*\rangle$ is fulfilled by direct inspection. Now suppose that there is another operator T'^* satisfying the requirement above. As a consequence, one finds

$$\langle E_{p,i_p} | (T^* - T'^*) z^* \rangle = 0$$

for all p and i_p. Since $(T^* - T'^*)z^* = \sum_{p=0}^{n} \sum_{i_p\in I_p} c_{p,i_p} E_p^{*i_p}$, where $c_{p,i_p} = \langle E_{p,i_p}|(T^* - T'^*)z^*\rangle$, we conclude that every c_{p,i_p} vanishes and thus $(T^* - T'^*)z^* = 0$ per all $z^* \in \Lambda(V^*)$. In other words $T^* = T'^*$. $\Box$

In view of the proved proposition we can give the following definition when, for $u \in \Lambda^p(V)$, $u\wedge : \Lambda(V) \to \Lambda(V)$ is defined by the linear extension to the whole $\Lambda(V)$ of the analogous operator $u\wedge : \Lambda^q(V) \to \Lambda^{p+q}(V)$ and the spaces $\Lambda^r(V)$ with $r > n$ integer are identified with $\{0\}$, 0 being the zero vector of $\bigoplus_{p=0}^{n} \Lambda^p(V)$

Definition 4.10. **(Interior product.)** Consider the exterior algebras $\Lambda(V)$ and $\Lambda(V^*)$ with $dimV = n$ and field $\mathbb{K}$. If $u \in \Lambda^p(V)$, $u\lrcorner : \Lambda(V^*) \to \Lambda(V^*)$ denotes the adjoint $(u\wedge)^*$ of the operator $u\wedge : \Lambda(V) \to \Lambda(V)$ with respect to $\langle\cdot|\cdot\rangle$. If $z^* \in \Lambda(V^*)$, $u\lrcorner\, z^*$ is called **interior product** of u and z^*. $\diamondsuit$

The most important properties of the interior product are encompassed by the following theorem.

Theorem 4.5. *Consider the exterior algebras $\Lambda(V)$ and $\Lambda(V^*)$ with $dimV = n$ and field $\mathbb{K}$. The interior product enjoys the following properties.*
(a) *If $w \in \Lambda^p(V)$ and $z^* \in \Lambda^q(V^*)$, then $w\lrcorner\, z^* \in \Lambda^{q-p}(V^*)$ (so that $w\lrcorner\, z^* = 0$ if $p > q$).*
(b) *If $w_1 \in \Lambda^p(V), w_2 \in \Lambda^q(V)$ and $z^* \in \Lambda^r(V^*)$, then*

$$(w_1 \wedge w_2)\lrcorner\, z^* = w_2\lrcorner\,(w_1\lrcorner\, z^*) .$$

(c) *If $x \in V$ and $z^{*1}, \ldots, y^{*p} \in V^*$, then*

$$x\lrcorner\,(y^{*1} \wedge \cdots \wedge y^{*p}) = \sum_{k=1}^{p} (-1)^{k+1} \langle x, y^{*k}\rangle y^{*1} \wedge \ldots \wedge y^{*k-1} \wedge y^{*k+1} \wedge \ldots \wedge y^{*p} .$$

(d) *If $x \in V$, $x\lrcorner$ is an* **anti-derivative** *on $\Lambda(V^*)$ with respect to $\wedge$, i.e.:*

$$x\lrcorner\,(z^{*1} \wedge z^{*2}) = (x\lrcorner\, z^{*1}) \wedge z^{*2} + (-1)^p z^{*1} \wedge (x\lrcorner\, z^{*2}) , \quad \text{for } z^{*1} \in \Lambda^p(V^*) \text{ and } z^{*2} \in \Lambda^q(V^*) .$$

Proof. (a) is valid by definition. (b) can be proved as follows:

$$\langle y|(w_1 \wedge w_2)\lrcorner\, z^*\rangle = \langle w_1 \wedge w_2 \wedge y|z^*\rangle = \langle w_2 \wedge y|w_1\lrcorner\, z^*\rangle = \langle y|w_2\lrcorner\,(w_1\lrcorner\, z^*)\rangle\ .$$

The arbitrariness of y implies the validity of the thesis.
Let us pass to prove (c).

$$\langle x_2 \wedge \cdots \wedge x_p|x_1\lrcorner\,(y^{*1} \wedge \cdots \wedge y^{*p})\rangle = \langle x_1 \wedge \cdots \wedge x_p|(y^{*1} \wedge \cdots \wedge y^{*p})\rangle = \det[\langle x_i, y^{*j}\rangle]\ ,$$

where we have used (4.7). But

$$\det[\langle x_i, y^{*j}\rangle] = \sum_{k=1}^{p}(-1)^{k+1}\langle x_1, y^{*k}\rangle \det[\langle x_r, y^{*s}\rangle]\ , \quad \text{for } r \neq 1 \text{ and } s \neq k\ .$$

The right-hand side can be re-written:

$$\sum_{k=1}^{p}(-1)^{k+1}\langle x_1, y^{*k}\rangle\langle x_2 \wedge \cdots \wedge x_p|y^{*1} \wedge \ldots \wedge y^{*k-1} \wedge y^{*k+1} \wedge \ldots \wedge z^{*p}\rangle$$

The arbitrariness of $x_2 \wedge \cdots \wedge x_p$ implies the validity of the thesis with $x = x_1$.
The proof of (d) straightforward consequence of (c). $\Box$

Comments 4.4. The use of linearity, (b) and (c) of theorem 4.5 allow one to compute the action of $x\lrcorner$ explicitly. For instance:

$$(u^{ij}e_i \wedge e_j)\lrcorner\,(\omega_{pqr}e^{*p} \wedge e^{*q} \wedge e^{*r}) = u^{ij}\omega_{pqr}e_i\lrcorner\,(\delta^p_j e^{*q} \wedge e^{*r} - \delta^q_j e^{*p} \wedge e^{*r} + \delta^r_j e^{*p} \wedge e^{*q})$$

$$= u^{ij}\,(\omega_{ijr}e^{*r} - \omega_{iqr}e^{*q} - \omega_{jir}e^{*r} + \omega_{pij}e^{*p} + \omega_{jqi}e^{*q} - \omega_{pji}e^{*p}) = 6u^{ij}\omega_{ijr}e^{*r}\ .$$

Chapter 5

Scalar Products and Metric Tools.

This section concerns the introduction of the notion of scalar product and several applications on tensors.

5.1 Scalar products.

First of all we give the definition of a *pseudo scalar product* and *semi scalar products* which differ from the notion of *scalar product* for the positivity and the non-degenerateness requirement respectively. In fact, a pseudo scalar product is a generalization of the usual definition of scalar product which has many applications in mathematical physics, relativistic theories in particular. Semi scalar products are used in several applications of quantum field theory (for instance in the celebrated *GNS theorem*).

Definition 5.1. **(Pseudo Scalar Product.)** Let V be a vector space on the field either $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
(a) A **pseudo scalar product** is a mapping $(\,|\,) : V \times V \to \mathbb{K}$ which is:
(i) **bi linear**, i.e., for all $u \in V$ both $(u|\cdot) : v \mapsto (u|v)$ and $(\cdot|u) : v \mapsto (v|u)$ are linear functionals on V;
(ii) **symmetric**, i.e., $(u|v) = (v|u)$ for all $u, v \in V$;
(iii) **non-degenerate**, i.e., $(u|v) = 0$ for all $v \in V$ implies $u = 0$.
(b) If $\mathbb{K} = \mathbb{C}$, a **Hermitian pseudo scalar product** is a mapping $(\,|\,) : V \times V \to \mathbb{K}$ which is:
(i) **sesquilinear**, i.e., for all $u \in V$, $(u|\cdot)$ and $(\cdot|u)$ are a linear functional and an anti-linear functional on V respectively;
(ii) **Hermitian**, i.e., $(u|v) = \overline{(v|u)}$ for all $u, v \in V$;
(iii) **non-degenerate**. $\diamondsuit$

Definition 5.2. **(Semi Scalar Product.)** Let V be a vector space on the field either $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) If $\mathbb{K} = \mathbb{R}$, a **semi scalar product** is a mapping $(\,|\,) : V \times V \to \mathbb{R}$ which satisfies (ai),(aii) above and is

(iv) **semi-defined positive**, i.e., $(u|u) \geq 0$ for all $u \in V$.

(b) *If* $\mathbb{K} = \mathbb{C}$, *a* **Hermitian semi scalar product** *is a mapping* $(\,|\,) : V \times V \to \mathbb{K}$ *which satisfies* (bi),(bii) *above and is*

(iv) **semi-defined positive.** $\diamondsuit$

Finally we give the definition of scalar product.

Definition 5.3. **(Scalar Product.)** Let V be a vector space on the field $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{C}$) endowed with a pseudo scalar product (resp. Hermitian pseudo scalar product) $(\,|\,)$. $(\,|\,)$ is called **scalar product (resp. Hermitian scalar product)** if $(\,|\,)$ is also a semi scalar product, i.e., if it is **semi-defined positive**. $\diamondsuit$

Comments 5.1.

(1) Notice that all given definitions do not require that V is finite dimensional.

(2) If $\mathbb{K} = \mathbb{C}$ and $(\,|\,)$ is not Hermitian, in general, any requirement on positivity of $(u|u)$ does not make sense because $(u|u)$ may *not* be real. If instead Hermiticity holds, we have $(u|u) = \overline{(u|u)}$ which assures that $(u|u) \in \mathbb{R}$ and thus positivity may be investigated.

(3) Actually, a (Hermitian) scalar product is *positive defined*, i.e,

$$(u|u) > 0 \quad \text{if} \quad u \in V \setminus \{0\} ,$$

because of Cauchy-Schwarz' inequality

$$|(u|v)|^2 \leq (u|u)(v|v) ,$$

which we shall prove below for semi scalar products.

(4) A **semi norm** on a vector space V with field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, is a mapping $||\ || : V \to \mathbb{K}$ such that the following properties hold:

(i) (**semi positivity**) $||v|| \in \mathbb{R}$ and in particular $||v|| \geq 0$ for all $v \in V$;

(ii) (**homogeneity**) $||\alpha v|| = |\alpha|||v||$ for all $\alpha \in \mathbb{K}$ and $v \in V$;

(iii) (**triangular inequality**) $||u + v|| \leq ||u|| + ||v||$ for all $u, v \in V$.

A semi norm $||\ || : V \to \mathbb{K}$ is a **norm** if

(iv) $||v|| = 0$ implies $v = 0$.

Notice that for semi norms it holds: $||0|| = 0$ because of (ii) above. With the given definitions, it quite simple to show (the reader might try to give a proof) that if V with field $\mathbb{K} = \mathbb{R}$ ($\mathbb{C}$) is equipped by a (Hermitian) semi scalar product then $||v|| := \sqrt{(v|v)}$ for all $v \in V$ defines a semi norm. Furthermore if $(\,|\,)$ is a scalar product, then the associated semi norm is a norm.

(5) If a vector space V is equipped with a norm $||\ ||$ it becomes a *metric space* by defining the *distance* $d(u, v) := ||u - v||$ for all $u, v \in V$. A **Banach space** $(V, ||\ ||)$ [Rudin] is a vector space equipped with a norm such that the associates metric space is *complete*, i.e, all Cauchy's sequences converge. A **Hilbert space** [Rudin] is a Banach space with norm given by a (Hermitean if the field is $\mathbb{C}$) scalar product as said above. Hilbert spaces are the central mathematical objects used in Quantum Mechanics.

Exercises 5.1.

1. Show that if $(\,|\,)$ is a (Hermitian) semi scalar product on V with field $\mathbb{R}$ ($\mathbb{C}$) then the mapping on V, $v \mapsto ||v|| := \sqrt{(v|v)}$, satisfies $||u+v|| \leq ||u|| + ||v||$ as a consequence of Cauchy-Schwarz' inequality

$$|(u|v)|^2 \leq (u|u)(v|v)\,,$$

which holds true by all (Hermitian) semi scalar product.
(Hint. Compute $||u+v||^2 = (u+v|u+v)$ using bi linearity or sesquilinearity property of $(\,|\,)$, then use Cauchy-Schwarz' inequality.)

Theorem 5.1. **(Cauchy-Schwarz' inequality.)** *Let V be a vector space with field $\mathbb{R}$ ($\mathbb{C}$) equipped with a (Hermitian) semi scalar product $(\,|\,)$. Then, for all $u, v \in V$, Cauchy-Schwarz' inequality holds:*

$$|(u|v)|^2 \leq (u|u)(v|v)\,.$$

Proof. Consider the complex case with a Hermitian semi scalar product. Take $u, v \in V$. For all $z \in \mathbb{C}$ it must hold $(zu+v|zu+v) \geq 0$ by definition of Hermitean semi scalar product. Using sesquilinearity and Hermiticity :

$$0 \leq \overline{z}z(u|u) + (v|v) + \overline{z}(u|v) + z(v|u) = |z|^2(u|u) + (v|v) + \overline{z}(u|v) + z\overline{(u|v)}\,,$$

which can be re-written as

$$|z|^2(u|u) + (v|v) + 2Re\{\overline{z}(u|v)\} \geq 0\,. \tag{5.1}$$

Then we pass to the polar representation of z, $z = re^{i\,\alpha}$ with $r, \alpha \in \mathbb{R}$ arbitrarily and independently fixed. Decompose also $(u|v)$, $(u|v) = |(u|v)|e^{i\ arg(u|v)}$. Inserting above we get:

$$F(r,\alpha) := r^2(u|u) + 2r|(u|v)|Re[e^{i(arg(u|v)\,-\,\alpha)}] + (v|v) \geq 0\,,$$

for all $r \in \mathbb{R}$ when $\alpha \in R$ is fixed arbitrarily. Since the right-hand side above is a second-order polynomial in r, the inequality implies that, for all $\alpha \in \mathbb{R}$,

$$\left\{2|(u|v)|Re[e^{i(arg(u|v)\,-\,\alpha)}]\right\}^2 - 4(v|v)(u|u) \leq 0\,,$$

which is equivalent to

$$|(u|v)|^2\cos(arg(u|v)\ -\ \alpha) - (u|u)(v|v) \leq 0\,,$$

for all $\alpha \in \mathbb{R}$. Choosing $\alpha = arg(u|v)$, we get Cauchy-Schwarz' inequality:

$$|(u|v)|^2 \leq (u|u)(v|v) .$$

The real case can be treated similarly, replacing $z \in \mathbb{C}$ with $x \in \mathbb{R}$. By hypotheses it must hold $(xu + v|xu + v) \geq 0$, which implies the analog of inequality (5.1)

$$x^2(u|u) + 2x(u|v) + (v|v) \geq 0 .$$

That inequality must be valid for every $x \in \mathbb{R}$. Since $2x(u|v)$ could have arbitrary sign and value fixing x opportunely, if $(u|u) = 0$, $x^2(u|u) + 2x(u|v) + (v|v) \geq 0$ is possible for every x only if $(u|v) = 0$. In that case $(u|v)^2 \leq (u|u)(v|v)$ is trivially true. If $(u|u) \neq 0$, and thus $(u|u) > 0$, the inequality $x^2(u|u) + 2x(u|v) + (v|v) \geq 0$ is valid for every $x \in \mathbb{R}$ if and only if $[2(u|u)]^2 - 4(u|u)(v|v) \leq 0$. That is $(u|v)^2 \leq (u|u)(v|v)$. $\Box$.

Corollary. *A bilinear symmetric (resp. sesquilinear Hermitian) mapping* $(\,|\,) : V \times V \to \mathbb{R}$ *(resp.* $\mathbb{C}$*) is a scalar product (resp. Hermitian scalar product) if and only if it is* **positive defined**, *that is* $(u|u) > 0$ *for all* $u \in V \setminus \{\mathbf{0}\}$.

Proof. Assume that $(\,|\,)$ is a (Hermitian) scalar product. Hence $(u|u) \geq 0$ by definition and $(\,|\,)$ is non-degenerate. Moreover it holds $|(u|v)|^2 \leq (u|u)(v|v)$. As a consequence, if $(u|u) = 0$ then $(u|v) = 0$ for all $v \in V$ and thus $u = 0$ because $(\,|\,)$ is non-degenerate. We have proved that $(u|u) > 0$ if $u \neq 0$. That is, a scalar product is a positive-defined bilinear symmetric (resp. sesquilinear Hermitian) mapping $(\,|\,) : V \times V \to \mathbb{R}$ (resp. $\mathbb{C}$). Now assume that $(\,|\,)$ is positive-defined bilinear symmetric (resp. sesquilinear Hermitian). By definition it is a semi scalar product since positive definiteness implies positive semi-definiteness. Let us prove that $(\cdot|\cdot)$ is non-degenerate and this concludes the proof. If $(u|v) = 0$ for all $v \in V$ then, choosing $v = u$, the positive definiteness implies $u = \mathbf{0}$. $\Box$

5.2 Natural isomorphism between V and V^* and metric tensor.

Let us show that if V is a finite-dimensional vector space endowed with a pseudo scalar product, V is isomorphic to V^*. That isomorphism is *natural* because it is built up using the structure of vector space with scalar product only, specifying nothing further.

Theorem 5.2. **(Natural (anti)isomorphism between V and V^*.)** *Let V be a finite-dimensional vector space with field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*

(a) *If V is endowed with a pseudo scalar product $(\,|\,)$ (also if $\mathbb{K} = \mathbb{C}$),*

(i) *the mapping defined on V, $h : u \mapsto (u|\cdot) \in V^*$ (where $(u|\cdot)$ is the linear functional $(u|\cdot) : v \mapsto (u|v)$) is an isomorphism;*

(ii) $(h(u)|h(v))^* := (u|v)$ *defines a pseudo scalar product on V^*.*

(b) *If $\mathbb{K} = \mathbb{C}$ and V is endowed with a Hermitean pseudo scalar product $(\,|\,)$,*

(i) *the mapping defined on* V, $h : u \mapsto (u|\cdot) \in V^*$ *(where* $(u|\cdot)$ *is the linear functional,* $(u|\cdot) : v \mapsto (u|v)$*) is an* anti *isomorphism;*

(ii) $(h(u)|h(v))^* := \overline{(u|v)}$ *(*$= (v|u)$*) defines a Hermitian pseudo scalar product on* V^**.*

Proof. First consider (i) in the cases (a) and (b). It is obvious that $(u|\cdot) \in V^*$ in both cases. Moreover the linearity or anti-linearity of the mapping $u \mapsto (u|\cdot)$ is a trivial consequence of the definition of pseudo scalar product and Hermitian pseudo scalar product respectively.
Then remind the well-known theorem, $dim(Ker f) + dim f(V) = dim V$, which holds true for linear and anti-linear mappings from some finite-dimensional vector space V to some vector space V'. Since $dim V = dim V^*$, it is sufficient to show that $h : V \to V^*$ defined by $u \mapsto (u|\cdot)$ has trivial kernel, i.e., is injective: this also assures the surjectivity of the map. Therefore, we have to show that $(u|\cdot) = (u'|\cdot)$ implies $u = u'$. This is equivalent to show that $(u - u'|v) = 0$ for all $v \in V$ implies $u - u' = 0$. This is nothing but the non-degenerateness property, which holds by definition of (Hermitean) scalar product.
Statements (ii) cases are obvious in both by definition of (Hermitian) pseudo scalar products using the fact that h is a (anti) isomorphism. $\square$

Comments 5.2.
(1) Notice that, with the definitions given above it holds also $(u|v)^* = (h^{-1}u|h^{-1}v)$ and, for the Hermitian case, $(u|v)^* = \overline{(h^{-1}u|h^{-1}v)}$ for all $u, v \in V^*$. This means that h and h^{-1} *(anti)preserve* the scalar products.
(2) The theorem above holds also considering a Hilbert space and its topological dual space (i.e., the subspace of the dual space consisting of continuous linear functionals on the Hilbert space). That is the mathematical content of celebrated Riesz' representation theorem.
(3) From theorem 5.2 it follows immediately that, if $u, v \in V$:

$$(u|v) = \langle u, h(v)\rangle = (h(v)|h(u))^* ,$$

either if $(\,|\,)$ is Hermitian or not.

From now on we specialize to the pseudo-scalar-product case dropping the Hermitian case. Suppose $(|)$ is a pseudo scalar product on a finite-dimensional vector space V with field $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$. The mapping $(u, v) \mapsto (u|v)$ belongs to $\mathscr{L}(V, V) = V^* \otimes V^*$ and thus it is a tensor $\mathbf{g} \in V^* \otimes V^*$. Fixing a canonical basis in $V^* \otimes V^*$ induced by a basis $\{e_i\}_{i\in I} \subset V$, we can write:

$$\mathbf{g} = g_{ij} e^{*i} \otimes e^{*j} ,$$

where, by theorem 2.6,

$$g_{ij} = (e_i|e_j) .$$

Definition 5.4. **((Pseudo) Metric Tensor.)** A pseudo scalar product $(\,|\,) = \mathbf{g} \in V^* \otimes V^*$ on a finite-dimensional vector space V with field $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$ is called **pseudo-metric tensor**. If $\mathbb{K} = \mathbb{R}$, a pseudo-metric tensor is called **metric tensor** if it defines a scalar product.

$\diamondsuit$

Comments 5.3.
(1) By theorem 3.2, the isomorphism $h : V \to V^*$ is represented by a tensor of $V^* \otimes V^*$ which acts on elements of V by means of a product of tensors and a contraction. The introduction of the pseudo-metric tensor allows us to represent the isomorphism $h : V \to V^*$ by means of the abstract index notation determining the tensor representing h as we go to illustrate. Since $h : u \mapsto (u|\ ) \in V^*$ and $(u|v) = (e_i|e_j)u^i v^j = g_{ij}u^i v^j$ we trivially have:

$$(hu)_j = g_{ij}u^i \,.$$

$h(u)$ is obtained by the product $\mathbf{g} \otimes u$ followed by a contraction. Hence, in the sense of theorem 3.2, the linear map $h : V \to V^*$ is represented by the tensor g itself.
(2) Pseudo metric tensors are *symmetric* because of the symmetry of pseudo scalar products:

$$\mathbf{g}(u, v) = (u|v) = (v|u) = \mathbf{g}(v, u) \,.$$

(3) The symmetry requirement on the metric tensor is not necessary to define an isomorphism between V and V^*. In Weyl spinor theory [Wald, Streater-Wightman], the space V is a two-dimensional complex vector space whose elements are called Weyl *spinors*. V is equipped with a fixed *antisymmetric* tensor $\epsilon \in V^* \otimes V^*$ (the so-called *metric spinor*) defining a non-degenerate linear map from V to V^* by contraction. Using abstract index notation

$$V \ni \xi^A \mapsto \epsilon_{AB}\xi^B \in V^* \,.$$

Notice that, differently from the metric tensor case, now $\epsilon_{AB}\xi^B = -\epsilon_{BA}\xi^B$. Non-degenerateness of ϵ entails that the map $\xi^A \mapsto \epsilon_{AB}\xi^B$ is an isomorphism from V to V^*.

Components of pseudo-metric tensors with respect to canonical basis enjoy some simple but important properties which are listed below.

Theorem 5.3. **(Properties of the metric tensor.)** *Referring to def. 5.4, the components of any pseudo-metric tensor* $\mathbf{g}$*,* $g_{ij} := \mathbf{g}(e_i, e_j)$ *with respect to the canonical basis induced in* $V^* \otimes V^*$ *by any basis* $\{e_i\}_{i=1,\dots,n} \subset V$*, enjoy the following properties:*
(1) *define a symmetric matrix* $[g_{ij}]$*, i.e,*

$$g_{ij} = g_{ji} \,;$$

(2) $[g_{ij}]$ *is non singular, i.e., it satisfies:*

$$det[g_{ij}] \neq 0 \,;$$

(3) *if* $\mathbb{K} = \mathbb{R}$ *and* g *is a scalar product, the matrix* $[g_{ij}]$ *is positive defined.*

Proof. **(1)** It is obvious: $g_{ij} = (e_i|e_j) = (e_j|e_i) = g_{ji}$.
(2) Suppose $det[g_{ij}] = 0$ and define $n = dimV$. The linear mapping $\mathbb{K}^n \to \mathbb{K}^n$ determined by the matrix $g := [g_{ij}]$ has a non-trivial kernel. In other words, there are n reals u^j, $j = 1, \ldots, n$ defining a $\mathbb{K}^n$ vector $[u] := (u^1, \ldots, u^n)^t$ with $g[u] = 0$ and $[u] \neq 0$. In particular $[v]^t g[u] = 0$ for whatever choice of $[v] \in \mathbb{K}^n$. Defining $u := u^j e_j$, the obtained result implies that there is $u \in V \setminus \{0\}$ with $(u|v) = (v|u) = 0$ for all $v \in V$. This is impossible because $(|)$ is non degenerate by hypothesis.
(3) The statement, $(u|u) > 0$ if $u \in V \setminus \{\mathbf{0}\}$, reads, in the considered canonical basis $[u]^t g[u] > 0$ for $[u] \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. That is one of the equivalent definitions of a positive defined matrix g. $\Box$

The following theorem shows that a (pseudo) scalar product can be given by the assignment of a convenient tensor which satisfies some properties when represented in some canonical bases. The important point is that there is no need to check on these properties for *all* canonical bases, verification for a single canonical basis is sufficient.

Theorem 5.4. **(Assignment of a (pseudo) scalar product.)** *Let V be a finite-dimensional vector space with field $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$. Suppose $\mathbf{g} \in V^* \otimes V^*$ is a tensor such that there is a canonical basis of $V^* \otimes V^*$ where the components g_{ij} of $\mathbf{g}$ define a symmetric matrix $g := [g_{ij}]$ with non-vanishing determinant. Then $\mathbf{g}$ is a pseudo-metric tensor, i.e. a pseudo scalar product. Furthermore, if $\mathbb{K} = \mathbb{R}$ and $[g_{ij}]$ is positive defined, the pseudo scalar product is a scalar product.*

Proof. If $\mathbf{g}$ is represented by a symmetric matrix of components in a canonical basis then it holds in all remaining bases and the tensor is symmetric (see exercise 4.31). This implies that $(u|v) := \mathbf{g}(u, v)$ is a bi-linear symmetric functional. Suppose $(\,|\,)$ is degenerate, then there is $u \in V$ such that $u \neq \mathbf{0}$ and $(u|v) = 0$ for all $v \in V$. Using notations of the proof of the item (2) of theorem 5.3, we have in components of the considered canonical bases, $[u]^t g[v] = 0$ for all $[v] = (v^1, \ldots, v^n)^t \in \mathbb{K}^n$ where $n = dimV$. Choosing $[v] = g[u]$, it also holds $[u]^t gg[u] = 0$. Since $g = g^t$, this is equivalent to $(g[u])^t g[u] = 0$ which implies $g[u] = 0$. Since $[u] \neq \mathbf{0}$, g cannot be injective and $detg = 0$. This is not possible by hypotheses, thus $(\,|\,)$ is non-degenerate. We conclude that $(u|v) := g(u, v)$ define a pseudo scalar product.
Finally, if $\mathbb{K} = \mathbb{R}$ and g is also positive defined, $(\,|\,)$ itself turns out to be positive defined, i.e., it is a scalar product since $(u|u) = [u]^t g[u] > 0$ if $[u] \neq \mathbf{0}$ (which is equivalent to $u \neq \mathbf{0}$). $\Box$

5.2.1 Signature of pseudo-metric tensor, pseudo-orthonormal bases and pseudo-orthonormal groups.

Let us introduce the concept of *signature* of a pseudo-metric tensor in a vector space with field $\mathbb{R}$ by reminding Sylvester's theorem whose proof can be found in any linear algebra textbook. The definition is interlaced with the definition of *pseudo-orthonormal basis* and *pseudo-orthonormal group*.

Theorem 5.5. **(Sylvester's theorem.)** *Let G be a real symmetric $n \times n$ matrix.*
(a) *There is a non-singular (i.e., with non vanishing determinant) real $n \times n$ matrix D such that:*

$$DGD^t = diag(0, \dots, 0, -1, \dots, -1, +1, \dots, +1) ,$$

where the reals $0, -1, +1$ appear $v \geq 0$ times, $m \geq 0$ times and $p \geq 0$ times respectively with $v + m + p = n$.
(b) *the triple (v, m, p) does not depend on D. In other words, if, for some non-singular real $n \times n$ matrix matrix $E \neq D$, EGE^t is diagonal and the diagonal contains reals $0, -1, +1$ only (in whatever order), then $0, -1, +1$ respectively appear v times, m times and p times.*

If $\mathbf{g} \in V^* \otimes V^*$ is a pseudo-metric tensor on the finite-dimensional vector space v with field $\mathbb{R}$, the transformation rule of the components of g with respect to canonical bases (see theorem 3.1) induced by bases $\{e_i\}_{i \in I}$, $\{e'_j\}_{j \in I}$ of V are

$$g'_{pq} = B_p{}^i B_q{}^j g_{ij} .$$

Defining $g' := [g_{pq}]$, $g := [g_{ij}]$, $B := [B_h{}^k]$, they can be re-written as

$$g' = BgB^t .$$

We remind (see theorem 3.1) that the non-singular matrices B are defined by $B = A^{-1t}$, where $A = [A^i{}_j]$ and $e_m = A^l{}_m e'_l$. Notice that the specification of B is completely equivalent to the specification of A because $A = B^{-1t}$.
Hence, since g is real and symmetric by theorem 5.3, Sylvester's theorem implies that, starting from any basis $\{e_i\}_{i \in I} \subset V$ one can find another basis $\{e'_j\}_{j \in I}$, which induces a canonical basis in $V^* \otimes V^*$ where the pseudo-metric tensor is represented by a diagonal matrix. It is sufficient to pick out a transformation matrix B as specified in **(a)** of theorem 5.5. In particular, one can find B such that each element on the diagonal of g' turns out to be either -1 or $+1$ only. The value 0 is not allowed because it would imply that the matrix has vanishing determinant and this is not possible because of theorem 5.3. Moreover the pair (m, p), where (m, p) are defined in theorem 5.5, does not depend on the basis $\{e'_j\}_{j \in I}$. In other words, it is an *intrinsic* property of the pseudo-metric tensor: that is the *signature* of the pseudo-metric tensor.

Definition 5.5. **(Pseudo Orthonormal Bases and Signature).** Let $\mathbf{g} \in V^* \otimes V^*$ be a pseudo-metric tensor on the finite-dimensional vector space V with field $\mathbb{R}$.
(a) A basis $\{e_i\}_{i \in I} \subset V$ is called **pseudo orthonormal** with respect to $\mathbf{g}$ if the components of $\mathbf{g}$ with respect to the canonical basis induced in $V^* \otimes V^*$ form a diagonal matrix with eigenvalues in $\{-1, +1\}$. In other words, $\{e_i\}_{i \in I}$ is pseudo orthonormal if

$$(e_i, e_j) = \pm\delta_{ij} .$$

If the pseudo-metric tensor is a metric tensor the pseudo-orthonormal bases are called orthonormal bases.

(b) The pair (m,p), where m is the number of eigenvalues -1 and p is the number of eigenvalues $+1$ of a matrix representing the components of $\mathbf{g}$ in an orthonormal basis is called **signature** of $\mathbf{g}$.
(c) $\mathbf{g}$ and its signature are said **elliptic** or **Euclidean** or **Riemannian** if $m = 0$, **hyperbolic** if $m > 0$ and $p \neq 0$, **Lorentzian** or **normally hyperbolic** if $m = 1$ and $p \neq 0$.
(d) If $\mathbf{g}$ is hyperbolic, an orthonormal basis $\{e_i\}_{i\in I}$ is said to be **canonical** if the matrix of the components of $\mathbf{g}$ takes the form:

$$diag(-1, ..., -1, +1, ..., +1) .$$

♢

Exercises 5.2.

1. Show that a *pseudo-metric* tensor $\mathbf{g}$ is a *metric* tensor if and only if its signature is elliptic.

Remark. If $\{e_i\}_{i\in I}$ is an orthonormal basis with respect to a hyperbolic pseudo-metric tensor $\mathbf{g}$, one can trivially re-order the vectors of the basis giving rise to a canonical orthonormal basis.

Let us consider a pseudo-metric tensor $\mathbf{g}$ in V with field $\mathbb{R}$. Let (m,p) be the signature of $\mathbf{g}$ and let $\mathcal{N}_g$ be the class of all of the canonical pseudo-orthonormal bases in V with respect to $\mathbf{g}$. In the following we shall indicate by η the matrix $diag(-1, ..., -1, +1, ..., +1)$ which represents the components of $\mathbf{g}$ with respect to each basis of $\mathcal{N}_{\mathbf{g}}$. If A is a matrix corresponding to a change of basis in $\mathcal{N}_{\mathbf{g}}$, and $B := A^{-1t}$ is the associated matrix concerning change of basis in V^*, it has to hold

$$\eta = B\eta B^t .$$

Conversely, each real $n \times n$ matrix B which satisfies the identity above determines $A := B^{-1t}$ which represents a change of basis in $\mathcal{N}_{\mathbf{g}}$. That A is well defined because every B satisfying $\eta = B\eta B^t$ is *non singular* and, more precisely, $detB = \pm 1$. Indeed, taking the determinant of both sides in $\eta = B\eta B^t$, using the fact that $det\eta = (-1)^m$ and $detB = detB^t$, we conclude that $(detB)^2 = 1$ and thus $\det B = \pm 1$. As a consequence, since $\det A = (\det B)^{-1}$, one also has $\det A = \pm 1$.
The identity $\eta = B\eta B^t$ can be equivalently re-written in terms of the matrix A. Since $A = B^{-1t}$, one has $B^{-1} = A^t$. Thus, we applying A^t on the left and A on the right of $\eta = B\eta B^t$, we get the equivalent relation in terms of the matrix A:

$$\eta = A^t\eta A .$$

That equation completely determines the set $O(m,p) \subset GL(n,\mathbb{R})$ $(n = m + p)$ of all real non-singular $n \times n$ matrices which correspond to changes of bases in $\mathcal{N}_{\mathbf{g}}$.
It is simply proved that $O(m,p)$ is a subgroup of $GL(n,\mathbb{R})$. We can state the following definition.

Definition 5.6. If η is the matrix $diag(-1, ..., -1, +1, ..., +1)$ where -1 occurs m times and $+1$ occurs p times, the subgroup of $GL(m+p, \mathbb{R})$:

$$O(m,p) := \{A \in GL(m+p, \mathbb{R}) \mid \eta = A^t \eta A\}$$

is called the **pseudo orthogonal** group of order (m, p). $\diamondsuit$

Notice that, if $m = 0$, $O(0,p) = O(n)$ reduces to the usual orthogonal group of order n. $O(1,3)$ is the celebrated **Lorentz group** which is the central mathematical object in relativistic theories. We shall come back on those issues in the last two chapters, focusing on the Lorentz group in particular.

Exercises 5.3.

1. Show that if $A \in O(m,p)$ then A^{-1} exists and

$$A^{-1} = \eta A^t \eta \, .$$

2. Show that $O(m,p)$ is a group with respect to the usual multiplication of matrices.
Note. This implies that $O(m,p)$ is a subgroup of $GL(n, \mathbb{R})$ with $n = p + m$.
(Hint. You have to prove that, (1) the identity matrix I belongs to $O(m,p)$, (2) if A and A' belong to $O(m,p)$, AA' belongs to $O(m,p)$, (3) if A belongs to $O(m,p)$, then A^{-1} exists and belongs to $O(m,p)$.)

3. Show that $SO(m,p) := \{A \in O(m,p) \mid det A = 1\}$ is not the empty set and is a subgroup of $O(m,p)$. $SO(m,p)$ is called the *special pseudo orthogonal group* of order (m,p).

4. Consider the special Lorentz group $SO(1,3)$ and show that the set (called the *special orthochronous Lorentz group*)

$$SO(m,p)\Uparrow := \{A \in SO(1,3) \mid A^1{}_1 > 0\}$$

is a not empty subgroup.

5.2.2 Raising and lowering indices of tensors.

Consider a finite dimensional vector space V with field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ endowed with a pseudo-metric tensor $\mathbf{g}$. As we said above, there is a natural isomorphism $h : V \to V^*$ defined by $h : u \mapsto (u|\cdot) = \mathbf{g}(u, \cdot)$. This isomorphism may be extended to the whole tensor algebra $\mathscr{A}_{\mathbb{K}}(V)$ using the universality theorem and def. 4.4.
Indeed, consider a space $S \in \mathscr{A}_{\mathbb{K}}(V)$ of the form $A \otimes V \otimes B$, where A and B are tensor spaces of the form $U_1 \otimes \ldots \otimes U_k$, and $U_{k+1} \otimes \ldots \otimes U_m$ respectively, U_i being either V or V^*. We may define the operators:

$$h^{\otimes} := I_1 \otimes \ldots \otimes I_k \otimes h \otimes I_{k+1} \otimes \ldots \otimes I_m : A \otimes V \otimes B \to A \otimes V^* \otimes B \, ,$$

and

$$(h^{-1})^{\otimes} := I_1 \otimes \ldots \otimes I_k \otimes h^{-1} \otimes I_{k+1} \otimes \ldots \otimes I_m : A \otimes V^* \otimes B \to A \otimes V \otimes B \,,$$

where $I_j : U_j \to U_j$ is the identity operator. Using the remark after def. 4.4, one finds

$$(h^{-1})^{\otimes} h^{\otimes} = I_1 \otimes \ldots \otimes I_k \otimes (h^{-1}h) \otimes I_{k+1} \otimes \ldots \otimes I_m = Id_{A\otimes V\otimes B} \,,$$

and

$$h^{\otimes}(h^{-1})^{\otimes} = I_1 \otimes \ldots \otimes I_k \otimes (hh^{-1}) \otimes I_{k+1} \otimes \ldots \otimes I_m = Id_{A\otimes V^*\otimes B} \,.$$

Therefore $h^{\otimes}$ is an isomorphism with inverse $(h^{-1})^{\otimes}$.

The action of $h^{\otimes}$ and $(h^{-1})^{\otimes}$ is that of **lowering** and **raising indices** respectively. In fact, in abstract index notation, one has:

$$h^{\otimes} : t^{AiB} \mapsto t^{A}{}_{j}{}^{B} := t^{AiB} g_{ij} \,,$$

and

$$(h^{-1})^{\otimes} : u^{A}{}_{i}{}^{B} \mapsto u^{AjB} := t^{A}{}_{i}{}^{B} \tilde{g}^{ij} \,.$$

Above g_{ij} represents the pseudo-metric tensor as specified in the remark 1 after def. 5.4. What about the tensor $\tilde{g} \in V \otimes V$ representing h^{-1} via theorem 3.2?

Theorem 5.6. *Let $h : V \to V^*$ be the isomorphism determined by a pseudo scalar product, i.e. a pseudo-metric tensor* $\mathbf{g}$ *on the finite-dimensional vector space V with field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*
(a) *The inverse mapping $h^{-1} : V^* \to V$ is represented via theorem 3.2 by a* **symmetric** *tensor $\tilde{\mathbf{g}} \in V \otimes V$ such that, if $\{e_i\}_{i\in I}$ is a basis of V, $\tilde{g}^{rs} := \tilde{g}(e^{*r}, e^{*s})$ and $g_{ij} := \mathbf{g}(e_i, e_j)$, then the matrix $[\tilde{g}^{ij}]$ is the* inverse matrix *of $[g_{ij}]$.*
(b) *The tensor $\tilde{\mathbf{g}}$ coincides with the pseudo-metric tensor* **with both indices raised**.

Proof. (a) By theorem 3.2, h^{-1} determines a tensor $\tilde{\mathbf{g}} \in V \otimes V$ with $h^{-1}(u^*) = \tilde{\mathbf{g}}(u^*, \cdot)$. In components $(h^{-1}u^*)^i = u^*_k \tilde{g}^{ki}$. On the other hand it must be

$$h(h^{-1}u^*) = u^*$$

or,

$$u^*_k \tilde{g}^{ki} g_{ir} = u^*_r \,,$$

for all $u^* \in V^*$. This is can be re-written

$$[u^*]^t(\tilde{g}g - I) = 0 \,,$$

for all $\mathbb{K}^n$ vectors $[u^*] = (u^*_1, \ldots, u^*_n)$. Then the matrix $(\tilde{g}g - I)^t$ is the null matrix. This implies that

$$\tilde{g}g = I \,,$$

which is the thesis. $\tilde{g}$ is symmetric because is the inverse of a symmetric matrix and thus also the tensor $\tilde{\mathbf{g}}$ is symmetric.
(b) Let g^{ij} be the pseudo-metric tensor with both indices raised, i.e.,

$$g^{ij} := g_{rk}\ \tilde{g}^{kj}\tilde{g}^{ri} .$$

By **(a)**, the right-hand side is equal to:

$$\delta_r^j \tilde{g}^{ri} = \tilde{g}^{ji} = \tilde{g}^{ij} .$$

That is the thesis. □

Remark. Another result which arises from the proof of the second part of the theorem is that

$$g_i^j = \delta_i^j .$$

Comments 5.4.
(1) When a vector space is endowed with a pseudo scalar product, tensors can be viewed as abstract objects which may be represented either as covariant or contravariant concrete tensors using the procedure of raising and lowering indices. For instance, a tensor t^{ij} of $V \otimes V$ may be viewed as a covariant tensor when "represented" in its covariant form $t_{pq} := g_{pi}\, g_{qj}\, t^{ij}$. Also, it can be viewed as a mixed tensor $t_p{}^j := g_{pi}\, t^{ij}$ or $t^i{}_q := g_{qj}\, t^{ij}$.
(2) Now consider a finite-dimensional vector space on $\mathbb{R}$, V, endowed with a metric tensor $\mathbf{g}$, i.e., with *elliptic signature*. In orthonormal bases the contravariant and covariant components numerically coincides because $g_{ij} = \delta_{ij} = g^{ij}$. This is the reason because, using the usual scalar product of vector spaces isomorphic to $\mathbb{R}^n$ and working in orthonormal bases, the difference between covariant and contravariant vectors does not arise.
Conversely, in relativistic theories where a Lorentzian scalar product is necessary, the difference between covariant and contravariant vectors turns out to be evident also in orthonormal bases, since the diagonal matrix $[g_{ij}]$ takes an eigenvalue -1.

Chapter 6

Pseudo tensors, Ricci's pseudotensor and tensor densities.

This section is devoted to introduce very important tools either in theoretical/mathematical physics and in pure mathematics: pseudo tensors and tensor densities.

6.1 Orientation and pseudo tensors.

The first example of "pseudo" object we go to discuss is a *orientation* of a *real* vector space.

Consider a finite-dimensional vector space V with field $\mathbb{R}$. In the following $\mathcal{B}$ indicates the set of all the vector bases of V. Consider two bases $\{e_i\}_{i\in I}$ and $\{e'_j\}_{j\in I}$ in $\mathcal{B}$. Concerning the determinant of the transformation matrix $A := [A^r{}_s]$, with $e_i = A^j{}_i e'_j$, there are two possibilities only: $detA > 0$ or $detA < 0$. It is a trivial task to show that the relation in $\mathcal{B}$:

$$\{e_i\}_{i\in I} \sim \{e'_j\}_{j\in I} \quad \text{iff} \quad detA > 0$$

where A indicates the transformation matrix as above, is an *equivalence relation.* Since there are the only two possibilities above, the partition of $\mathcal{B}$ induced by $\sim$ is made of two *equivalence classes* $\mathcal{B}_1$ and $\mathcal{B}_2$. Hence if a basis belongs to $\mathcal{B}_1$ or $\mathcal{B}_2$ any other basis belongs to the same set if and only if the transformation matrix has positive determinant.

Definition 6.1. **(Orientation of a vector space.)** Consider a finite-dimensional vector space V with field $\mathbb{R}$, an **orientation** of V is a bijective mapping $\mathcal{O} : \{\mathcal{B}_1, \mathcal{B}_2\} \to \{-1, +1\}$. If V has an orientation $\mathcal{O}$, is said to be **oriented** and a basis $\{e_i\}_{i\in I} \in \mathcal{B}_k$ is said to be **positive oriented** if $\mathcal{O}(\mathcal{B}_k) = +1$ or **negative oriented** if $\mathcal{O}(\mathcal{B}_k) = -1$. $\diamondsuit$

Comment. The usual physical vector space can be oriented "by hand" using the natural basis given by our own right hand. When we use the right hand to give an orientation we determine

$\mathcal{O}^{-1}(+1)$ by the exhibition of a basis contained therein.

The given definition can be, in some sense, generalized with the introduction of the concept of *pseudo tensor*.

Definition 6.2. **(Pseudotensors.)** Let V be a finite-dimensional vector space with field $\mathbb{R}$. Let S be a tensor space of $\mathscr{A}_{\mathbb{R}}(V)$. A **pseudo tensor of** S is a bijective mapping $t_s : \{\mathcal{B}_1, \mathcal{B}_2\} \to \{s, -s\}$, where $s \in S$. Moreover:
(a) the various tensorial properties enjoyed by both s and $-s$ are attributed to t_s. (So, for instance, if s, and thus $-s$, is symmetric, t_s is said to be symmetric);
(b) if $\{e_i\}_{i\in I} \in \mathcal{B}_i$, the **components of** t_s with respect to the canonical bases induced by that basis are the components of $t_s(\mathcal{B}_i)$. $\diamondsuit$

Remarks.
(1) The given definition encompasses the definition of *pseudo scalar*.
(2) It is obvious that the assignment of a pseudo tensor t_s of, for instance, $V^{n\otimes} \otimes V^{*m\otimes}$, is equivalent to the assignment of components

$$t^{i_1 \ldots i_n}{}_{j_1 \ldots j_m}$$

for each canonical basis

$$\{e_{i_1} \otimes \ldots e_{i_n} \otimes e^{*j_1} \otimes \cdots \otimes e^{*j_m}\}_{i_1,\ldots,i_n,j_1,\ldots,j_m \in I}$$

such that the transformation rules passing to the basis

$$\{e'_{r_1} \otimes \ldots e'_{r_n} \otimes e'^{*l_1} \otimes \cdots \otimes e'^{*l_m}\}_{r_1,\ldots,r_n,l_1,\ldots,l_m \in I} \ ,$$

are given by:

$$t'^{k_1\ldots k_n}{}_{h_1\ldots h_m} = \frac{detA}{|detA|} \ A^{k_1}{}_{i_1} \cdots A^{k_n}{}_{i_n} \, B_{h_1}{}^{j_1} \cdots B_{h_m}{}^{j_m} \ t^{i_1\ldots i_n}{}_{j_1\ldots j_n} \ ,$$

where $e_l = A^m{}_l e'_m$ and $B = A^{-1t}$ with $B := [B_k{}^j]$ and $A := [A^p{}_q]$.
In fact, $t^{i_1\ldots i_n}{}_{j_1\ldots j_m} = s^{i_1\ldots i_n}{}_{j_1\ldots j_m}$ if the considered base is in $t_s^{-1}(+1)$ or $t^{i_1\ldots i_n}{}_{j_1\ldots j_m} = (-s)^{i_1\ldots i_n}{}_{j_1\ldots j_m}$ if the considered base is in $t_s^{-1}(-1)$.

Examples 6.1.
1. Consider the **magnetic field** $B = B^i e_i$ where e_1, e_2, e_3 is a right-hand orthonormal basis of the space V_3 of the vectors with origin in a point of the physical space E_3. Actually, as every physicist knows, changing basis, the components of B changes as usual only if the new basis is a right-hand basis, otherwise a sign $-$ appears in front of each component. That is a physical requirement due to the Lorentz law. This means that the magnetic field has to be represented in terms of **pseudo vectors**.

6.2 Ricci's pseudo tensor.

A particular pseudo tensor is Ricci's one which is very important in physical applications. The definition of this pseudo tensor requires a preliminary discussion.
Consider a finite-dimensional vector space V with a pseudo-metric tensor $\mathbf{g}$. We know that, changing basis $\{e_i\}_{i\in I} \to \{e'_j\}_{j\in I}$, the components of the pseudo-metric tensor referred to the corresponding canonical bases, transform as:

$$g' = BgB^t ,$$

where $g = [g_{ij}]$, $g' = [g'_{pq}]$ and $B = A^{-1t}$, $A := [A^p{}_q]$, $e_l = A^m{}_l e'_m$. This implies that

$$detg' = (detB)^2 \ detg , \tag{6.1}$$

which is equivalent to

$$\sqrt{|detg'|} = |detA|^{-1} \ \sqrt{|detg|} \tag{6.2}$$

Now fix a basis $\{e_i\}_{i\in I}$ in V and consider the canonical basis induced in $V^{*n\otimes}$ where $n = dimV$, $\{e^{*i_1} \otimes \ldots \otimes e^{*i_n}\}_{i_1,\ldots,i_n\in I}$. Then consider components $\eta_{i_1\ldots i_n}$ referred to the considered basis, given by:

$$\eta_{i_1\ldots i_n} = \epsilon_{\sigma_{i_1\ldots i_n}} ,$$

if $(i_1, \ldots, i_n)$ is a permutated string of $(1, 2, \ldots, n)$, or

$$\eta_{i_1\ldots i_n} = 0 ,$$

otherwise; where $\epsilon_{\sigma_{i_1\ldots i_n}}$ is the *parity* of the *permutation* $\sigma_{i_1\ldots i_n} \in \mathscr{P}_n$ defined by:

$$(\sigma(1), \ldots, \sigma(n)) = (i_1, \ldots, i_n) .$$

In other words, in the linguage of the forms of section 4.4, $\eta_{i_1\ldots i_n}$ are the components in the basis $\{e^{*i_1} \otimes \ldots \otimes e^{*i_n}\}_{i_1,\ldots,i_n\in I}$ of the form $\eta \in \Lambda(V^*)$:

$$\eta = n!\, e^{*1} \wedge \cdots \wedge e^{*n} , \tag{6.3}$$

Finally define the components:

$$\varepsilon_{i_1\ldots i_n} := \sqrt{|detg|} \ \eta_{i_1\ldots i_n} .$$

We want to show that, if we define analogous components in each canonical basis of $V^{*n\otimes}$, an anti-symmetric $(0, n)$-order pseudo tensor turns out to be defined by the whole assignment of components.
Taking the above remark 2 into account, it is sufficient to show that, under a change of basis,

$$\sqrt{|detg'|} \ \eta_{i_1\ldots i_n} = \frac{detA}{|detA|} \ B_{i_1}^{\ j_1} \cdots B_{i_n}^{\ j_n} \ \sqrt{|detg|} \ \eta_{j_1\ldots j_n} . \tag{6.4}$$

We start by noticing that:

$$B_{i_1}^{\ j_1} \cdots B_{i_n}^{\ j_n} \ \eta_{j_1 \cdots j_n} = \sum_{(j_1,\ldots,j_n)} B_{i_1}^{\ j_1} \cdots B_{i_n}^{\ j_n} \epsilon_{\sigma_{j_1 \cdots j_n}} ,$$

where $(j_1, \ldots, j_n)$ ranges over the set of all the permutated strings of $(1, 2, \ldots, n)$. We consider the various cases separately.

(1) Suppose that $i_p = i_q$ with $p \neq q$. To fix the situation, assume $i_1 = i_2$. Therefore

$$\sum_{(j_1,\ldots,j_n)} B_{i_1}^{\ j_1} B_{i_2}^{\ j_2} \cdots B_{i_n}^{\ j_n} \epsilon_{\sigma_{j_1 j_2 \cdots j_n}} = \sum_{(j_1,\ldots,j_n)} B_{i_2}^{\ j_1} B_{i_1}^{\ j_2} \cdots B_{i_n}^{\ j_n} \epsilon_{\sigma_{j_1 j_2 \cdots j_n}} .$$

The right-hand side can be re-written

$$\sum_{(j_1,\ldots,j_n)} B_{i_1}^{\ j_2} B_{i_2}^{\ j_1} \cdots B_{i_n}^{\ j_n} \epsilon_{\sigma_{j_1 j_2 \cdots j_n}} ,$$

that is, interchanging the names of j_1 and j_2,

$$\sum_{(j_1,\ldots,j_n)} B_{i_1}^{\ j_2} B_{i_2}^{\ j_1} \cdots B_{i_n}^{\ j_n} \epsilon_{\sigma_{j_2 j_1 \cdots j_n}} ,$$

But

$$\epsilon_{\sigma_{j_2 j_1 \cdots j_n}} = -\epsilon_{\sigma_{j_1 j_2 \cdots j_n}} ,$$

so that we finally get

$$\sum_{(j_1,\ldots,j_n)} B_{i_1}^{\ j_1} B_{i_2}^{\ j_2} \cdots B_{i_n}^{\ j_n} \epsilon_{\sigma_{j_1 j_2 \cdots j_n}} = - \sum_{(j_1,\ldots,j_n)} B_{i_1}^{\ j_1} B_{i_2}^{\ j_2} \cdots B_{i_n}^{\ j_n} \epsilon_{\sigma_{j_1 j_2 \cdots j_n}} .$$

In other words, *if* $i_1 = i_2$,

$$B_{i_1}^{\ j_1} \cdots B_{i_n}^{\ j_n} \ \sqrt{|detg|} \ \eta_{j_1 \cdots j_n} = 0$$

and consequently (6.4) holds:

$$\sqrt{|detg'|} \ \eta_{i_1 \ldots i_n} = \frac{detA}{|detA|} \ B_{i_1}^{\ j_1} \cdots B_{i_n}^{\ j_n} \ \sqrt{|detg|} \ \eta_{j_1 \ldots j_n} .$$

because the left-hand side vanishes too. The remaining subcases of $i_p = i_q$ with $p \neq q$, can be treated analogously.

(2) Next we pass to consider the case of $i_k = k$ for $k = 1, 2, \ldots, n$. In that case (6.4), that we need to prove, reduces to

$$\sqrt{|detg'|} = \sqrt{|detg|} \frac{detA}{|detA|} \ B_1^{\ j_1} \cdots B_n^{\ j_n} \quad \eta_{j_1 \cdots j_n} ,$$

that is

$$\frac{1}{detA} = B_1^{\ j_1} \cdots B_n^{\ j_n} \quad \eta_{j_1 \ldots j_n} , \tag{6.5}$$

where we have used (6.2). (6.5) can equivalently be re-written

$$detB = B_1^{\ j_1} \cdots B_n^{\ j_n} \quad \eta_{j_1 \ldots j_n} \ ,$$

which is trivially true by the properties of the determinant.
(3) To conclude, it remains to consider the case of a permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$. In that case, taking (6.2) into account, the identity (6.4) which has to be proved, reduces to

$$detB \epsilon_{\sigma_{i_1 \ldots i_n}} = B_{i_1}^{\ j_1} \cdots B_{i_n}^{\ j_n} \quad \eta_{j_1 \ldots j_n} \ . \tag{6.6}$$

Let us prove (6.6). Suppose for instance that $i_1 = 2$ and $i_2 = 1$ while $i_k = k$ if $k > 2$. In that case (6.6) holds true because the left-hand side is noting but $-detB$ and the right-hand side can be re-written

$$\sum_{(j_1, \ldots, j_n)} B_2^{\ j_1} B_1^{\ j_2} \cdots B_n^{\ j_n} \epsilon_{\sigma_{j_1 j_2 \cdots j_n}} = \sum_{(j_1, \ldots, j_n)} B_1^{\ j_2} B_2^{\ j_1} \cdots B_n^{\ j_n} \epsilon_{\sigma_{j_1 j_2 \cdots j_n}} \ ,$$

that is, interchanging the names of j_1 and j_2,

$$\sum_{(j_1, \ldots, j_n)} B_1^{\ j_1} B_2^{\ j_2} \cdots B_n^{\ j_n} \epsilon_{\sigma_{j_2 j_1 \cdots j_n}}$$

which, in turn, equals just

$$- \sum_{(j_1, \ldots, j_n)} B_1^{\ j_1} B_2^{\ j_2} \cdots B_n^{\ j_n} \epsilon_{\sigma_{j_1 j_2 \cdots j_n}} = -detB \ .$$

If $(i_1, \ldots, i_n)$ is different from $(1, \ldots, 2)$ just for a transposition only, the same procedure can be adapted trivially. If $(i_1, \ldots, i_n)$ is a proper permutation of $(1, \ldots, 2)$, it can be decomposed as a product of N transpositions. Concerning the right hand side of (6.6), using the procedure above for each transposition, we get in the end that it can be re-written :

$$B_{i_1}^{\ j_1} \cdots B_{i_n}^{\ j_n} \quad \eta_{j_1 \ldots j_n} = detB(-1)^N \ .$$

On the other hand, in the considered case, the left-hand side of (6.6) equals

$$detB(-1)^N \ ,$$

so that (6.6) holds true once again. This concludes the proof.

Definition 6.3. **(Ricci's Pseudo tensor).** Let V be a vector space with field $\mathbb{R}$ and dimension $n < +\infty$, endowed with a pseudo-metric tensor **g**. **Ricci's pseudo tensor** is the anti-symmetric $(0, n)$ pseudo tensor ε represented in each canonic basis by components

$$\varepsilon_{i_1 \ldots i_n} := \sqrt{|detg|} \ \eta_{i_1 \ldots i_n} \ ,$$

where $g = [g_{ij}]$, g_{ij} being the components of g in the considered basis and

$$\eta_{i_1 \dots i_n} = \epsilon_{\sigma_{i_1 \dots i_n}} ,$$

if $(i_1, \dots, i_n)$ is a permutated string of $(1, 2, \dots, n)$, otherwise

$$\eta_{i_1 \dots i_n} = 0 ,$$

and $\epsilon_{\sigma_{i_1 \dots i_n}}$ is the parity of the permutation $\sigma_{i_1 \dots i_n} \in \mathscr{P}_n$ defined by:

$$(\sigma(1), \dots, \sigma(n)) = (i_1, \dots, i_n) .$$

Equivalently, in the language of the forms:

$$\varepsilon = n! \sqrt{|detg|}\, e^{*1} \wedge \cdots \wedge e^{*n} . \tag{6.7}$$

♢

Remark. If V is oriented, it is possible to define ε as a *proper tensor* instead a pseudo tensor. In this case one defines the components in a canonical basis associated with a positive-oriented basis of V as

$$\varepsilon_{i_1 \dots i_n} := \sqrt{|detg|}\ \eta_{i_1 \dots i_n} \text{ , that is } \varepsilon = n! \sqrt{|detg|}\, e^{*1} \wedge \cdots \wedge e^{*n}$$

and

$$\varepsilon_{i_1 \dots i_n} := -\sqrt{|detg|}\ \eta_{i_1 \dots i_n} \text{ , that is } \varepsilon = -n! \sqrt{|detg|}\, e^{*1} \wedge \cdots \wedge e^{*n}$$

if the used basis is associated with a basis of V which is negative-oriented. One can prove straightforwardly that the defined components give rise to a tensor of $V^{*n\otimes}$ called **Ricci tensor**. This alternative point of view is equivalent, in the practice, to the other point of view corresponding to definition 6.3.

Ricci's pseudo tensor has various applications in mathematical physics in particular when it is used as a linear operator which produces pseudo tensors when acts on tensors. In fact, consider $t \in V^{n\otimes}$ and take an integer $m \leq n$. Fix a basis in V and, in the canonical bases induced by that basis, consider the action of ε on t:

$$t^{i_1 \dots i_n} \mapsto \tilde{t}_{j_1 \dots j_{n-m}} := \varepsilon_{j_1 \dots j_{n-m} i_1 \dots i_n} t^{i_1 \dots i_n} .$$

We leave to the reader the proof of the fact that the components $\tilde{t}_{j_1, \dots, j_{n-m}}$ define a anti-symmetric pseudo tensor of order $(0, n-m)$ which is called the **conjugate** pseudo tensor of t. This correspondence from tensors $(0, n)$ and pseudotensors of order $(0, n-m)$ is bijective, due to the formula:

$$t_{i_1 \dots i_p} = \frac{|detg|}{detg}\ \frac{(-1)^{p(n-p)}}{p!(n-p)!}\ \varepsilon_{i_1 \dots i_p j_1 \dots j_n} \varepsilon^{j_1 \dots j_n r_1 \dots r_p}\ t_{r_1 \dots t_p} ,$$

which holds for *anti-symmetric* tensors $t \in V^{*p\otimes}$, $0 \leq p \leq n = dimV$.
This implies that, if the tensor t is anti symmetric, then its conjugate pseudo tensor $\tilde{t}$ takes the same information than t itself.

Examples 6.2. **1**. As a trivial but very important example consider the **vector product** in a three-dimensional vector space V on the field $\mathbb{R}$ endowed with a metric tensor g. If $u, v \in V$ we may define the pseudo vector of order $(1, 0)$:

$$(u \wedge v)^r := g^{ri} \varepsilon_{ijk} u^j v^k .$$

If $\{e_i\}_{i=1,2,3}$ is an orthonormal basis in V, everything strongly simplifies. In fact, the Ricci tensor is represented by components

$$\varepsilon_{ijk} := 0$$

if (i, j, k) is not a permutation of $(1, 2, 3)$ and, otherwise,

$$\varepsilon_{ijk} := \pm 1 ,$$

where $+1$ corresponds to *cyclic* permutations of $1, 2, 3$ and -1 to *non-cyclic* permutations (see **Examples 3.1.2**). In such a basis:

$$(u \wedge v)_i = \varepsilon_{ijk}\, u^j v^k ,$$

because $g^{ij} = \delta^{ij}$ in each orthonormal bases.

Exercises 6.1.

1. Often, the definition of vector product in $\mathbb{R}^3$ is given, in orthonormal basis, as

$$(u \wedge v)_i = \varepsilon_{ijk}\, u^i v^k ,$$

where it is assumed that the basis is *right oriented*. Show that it defines a proper vector (and not a pseudo vector) if a convenient definition of $\wedge$ is given in *left oriented* basis.

2. Is it possible to define a sort of vector product (which maps pair of vectors in vectors) in $\mathbb{R}^4$ generalizing the vector product in $\mathbb{R}^3$?

3. In physics literature one may find the statement "Differently from the impulse $\vec{p}$ which is a **polar vector**, the angular momentum $\vec{l}$ is an **axial vector**". What does it mean?
(Solution. Polar vector = vector, Axial vector = pseudo vector.)

4. Consider the **parity inversion**, $P \in O(3)$, as the active transformation of vectors of physical space defined by $P := -I$ when acting in components of vectors in any orthonormal basis. What do physicists mean when saying "Axial vectors transform differently from polar vectors under parity inversion"?
(Hint. interpret P as a passive transformation, i.e. a changing of basis and extend the result to the active interpretation.)

5. Can the formula defining the conjugate pseudo tensor of t:

$$t^{i_1 \ldots i_n} \mapsto \tilde{t}_{j_1 \ldots j_{n-m}} := \varepsilon_{j_1 \ldots j_{n-m} i_1 \ldots i_m} t^{i_1 \ldots i_n} ,$$

be generalized to the case where t is a pseudo tensor? If yes, what sort of geometric object is $\tilde{t}$?

6. Consider a vector product $u \wedge v$ in $\mathbb{R}^3$ using an orthonormal basis. In that basis there is an anti-symmetric matrix which takes the same information as $u \wedge v$ and can be written down using the components of the vectors u and v. Determine that matrix and explain the tensorial meaning of the matrix.

7. Prove the formula introduced above:

$$t_{i_1 \ldots i_p} = \frac{|detg|}{detg} \frac{(-1)^{p(n-p)}}{p!(n-p)!} \varepsilon_{i_1 \ldots i_p j_1 \ldots j_n} \varepsilon^{j_1 \ldots j_n r_1 \ldots r_p} \ t_{r_1 \ldots t_p} \ ,$$

for anti-symmetric tensors $t \in V^{*p\otimes}$, $0 \leq p \leq n = dimV$.

6.3 Tensor densities.

In section 5.2 we have seen that the determinant of the matrix representing a pseudo-metric tensor g transforms, under change of basis with the rule

$$detg' = |detA|^{-2} detg$$

where the pseudo-metric tensor is $g'_{ij} e'^{*i} \otimes e'^{*i} = g_{pq} e^{*p} \otimes e^{*q}$ and $A = [A^i{}_j]$ is the matrix used in the change of basis for contravariant vectors $t^i e_i = t'^p e'_p$, that is $t^i = A^i{}_p t^p$. Thus the assignment of the numbers $detg$ for each basis in $\mathcal{B}$ does not define a scalar because of the presence of the factor $|detA|^{-1}$. Similar mathematical objects plays a relevant role in mathematical/theoretical physics and thus deserve a precise definition.

Definition 6.4. **(Tensor densities.)** Let V be a finite-dimensional vector space with field $\mathbb{R}$ and $\mathcal{B}$ the class of all bases of V.
If $S \in \mathscr{A}_{\mathbb{R}}(V)$, a **tensor density of** S **with weight** $w \in \mathbb{Z} \setminus \{0\}$ is a mapping $d : \mathcal{B} \to S$ such that, if $B = \{e_i\}_{i \in I}$ and $B' = \{e'_j\}_{j \in I}$ are two bases in $\mathcal{B}$ with $e_k = A^i{}_k e'_i$ then

$$d(B') = |detA|^w d(B) \ .$$

where $A = [A^i{}_k]$. Furthermore:
(a) the various tensorial properties enjoyed by all $d(B)$ are attributed to d. (So, for instance, if a $d(B)$ is symmetric (and thus all $d(B)$ with $B \in \mathcal{B}$ are symmetric), d is said to be symmetric);
(b) if $B \in \mathcal{B}$, the **components of** d with respect to the canonical bases induced by B are the components of $d(B)$ in those bases. ♢

If **g** is a pseudo-metric tensor on V a trivial example of a density with weight w in, for instance $S = V \otimes V^* \otimes V$, can be built up as follows. Take $t \in V \otimes V^* \otimes V$ and define

$$d_t(\{e_i\}_{i \in I}) := (\sqrt{|detg|})^{-w} t \ ,$$

where g is the matrix of the coefficients of $\mathbf{g}$ in the canonical basis associated with $\{e_i\}_{i\in I}$. In components, in the sense of (b) of the definition above:

$$(d_t)^i{}_j{}^k = (\sqrt{|detg|})^{-w} t^i{}_j{}^k \, .$$

To conclude we give the definition of pseudo tensor density which is the straightforward extension of the definition given above.

Definition 6.5. **(Pseudo-tensor densities.)** Let V be a finite-dimensional vector space with field $\mathbb{R}$ and $\mathcal{B}$ the class of all bases of V.
If $S \in \mathscr{A}_{\mathbb{R}}(V)$, a **pseudo-tensor density of** S **with weight** $w \in \mathbb{Z}\setminus\{0\}$ is a mapping $d : \mathcal{B} \to S$ such that, if $B = \{e_i\}_{i\in I}$ and $B' = \{e'_j\}_{j\in I}$ are two bases in $\mathcal{B}$ with $e_k = A^i{}_k e'_i$ then

$$d(B') = \frac{detA}{|detA|}|detA|^w d(B) \, .$$

where $A = [A^i{}_k]$. Furthermore:
(a) the various tensorial properties enjoyed by all $d(B)$ are attributed to d. (So, for instance, if a $d(B)$ is symmetric (and thus all $d(B)$ with $B \in \mathcal{B}$ are symmetric), d is said to be symmetric);
(b) if $B \in \mathcal{B}$, the **components of** d with respect to the canonical bases induced by B are the components of $d(B)$ in those bases. $\diamondsuit$

Remark. It is obvious that the sets of tensor densities and pseudo-tensor densities of a fixed space S and with fixed weight form linear spaces with composition rule which reduces to usual linear composition rule of components.

There is an important property of densities with weight -1 which is very useful in integration theory on manifolds. The property is stated in the following theorem.

Theorem 6.1. *Let V be a vector space on $\mathbb{R}$ with dimension $n < +\infty$. There is a natural isomorphism G from the space of scalar densities of weight -1 and the space of antisymmetric covariant tensors of order n. In components, using notation as in Definition 5.3,*

$$G : \alpha \mapsto \alpha\eta_{i_1\ldots i_n} \, .$$

Proof. Fix a canonical basis of $V^{*n\otimes}$ associated with a basis of V. Any non-vanishing tensor t in space, $\Lambda^n(V^*)$, of antisymmetric covariant tensors of order $n = dimV$ must have the form $t_{i_1\ldots i_n} = \alpha\eta_{i_1\ldots i_n}$ in components because different non-vanishing components can be differ only for the sign due to antisymmetry properties of t. Therefore the dimension of $\Lambda_n(V)$ is 1 which is also the dimension of the space of scalar densities of weight -1. The application G defined above in components from $\mathbb{R}$ to $\Lambda^n(V^*)$ is linear and surjective and thus is injective. Finally, re-adapting straightforwardly the relevant part of the discussion used to define ϵ, one finds that the coefficient α in $t_{i_1\ldots i_n} = \alpha\eta_{i_1\ldots i_n}$ transforms as a scalar densities of weight -1 under change of basis. $\Box$

Chapter 7

Polar Decomposition Theorem in the finite-dimensional case.

The goal of this chapter is to introduce the polar decomposition theorem of operators in finite dimensional spaces which has many applications in mathematics and physics.

7.1 Operators in spaces with scalar product.

We recall here some basic definitions and results which should be known by the reader from elementary courses of linear algebra [Lang, Sernesi].

If $A \in \mathscr{L}(V|V)$, that is $A : V \to V$ is a (linear) operator on any finite-dimensional vector space V with field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, an **eigenvalue** of A is an element $\lambda \in \mathbb{K}$ such that

$$(A - \lambda I)u = 0$$

for some $u \in V \setminus \{0\}$. In that case u is called **eigenvector** associated with λ. The set $\sigma(A)$ containing all of the eigenvalues of A is called the **spectrum** of A. The **eigenspace** E_λ associated with $\lambda \in \sigma(A)$ is the subspace of V spanned by the eigenvectors associated with λ.

Proposition 7.1. *Let V be a real (complex) finite-dimensional vector space equipped with a (resp. Hermitean) scalar product $(\cdot|\cdot)$. For every operator $A : V \to V$ there exists exactly one of operator $A^\dagger : V \to V$, called the* **adjoint operator** *of A, such that*

$$(A^\dagger u|v) = (u|Av)\ ,$$

for all $u, v \in V$.

Proof. Fix $u \in V$, the mapping $v \mapsto (u|Av)$ is a linear functional and thus an element of V^*. By theorem 5.2 there is a unique element $w_{u,A} \in V$ such that $(u|Av) = (w_{u,A}|v)$ for all $v \in V$. Consider the map $u \mapsto w_{u,A}$. It holds, if a, b are scalars in the field of V and $u, u' \in V$

$$(w_{au+bu',A}|v) = (au+bu'|Av) = \overline{a}(u|Av)+\overline{b}(u'|Av) = \overline{a}(w_{u,A}|v)+\overline{b}(w_{u',A}|v) = (aw_{u,A}+bw_{u',A}|v).$$

Hence, for all $v \in V$:

$$(w_{au+bu',A} - aw_{u,A} - bw_{u',A}|v) = 0 ,$$

The scalar product is non-degenerate by definition and this implies

$$w_{au+bu',A} = aw_{u,A} + bw_{u',A} .$$

We have obtained that the mapping $A^\dagger : u \mapsto w_{u,A}$ is linear, in other words it is an operator. The uniqueness is trivially proved: if the operator B satisfies $(Bu|v) = (u|Av)$ for all $u, v \in V$, it must holds $((B - A^\dagger)u|v) = 0$ for all $u, v \in V$ which, exactly as we obtained above, entails $(B - A^\dagger)u = 0$ for all $u \in V$. In other words $B = A^\dagger$. $\Box$

Comments 7.1. There is an interesting interplay between this notion of adjoint operator and that given in definition 2.4. Let $h : V \to V^*$ be the natural (anti)isomorphism induced by the scalar product as seen in theorem 5.2. If $T \in \mathscr{L}(V|V)$ and $T^*, T^\dagger$ denotes, respectively the two adjoint operators of T defined in definition 2.4 and in proposition 7.1, one has:

$$h \circ T^\dagger = T^* \circ h . \tag{7.1}$$

The proof of this fact is an immediate consequence of (3) in comments 5.1.

There are a few simple properties of the adjoint operator whose proofs are straightforward. Below A, B are operators in a complex (resp. real) finite-dimensional vector space V equipped with a Hermitean (resp. real) scalar product $(\cdot|\cdot)$ and a, b belong to the field of V.
(1) $(A^\dagger)^\dagger = A$,
(2) $(aA + bB)^\dagger = \overline{a}A^\dagger + \overline{b}B^\dagger$ (resp. $(aA + bB)^\dagger = aA^\dagger + bB^\dagger$),
(3) $(AB)^\dagger = B^\dagger A^\dagger$,
(4) $(A^\dagger)^{-1} = (A^{-1})^\dagger$ (if A^{-1} exists).

Take a finite-dimensional vector space V equipped with a scalar product $(\cdot|\cdot)$ and consider $A \in \mathscr{L}(V|V)$. We consider two cases: the *real case* where the field of V and the scalar product are real, and the *complex case*, where the field of V is $\mathbb{C}$ and the scalar product is Hermitean. We have the following definitions.
(1) In the real case, A is said to be **symmetric** if $(Au|v) = (u|Av)$ for all $u, v \in V$. In the complex case, A is said to be **Hermitean** if $(Au|v) = (u|Av)$ for all $u, v \in V$. It is simply proved that, in the respective cases, A is symmetric or Hermitean if and only if $A = A^\dagger$. In both cases $\sigma(A) \subset \mathbb{R}$.
(2) In the real case, A is said to be **orthogonal** if $(Au|Av) = (u|v)$ for all $u, v \in V$. In the complex case, A is said to be **unitary** if $(Au|Av) = (u|v)$ for all $u, v \in V$. It is simply proved that, in the respective cases, A is orthogonal or unitary if and only if A is bijective and $A^{-1} = A^\dagger$. In both cases if $\lambda \in \sigma(A)$ then $|\lambda| = 1$.
(3) In both cases A is said to be **normal** if $AA^\dagger = A^\dagger A$. It is obvious that symmetric, Hermitean, unitary, orthogonal operators are normal.

An important and straightforwardly-proved result is that, if $A \in \mathscr{L}(V|V)$ is normal, then:
(a) $u \in V$ is eigenvector of A with eigenvalue λ if and only if the same u is eigenvector of $A^\dagger$ with eigenvalue $\overline{\lambda}$ (which coincides with λ in the real case);
(b) If $\lambda, \mu \in \sigma(A)$ and $\lambda \neq \mu$ then $(u_\lambda|u_\mu) = 0$ if u_λ and u_μ are eigenvectors with eigenvalue λ and μ respectively for A.

In either the real or complex case, if $U \subset V$ is a subspace, the **orthogonal** of U, $U^\perp$, is the subspace of V made of all the vectors which are orthogonal to U, i.e., $v \in U^\perp$ if and only if $(u|v) = 0$ for all $u \in U$. As is well-known (see [Lang, Sernesi]), if $w \in V$, the decomposition $w = u + v$ with $u \in U$ and $v \in U^\perp$ is uniquely determined, and the map $P_U : w \mapsto u$ is linear and it is called **orthogonal projector** onto U.
If V is a finite-dimensional vector space (either real or complex), and $U \subset V$ is a subspace, the subspace $U^*_\perp \subset V^*$ defined as

$$U^*_\perp := \{v \in V^* \ \mid \ \langle u, v\rangle = 0\,, \quad \text{for all } u \in U\}$$

gives another and more general notion of "orthogonal space", in the absence of a scalar product in V. It is simply proved that, in the presence od a (pseudo)scalar product and where $h : V \to V^*$ is the natural (anti)isomorphism induced by the scalar product as seen in theorem 5.2, one has

$$h^{-1}(U^*_\perp) = U^\perp\,.$$

We leave to the reader the simple proof of the fact that an operator $P : V \to V$ is an orthogonal projector onto some subspace $U \subset V$ if and only if both the conditions below hold
(1) $PP = P$,
(2) $P = P^\dagger$.
In that case P is the orthogonal projector onto $U = \{Pv \mid v \in V\}$.
Another pair of useful results concerning orthogonal projectors is the following. Let V be a space as said above, let U, U' be subspaces of V, with P, P' are the corresponding orthogonal projectors P, P'.
(a) U and U' are **orthogonal** to each other, i.e., $U' \subset U^\perp$ (which is equivalent to $U \subset U'^\perp$) if and only if $PP' = P'P = 0$.
(b) $U \subset U'$ if and only if $PP' = P'P = P$.

If V is as above and it has finite dimension n and $A : V \to V$ is normal, there exist a well-known spectral decomposition theorem [Lang, Sernesi]: (the finite-dimensional version of the "spectral theorem").

Proposition 7.2. (**Spectral decomposition for normal operators in complex spaces**.) *Let V be a complex finite-dimensional vector space equipped with a Hermitean scalar product $(\cdot|\cdot)$. If $A \in \mathscr{L}(V|V)$ is normal (i.e., $A^\dagger A = AA^\dagger$), the following expansion holds:*

$$A = \sum_{\lambda \in \sigma(A)} \lambda P_\lambda\,,$$

where P_λ is the orthogonal projector onto the eigenspace associated with λ. Moreover the mapping $\sigma(A) \ni \lambda \mapsto P_\lambda$ satisfies the following two properties:
(1) $I = \sum_\lambda P_\lambda$,
(2) $P_\lambda P_\mu = P_\mu P_\lambda = 0$ *for* $\mu \neq \lambda$.
A **spectral measure**, *i.e. a mapping* $B \ni \mu \mapsto P'_\mu$ *with* $B \subset \mathbb{C}$ *finite,* P'_μ *non-vanishing orthogonal projectors and:*
(1)' $I = \sum_{\mu \in B} P'_\mu$,
(2)' $P'_\lambda P'_\mu = P'_\mu P'_\lambda = 0$ *for* $\mu \neq \lambda$,
coincides with $\sigma(A) \ni \lambda \mapsto P_\lambda$ *if*
(3)' $A = \sum_{\mu \in B} \mu P'_\mu$.

Proof. The equation $\det(A - \lambda I) = 0$ must have a (generally complex) solution λ due to fundamental theorem of algebra. As a consequence, A admits an eigenspace $E_{\lambda_1} \subset V$. From the properties (a) and (b) of normal operators one obtains that $A(E_\lambda^\perp) \subset E_{\lambda_1}^\perp$. Moreover $A \restriction_{E_{\lambda_1}^\perp}$ is normal. Therefore the procedure can be iterated obtaining a new eigenspace $E_{\lambda_2} \subset E_{\lambda_1}^\perp$. Notice that $\lambda_2 \neq \lambda_1$ because every eigenvector of $A \restriction_{E_{\lambda_1}^\perp}$ is also eigenvector for A with the same eigenvalue. Since eigenspaces with different eigenvalues are orthogonal, one finally gets a sequence of pairwise orthogonal eigenspaces $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_k}$, with $E_{\lambda_l} \subset E_{\lambda'_l}^\perp$ if $l' < l$. This sequence must be finite because the dimension of V is finite and orthogonal (eigen)vectors are linearly independent. Therefore $E_{\lambda_k}^\perp = \{0\}$ and so $\oplus_{l=1}^k E_{\lambda_l} = V$. On the other hand, the set $\{\lambda_1, \lambda_2, \ldots \lambda_k\}$ must coincide with the whole $\sigma(A)$ because eigenspaces with different eigenvalues are orthogonal (and thus any other eigenspace E_{λ_0} with $\lambda_0 \notin \{\lambda_1, \lambda_2, \ldots \lambda_k\}$ would be included in $(\oplus_{l=1}^k E_{\lambda_l})^\perp = \{0\}$). If $\{e_j^{(l)}\}_{j \in I_l}$ is an orthonormal basis for E_{λ_l}, the orthogonal projector on E_{λ_l} is

$$P_\lambda := \sum_{j \in I_l} (e_j^{(l)} | \quad) e_j^{(l)} .$$

Property (1) is nothing but the fact that $\{e_j^{(l)}\}_{j \in I_l, l=1,\ldots,k}$ is an orthonormal basis for V. Property (2) is another way to say that eigenspaces with different eigenvalues are orthogonal. Finally, since $e_j^{(l)}$ is an eigenvector of A with eigenvalue λ_l, it holds

$$A = \sum_{j \in I_l, l=1,\ldots,k} \lambda_l (e_j^{(l)} | \quad) e_j^{(l)} ,$$

which can be re-written as:

$$A = \sum_{\lambda \in \sigma(A)} \lambda P_\lambda .$$

The uniqueness property of spectral measures is proved as follows. It holds

$$\sum_{\mu \in B} \mu P'_\mu = \sum_{\lambda \in \sigma(A)} \lambda P_\lambda .$$

Applying P'_ν on the left and P_τ on the right, and using properties (1)' and (1) one finds

$$\nu P'_\nu P_\tau = \tau P'_\nu P_\tau \,,$$

so that

$$(\nu - \tau) P'_\nu P_\tau = 0 \,.$$

This is possible if $\nu = \tau$ or $P'_\nu P_\tau = 0$. For a fixed P'_ν it is not possible that $P'_\nu P_\tau = 0$ for all P_τ, because, using (1), it would imply

$$0 = P'_\nu \sum_{\tau \in \sigma(A)} P_\tau = P'_\nu I = P'_\nu$$

but $P'_\nu \neq 0$ hypotheses. Therefore $\nu = \tau$ for some $\tau \in \sigma(A)$ and hence $B \subset \sigma(A)$. By means of the analogous procedure one sees that $\sigma(A) \subset B$. We conclude that $B = \sigma(A)$. Finally, the decomposition:

$$A = \sum_{\lambda \in \sigma(A)} \lambda P'_\lambda$$

implies that each P'_λ is an orthogonal projector on a subspace of E_λ. As a consequence, if u_λ is an eigenvector of A with eigenvalue λ, it must be $P_{\lambda'} u_\lambda = 0$ for $\lambda' \neq \lambda$. Since (1)' is valid, one concludes that $P'_\lambda u_\lambda = u_\lambda$ and thus not only $P'_\lambda(V) \subset E_\lambda$, but, more strongly, $P'_\lambda(V) = E_\lambda$. Since the orthogonal projector onto a subspace is biunivocally determined by the subspace itself, we have proved that $P'_\lambda = P_\lambda$ for every $\lambda \in \sigma(A)$. □

7.1.1 Positive operators

Definition 7.1. If V is a real (complex) vector space equipped with a (resp. Hermitean) scalar product $(\cdot|\cdot)$, an operator $A : V \to V$ is said to be **positive** (or *positive semidefined*) if

$$(u|Au) \geq 0 \quad \text{for all } u \in V \,.$$

A positive operator A is said to be **strictly positive** (or *positive defined*) if

$$(u|Au) = 0 \quad \text{entails } u = 0 \,.$$

♢

A straightforward consequence of the given definition is the following lemma.

Lemma 7.1. *Let V be a complex vector space equipped with a Hermitean scalar product $(\cdot|\cdot)$. Any positive operator $A : V \to V$, is Hermitean.*
Moreover, if $dimV < \infty$, a normal operator $A : V \mapsto V$
(a) *is positive if and only if $\sigma(A) \subset [0, +\infty)$;*

(b) *is strictly positive if and only if* $\sigma(A) \subset (0, +\infty)$.

Proof. As $(v|Av) \geq 0$, by complex conjugation $(Av|v) = \overline{(v|Av)} = (v|Av)$ and thus

$$((A^\dagger - A)v|v) = 0$$

for all $v \in V$. In general we have:

$$2(Bu|w) = (B(u+w)|(u+w)) + i(B(w+iu)|(w+iu)) - (1+i)(Bw|w) - (1+i)(Bu|u)\,.$$

So that, taking $B = A^\dagger - A$ we get $(Bu|w) = 0$ for all $u, w \in V$ because $(Bv|v) = 0$ for all $v \in V$. $(Bu|w) = 0$ for all $u, w \in V$ entails $B = 0$ or $A^\dagger = A$.
Let us prove (a). Suppose A is positive. We know that $\sigma(A) \subset \mathbb{R}$. Suppose there is $\lambda < 0$ in $\sigma(A)$. Let u be an eigenvector associated with λ. $(u|Au) = \lambda(u|u) < 0$ because $(u|u) > 0$ since $u \neq 0$. This is impossible.
Now assume that A is normal with $\sigma(A) \subset [0, +\infty)$. By proposition 7.2:

$$(u|Au) = \left(\sum_\mu P_\mu u \middle| \sum_\lambda \lambda P_\lambda \sum_\nu P_\nu u\right) = \sum_{\mu,\lambda,\nu} \lambda \left(P_\nu^\dagger P_\lambda^\dagger P_\mu u \middle| u\right) = \sum_{\mu,\lambda,\nu} \lambda \left(P_\nu P_\lambda P_\mu u \middle| u\right)$$

because, if P is an orthogonal projector, $P = P^\dagger$. Using Proposition A.2 once again, $P_\nu P_\lambda P_\mu = \delta_{\nu\mu}\delta_{\mu\lambda}P_\lambda$ and thus

$$(u|Au) = \sum_\lambda \lambda \left(P_\lambda u \middle| u\right) = \sum_\lambda \lambda \left(P_\lambda u \middle| P_\lambda u\right)\,,$$

where we have used the property of orthogonal projectors $PP = P$. $\lambda \left(P_\lambda u \middle| P_\lambda u\right) \geq 0$ if $\lambda \geq 0$ and thus $(u|Au) \geq 0$ for all $u \in V$.
Concerning (b), assume that A is strictly positive (so it is positive and Hermitean). If $0 \in \sigma(A)$ there must exist $u \neq 0$ with $Au = 0u = 0$. That entails $(u, Au) = (u, 0) = 0$ which is not allowed. Therefore $\sigma(A) \subset [0, +\infty)$. Conversely if A is normal with $\sigma(A) \subset (0, +\infty)$, A is positive by (a). If A is not strictly positive, there is $u \neq 0$ such that $(u|Au) = 0$ and thus, using the same procedure as above,

$$(u|Au) = \sum_\lambda \lambda \left(P_\lambda u \middle| P_\lambda u\right) = 0\,.$$

Since $\lambda > 0$ and $\left(P_\lambda u \middle| P_\lambda u\right) \geq 0$, it must be $\left(P_\lambda u \middle| P_\lambda u\right) = 0$ for all $\lambda \in \sigma(A)$. This means $P_\lambda u = 0$ for all $\lambda \in \sigma(A)$. This is not possible because, using (1) in proposition 7.2, $0 \neq u = Iu = \sum_\lambda P_\lambda u = 0$. $\square$

7.1.2 Complexification procedure.

There is a standard and useful way to associate a complex vector space endowed with Hermitian scalar product to a given real vector space equipped with a scalar product. This correspondence allows one to take advantage of some results valid for complex vector spaces also in the case of real vector spaces. We shall employ that possibility shortly.

Definition 7.2. If V is a real vector space equipped with the scalar product $(\cdot|\cdot)$, the **complexification** of V, $V \oplus iV$, is the complex vector space with Hermitean scalar product $(\cdot|\cdot)_{\mathbb{C}}$ defined as follows. The elements of $V \oplus iV$, are the corresponding pairs $(u, v) \in V \times V$ denotes with $u + iv := (u, v)$ moreover,
(1) the product of scalar in $\mathbb{C}$ and vectors in $V \oplus iV$ is defined as:

$$(a + ib)(u + iv) := au - bv + i(bu + av) \quad \text{for } a + ib \in \mathbb{C} \text{ and } u + iv \in V \times V \ ,$$

(2) the sum of vectors is defined as:

$$(u + iv) + (x + iy) := (u + x) + i(v + y) \ , \quad \text{for } u + iv, x + iy \in V \times V \ ,$$

(3) the Hermitean scalar product is defined as:

$$(u + iv|w + ix)_{\mathbb{C}} := (u|v) + (v|x) + i(u|x) - i(v|w) \ , \quad \text{for } u + iv, w + ix \in V \times V \ .$$

$\diamondsuit$

Let us introduce a pair of useful operators from $V \oplus iV$ to $V \oplus iV$.
The **complex conjugation** is defined as

$$J : u + iv \mapsto u - iv \ .$$

It is anti linear and satisfies $(J(u + iv)|J(w + ix))_{\mathbb{C}} = (w + ix|u + iv)_{\mathbb{C}}$ and $JJ = I$.
An important property of J is the following: $s \in V \oplus iV$ satisfies $s \in V$ (i.e. $s = (u, 0) \in V \times V$) if and only if $Js = s$, whereas it satisfies $s \in iV$ (i.e. $s = (0, v) \in V \times V$) if and only if $Js = -s$.
The proofs of these features follows trivially by the definition.
The second interesting operator is the **flip operator**, defined as

$$C : u + iv \mapsto v - iu \ .$$

It satisfies $CC = I$ and $C^\dagger = C$, where the adjoint is refereed to the Hermitean scalar product.
Also in this case the proof is straightforward using the given definition.
A linear operator $A : V \oplus iV \to V \oplus iV$ is said to be **real** if $JA = AJ$.

Proposition 7.3. *Referring to definition 7.2 let $A \in \mathscr{L}(V \oplus iV|V \oplus iV)$. The following facts hold.*
(a) *A is real if and only if there is a uniquely-determined pair of $\mathbb{R}$-linear operators $A_j : V \to V$, $j = 1, 2$, such that*

$$A(u + iv) = A_1 u + iA_2 v \ , \quad \textit{for all } u + iv \in V \oplus iV \ .$$

(b) *A is real and $AC = CA$, if and only if there is a uniquely-determined $\mathbb{R}$-linear operator $A_0 : V \to V$, such that*

$$A(u + iv) = A_0 u + iA_0 v\,, \quad \textit{for all } u + iv \in V \oplus iV\,.$$

Proof. The proof of uniqueness is trivial in both cases and it is based on linearity and on the following fact. If $T, T' \in \mathscr{L}(V|V)$, $Tu + iT'v = 0$ for all $u, v \in V$ entails $T = T' = 0$. Moreover, by direct inspection one sees that A is real if $A(u + iv) = A_1 u + iA_2 v$ for all $u + iv \in V \oplus iV$ and A is real and commute with C if $A(u+iv) = A_0 u + iA_0 v$ for all $u + iv \in V \oplus iV$, where the operators A_j are $\mathbb{R}$-linear operators from V to V.
Let us conclude the proof of (a) proving that if A is real, there must be A_1 and A_2 as in (a). Take $s \in V$, as a consequence of $AJ = JA$ one has: $JAs = AJs = As$, where we used the fact that $Js = s$ if and only if $s \in V$. Since $JAs = s$ we conclude that $As \in V$ if $s \in V$ and so $A \restriction_V$ is well defined as $\mathbb{R}$-linear operator from V to V. With an analogous procedure one sees that $A \restriction_{iV}$ is well defined as $\mathbb{R}$-linear operator from iV to iV. Defining $A_1 := A \restriction_V : V \to V$ and $A_2 := -iA \restriction_{iV} i : V \to V$ one has:

$$A(u + iv) = A \restriction_V u + A \restriction_{iV} iv = A_1 u + iA_2 v\,.$$

We have proved that (a) is true.
Let us conclude the proof of (b). If A is real and commute with C, by (a) there must be A_1 and A_2 satisfying the condition in (a) and, furthermore, the condition $AC = CA$ reads $A_1 v = A_2 v$ and $A_2 u = A_1 v$ for all $u, v \in V$. Therefore $A_0 := A_1 = A_2$ verifies (b). $\Box$

7.1.3 Square roots.

Definition 7.3. If V is a complex (real) vector space equipped with a Hermitean (respectively real) scalar product $(\cdot|\cdot)$, Let $A : V \to V$ a (symmetric in the real case) positive operator. If $B : V \to V$ is another (symmetric in the real case) positive operator such that $B^2 = A$, B is called a **square root of** A. $\diamondsuit$

Notice that square roots, if they exist, are Hermitean by lemma 7.1. The next theorem proves that the square root of a positive operator exist and is uniquely determined.

Theorem 7.1. *Let V be a finite-dimensional either complex or real vector space equipped with a Hermitean or, respectively, real scalar product $(\cdot|\cdot)$. Every positive operator $A : V \to V$ admits a unique square root indicated by $\sqrt{A}$. $\sqrt{A}$ is Hermitian or, respectively, symmetric and $\sqrt{A}$ is bijective if and only if A is bijective.*

Proof. First consider the complex case. A is Hermitean by lemma 7.1. Using proposition 7.2,

$$A = \sum_{\lambda \in \sigma(A)} \lambda P_\lambda\,.$$

Since $\lambda \geq 0$ we can define the linear operator

$$\sqrt{A} := \sum_{\lambda \in \sigma(A)} \sqrt{\lambda} P_\lambda \, .$$

By proposition 7.2 we have

$$\sqrt{A}\sqrt{A} = \sum_{\lambda \in \sigma(A)} \sqrt{\lambda} P_\lambda \sum_{\mu \in \sigma(A)} \sqrt{\mu} P_\mu = \sum_{\lambda\mu} \sqrt{\lambda\mu} P_\lambda P_\mu \, .$$

Using property (2)

$$\sqrt{A}\sqrt{A} = \sum_{\lambda\mu} \sqrt{\lambda\mu}\delta_{\mu\nu} P_\lambda = \sum_{\lambda} (\sqrt{\lambda})^2 P_\lambda = \sum_{\lambda} \lambda P_\lambda = A \, .$$

Notice that $\sqrt{A}$ is Hermitean by construction:

$$\sqrt{A}^\dagger := \sum_{\lambda \in \sigma(A)} \sqrt{\lambda} P_\lambda^\dagger = \sum_{\lambda \in \sigma(A)} \sqrt{\lambda} P_\lambda = \sqrt{A} \, .$$

Moreover, if $B = \{\mu = \sqrt{\lambda} \;|\lambda \in \sigma(A)\}$ and $P'_\mu := P_\lambda$ with $\mu = \sqrt{\lambda}$, it holds

$$\sqrt{A} := \sum_{\mu \in B} \mu P'_\mu \, ,$$

and $B \ni \mu \mapsto P'_\mu$ satisfy the properties (1)',(2)',(3)' in proposition 7.2 As a consequence it coincides with the spectral measure associated with $\sqrt{A}$,

$$\sqrt{A} := \sum_{\lambda \in \sigma(A)} \sqrt{\lambda} P_\lambda$$

is the unique spectral decomposition of A,

$$\sigma(\sqrt{A}) = \{\mu = \sqrt{\lambda} \;|\lambda \in \sigma(A)\} \, ,$$

and thus $\sqrt{A}$ is positive by lemma 7.1, $\sqrt{A}$ is a Hermitean square root of A. Notice that, by construction $0 \in \sigma(\sqrt{A})$ if and only if $0 \in \sigma(A)$ so that $\sqrt{A}$ is bijective if and only if A is bijective.
Let us pass to prove the uniqueness property. Suppose there is another square root S of A. As S is positive, it is Hermitean with $\sigma(S) \subset [0, +\infty)$ and it admits a (unique) spectral decomposition

$$S = \sum_{\nu \in \sigma(S)} \nu P'_\mu \, .$$

Define $B := \{\nu^2 \mid \nu \in \sigma(S)\}$. It is simply proved that the mapping $B \ni \lambda \mapsto P'_{\sqrt{\lambda}}$ satisfy:
(1)' $I = \sum_{\lambda \in B} P'_{\sqrt{\lambda}}$,

(2)' $P'_{\sqrt{\lambda}}P'_{\sqrt{\mu}} = P'_{\sqrt{\mu}}P'_{\sqrt{\lambda}} = 0$ for $\mu \neq \lambda$,
(3)' $A = S^2 = \sum_{\lambda\in B} \lambda P'_{\sqrt{\lambda}}$.
proposition 7.2 entails that the spectral measure of A and $B \ni \lambda \mapsto P'_{\sqrt{\lambda}}$ coincides: $P'_{\sqrt{\lambda}} = P_\lambda$ for all $\lambda \in \sigma(A) = B$. In other words

$$S = \sum_{\nu\in\sigma(S)} \nu P'_\nu = \sum_{\lambda\in\sigma(A)} \lambda P_\lambda = \sqrt{A}\,.$$

Let us finally consider the case of a real space V. If $A \in \mathscr{L}(V|V)$ is positive and symmetric, the operator on $V \oplus iV$, $S : u + iv \mapsto Au + iAv$ is positive and Hermitean. By the first part of this proof, there is only one Hermitean positive operator $B \in \mathscr{L}(V \oplus iV|V \oplus iV)$ with and $B^2 = S$, that is the square root of S which we indicate by $\sqrt{S}$. Since S commutes with both J and C and $CC = JJ = I$, one has

$$\sum_{\lambda\in\sigma(S)} \lambda J P^{(S)}_\lambda J = S\,, \quad \text{and} \sum_{\lambda\in\sigma(S)} \lambda C P^{(S)}_\lambda C = S\,.$$

Since J and C (anti-) preserve the scalar product and $JJ = CC = I$, one straightforwardly proves that $\sigma(S) \ni \lambda \mapsto JP^{(S)}_\lambda J$ and $\sigma(S) \ni \lambda \mapsto JP^{(S)}_\lambda J$ are spectral measures. By the uniqueness property in proposition 7.2 one concludes that these spectral measures coincide with $\sigma(S) \ni \lambda \mapsto P^{(S)}_\lambda$ and thus, in particular, each projector of the spectral measure commutes with both J and C. Hence $\sqrt{S} = \sum_{\lambda\in\sigma(S)} \sqrt{\lambda} P^{(S)}_\lambda$ commutes with both J and C. We conclude that $\sqrt{S}$ is real with the form $\sqrt{S} : u + iv \mapsto Ru + iRv$. The operator $\sqrt{A} := R$ fulfills all of requirements of a square root it being symmetric and positive because $\sqrt{S}$ is Hermitean and positive, and $R^2 = A$ since $(\sqrt{S})^2 = S : u + iv \mapsto Au + iAv$. If A is bijective, S is such by construction and thus its kernel is trivial. Since $\sqrt{S} = \sum_{\lambda\in\sigma(S)} \sqrt{\lambda} P^{(S)}_\lambda$, its kernel is trivial too and $\sqrt{S}$ is bijective. In turn, R is bijective by construction. If A is not bijective, and $0 \neq u \in KerA$, $u + iu \in KerR$ so that R is not bijective.
Let us consider the uniqueness of the found square root. If R' is another positive symmetric square root of A, $B : u + iv \to R'u + iR'v$ is a Hermitean positive square root of S and thus it must coincide with $\sqrt{S}$. This implies that $R = R'$.
□

7.2 Polar Decomposition.

The notions and the results obtained above allow us to state and prove the polar decomposition theorem for operators in finite dimensional vector spaces equipped with scalar product.

Theorem 7.2. (**Polar Decomposition of operators**.) *If $T \in \mathscr{L}(V|V)$ is a bijective operator where V is a real (resp. complex), finite-dimensional vector space equipped with a real*

(resp. Hermitean) scalar product space:
(a) *there is a unique decomposition $T = UP$, where U is orthogonal (resp. unitary) and P is bijective, symmetric (resp. Hermitean), and positive. In particular $P = \sqrt{T^\dagger T}$ and $U = T(\sqrt{T^\dagger T})^{-1}$;*
(b) *there is a unique decomposition $T = P'U'$, where U' is orthogonal (resp. unitary) and P' is bijective, symmetric (resp. Hermitean), and positive. In particular $U' = U$ and $P' = UPU^\dagger$.*

Proof. (a) Consider $T \in \mathscr{L}(V|V)$ bijective. $T^\dagger T$ is symmetric/self-adjoint, positive and bijective by construction. Define $P := \sqrt{T^\dagger T}$, which exists and is symmetric/self-adjoint, positive and bijective by theorem 7.1, and $U := TP^{-1}$. U is orthogonal/unitary because

$$U^\dagger U = P^{-1}T^\dagger T P^{-1} = P^{-1}P^2P^{-1} = I ,$$

where we have used $P^\dagger = P$. This proves that a polar decomposition of T exists because $UP = T$ by construction. Let us pass to prove the uniqueness of the decomposition. If $T = U_1P_1$ is a other polar decomposition, $T^\dagger T = P_1U_1^\dagger U_1P_1 = PU^\dagger UP$. That is $P_1^2 = P^2$. Theorem 7.1 implies that $P = P_1$ and $U = T^{-1}P = T^{-1}P_1 = U_1$.
(b) $P' := UPU\dagger$ is symmetric/self-adjoint, positive and bijective since $U^\dagger$ is orthogonal/unitary and $P'U' = UPU^\dagger U = UP = T$. The uniqueness of the decomposition in (b) is equivalent to the uniqueness of the polar decomposition $U'^\dagger P'^\dagger = T^\dagger$ of $T^\dagger$ which holds true by (a) replacing T by $T^\dagger$. $\Box$

Chapter 8

Minkowski spacetime and Poincaré-Lorentz group.

8.1 Minkowski spacetime.

This chapter is a pedagogical review on basic geometric notions of Special Relativity theory. We employ definitions and results presented in section 3.4. In the last part we shall employ also some standard results of Lie group theory.

8.1.1 General Minkowski spacetime structure.

As is well known, Poincaré group encodes all coordinate transformations between a pair of inertial frames moving in Minkowski spacetime. Let us recall some features of that from a mathematical point of view.

Definition 8.1. Minkowski spacetime $\mathbb{M}^4$ is a four-dimensional affine space whose space equipped with a pseudo-scalar product in the real four-dimensional vector space of translations T^4 (identified to $\mathbb{R}^4$) defined by a metric tensor $\mathbf{g}$, with signature $(1,3)$ (i.e. $(-1,+1,+1,+1)$). The following further definitions hold.
(a) The points of Minkowski spacetime are called **events**.
(b) The Cartesian coordinate systems x^0, x^1, x^2, x^3 induced from the affine structure by arbitrarily fixing any $\mathbf{g}$-orthonormal basis $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in T^4 (with $\mathbf{g}(\mathbf{e}_0, \mathbf{e}_0) = -1$, $\mathbf{g}(\mathbf{e}_1, \mathbf{e}_1) = 1$, $\mathbf{g}(\mathbf{e}_2, \mathbf{e}_2) = 1$, $\mathbf{g}(\mathbf{e}_3, \mathbf{e}_3) = 1$) and any origin $O \in \mathbb{M}^4$, are called **Minkowskian coordinate frames**.
(c) The elements of T^4 (as well as the vectors applied at every event) are called **four-vectors**. $\diamondsuit$

In practice, exploiting the affine structure and using standard notation for affine spaces, Minkowskian coordinates x^0, x^1, x^2, x^3 are defined in order that the map $\mathbb{M}^4 \ni p \mapsto (x^0(p), x^1(p), x^2(p), x^3(p)) \in$

$\mathbb{R}^4$ satisfies (where we are using the convention of summation over repeated indices):

$$T^4 \ni p - O = x^\mu(p)\mathbf{e}_\mu$$

for every event $p \in \mathbb{M}^4$. The pseudo-scalar product $(\cdot|\cdot)$, that is the metric tensor $\mathbf{g}$ in T^4, has form constant and diagonal in Minkowskian coordinates:

$$\mathbf{g} = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 \,. \tag{8.1}$$

Physically speaking the events are the minimal space-temporal characterization of everything occurs in the universe. Modulo technicalities we shall examine shortly, Minkowskian coordinate frames individuate (not biunivocally) the the class of coordinate system of all *inertial observers*. Referring to the decomposition (8.1) of $\mathbf{g}$, the coordinates x^1, x^2, x^3, are thought as "spatial coordinates", whereas the coordinate x^0 is a temporal coordinate. The metric g is used to perform measurements either in time and in space as we clarify in the rest of this section. If $X_p \neq 0$ is a vector in $T_p\mathbb{M}^4$, it may represent either infinitesimal temporal displacements if $\mathbf{g}_p(X_p, X_p) \leq 0$ or infinitesimal spatial displacements if $\mathbf{g}_p(X_p, X_p) > 0$. In both cases $|\mathbf{g}_p(X_p, X_p)|$ has the physical meaning of the length (duration) of V_p. Actually the distinguishable case $\mathbf{g}(X_p, X_p) = 0$ (but $X_p \neq 0$) deserves a particular comment. These vectors represent an infinitesimal part of the story of a light particle.

Definition 8.2. For every event $p \in \mathbb{M}^4$, $T_p\mathbb{M}^4 \setminus \{0\}$ is decomposed in three pair-wisely disjoint subsets:
(i) the set of **spacelike** vectors which satisfy: $\mathbf{g}_p(X_p, X_p) > 0$,
(ii) the set of **timelike** vectors which satisfy: $\mathbf{g}_p(X_p, X_p) < 0$,
(iii) the set of **lightlike**, also called **null**, four-vectors which satisfy: $\mathbf{g}_p(X_p, X_p) = 0$.
The following further definitions hold.
(a) The union of the sets $\{0\}$ and timelike and lightlike four-vectors is a closed cone, $\overline{V_p}$, called **closed light cone** at p. Its non-vanishing element are called **causal** four-vectors.
(b) The interior V_p of $\overline{V_p}$ is called **open light cone** at p.
(c) The boundary ∂V_p is called **light cone** at p.$\diamondsuit$

8.1.2 Time orientation.

Definition 8.3. Two smooth timelike four-vector fields T, T' on $\mathbb{M}^4$ are said to **have the same time orientation** if $\mathbf{g}_p(T_p, T'_p) < 0$ for every $p \in \mathbb{M}^4$. $\diamondsuit$

To go on we notice the following fact. Using Minkowskian coordinates and referring to the base of $T_p\mathbb{M}^4$ associated with these coordinates, one sees that the open light cone is pictured as the set

$$V_p = \{(X^0, X^1, X^2, X^3) \in \mathbb{R}^4 \setminus \{0\} \mid (X^0)^2 > (X^1)^2 + (X^2)^2 + (X^3)^2\} \,.$$

Thus, in those Minkowskian coordinates, V_p is made of two disjoint halves

$$V_p^{(>)} := \{(X^0, X^1, X^2, X^3) \in V_p \mid X^0 > 0\} \,, \quad V_p^{(<)} := \{(X^0, X^1, X^2, X^3) \in V_p \mid X^0 < 0\} \,. \tag{8.2}$$

We remark that, *a priori*, this decomposition depends on the used coordinate system.

Proposition 8.1. *The class $\mathscr{T}(\mathbb{M}^4)$ of smooth (i.e. C^∞) timelike four-vector fields on $\mathbb{M}^4$ satisfies following.*
(a) *$\mathscr{T}(\mathbb{M}^4)$ is not empty.*
(b) *Referring to a Minkowskian coordinate frame and decomposing every V_p into the two disjoint halves (8.2), if $T, T' \in \mathscr{T}(\mathbb{M}^4)$ it holds constantly in $p \in \mathbb{M}^4$, $\mathbf{g}(T_p, T'_p) < 0$ or $\mathbf{g}(T_p, T'_p) > 0$. The former happens if and only if both T_p, T'_p belong to the same half of V_p, the latter holds if and only if they belong to different halves.*
(c) *"to have the same time orientation" is a equivalence relation in $\mathscr{T}(\mathbb{M}^4)$ and it admits two equivalence classes only.*

Proof. (a) $\mathscr{T}(\mathbb{M}^4)$ is not empty since it includes the vector field ∂_{x^0} associated to any Minkowskian coordinate frame on $\mathbb{M}^4$. Let us show (b). Consider a smooth timelike vector field S. Using Minkowskian coordinates x^0, x^1, x^2, x^3 and the associated orthonormal bases of elements $\mathbf{e}_{\alpha,p} = \partial_{x^\alpha}|_p \in T_p\mathbb{M}^4$, $\alpha = 0, 1, 2, 3$, one has:

$$(S_p^0)^2 > \sum_{i=1}^{3}(S_p^i)^2 \, . \tag{8.3}$$

Consider the two halves $V_p^{(>)}$ and $V_p^{(<)}$ said above. S_p^0 cannot change it sign varying $p \in \mathbb{M}$ because it would imply that $S_p^0 = 0$ which is not allowed. Therefore it holds $S_p \in V_p^{(>)}$ constantly in $p \in \mathbb{M}^4$, that is

$$S_p^0 > \sqrt{\sum_{i=1}^{3}(S_p^i)^2} \, , \quad \text{for all } p \in \mathbb{M}^4 \, , \tag{8.4}$$

or $S_p \in V_p^{(<)}$ constantly in $p \in \mathbb{M}^4$, that is

$$S_p^0 < -\sqrt{\sum_{i=1}^{3}(S_p^i)^2} \, , \quad \text{for all } p \in \mathbb{M}^4 \, . \tag{8.5}$$

Now consider two timelike smooth vector fields T and T'. One has

$$\mathbf{g}_p(T, T') = -T^0 T'^0 + \sum_{i=1}^{3} T'^i T^i \, . \tag{8.6}$$

On the other hand, by Cauchy-Schwartz inequality

$$\left|\sum_{i=1}^{3} T^i T'^i\right| \leq \sqrt{\sum_{i=1}^{3}(T_p^i)^2}\sqrt{\sum_{i=1}^{3}(T_p'^i)^2} \, .$$

Now, taking into account that it must hold either (8.4) or (8.5) for T and T' in place of S, we conclude from (8.6) that, constantly in $p \in \mathbb{M}^4$, $\mathbf{g}_p(T_p, T'_p) < 0$ or $\mathbf{g}_p(T_p, T'_p) > 0$ and the former happens if and only if both T, T' belong to the same half of V_p, whereas the latter arises if and only if they belong to different halves. The proof of (b) is concluded.
Using (b) result we can discuss the situation in a single tangent space fixing $p \in \mathbb{M}^4$ arbitrarily and we can prove (c), that "to have the same time orientation" is a equivalence relation. By definition of timelike four-vector $\mathbf{g}(T_p, T_p) < 0$, so T has the same time orientation as T itself. If $\mathbf{g}(T_p, T'_p) < 0$ then $\mathbf{g}(T'_p, T_p) = \mathbf{g}(T_p, T'_p) < 0$ so that the symmetric property is satisfied. Let us come to transitivity. Suppose that $\mathbf{g}(T_p, T'_p) < 0$ so that T_p and T'_p belong to the same half of V_p, and $\mathbf{g}(T'_p, S_p) < 0$ so that T'_p and S_p belong to the same half of V_p. We conclude that T_p and S_p belong to the same half of V_p and thus $\mathbf{g}(T_p, S_p) < 0$. This proves transitivity.
To conclude, notice that, if T is a smooth timelike vector field on $\mathbb{M}^4$, T and $-T$ belong to different equivalence classes and if $g(T, S) > 0$ then $\mathbf{g}(-T, S) < 0$ so that, every timelike smooth vector field belongs to the equivalence class of T or to the equivalence class of $-T$. $\square$

Corollary. *The decomposition of V_p into two disjoint halves does not depend on used Minkowskian coordinate frame. In other words, considering two Minkowskian coordinate frames x^0, x^1, x^2, x^3 and $x_1^0, x_1^1, x_1^2, x_1^3$, $p \in \mathbb{M}^4$ and and the associated decompositions (8.2), $V_p = V_p^{(>)} \cup V_p^{(<)}$ and $V_{1,p}^{(>)} \cup V_{1,p}^{(<)}$, one of the following exclusive cases must occur:*

(1) $V_p^{(>)} = V_{1,p}^{(>)}$ and $V_p^{(<)} = V_{1,p}^{(<)}$ for every $p \in \mathbb{M}^4$ or (2) $V_p^{(>)} = V_{1,p}^{(<)}$ and $V_p^{(<)} = V_{1,p}^{(>)}$ for every $p \in \mathbb{M}^4$.

Proof. By (b) of proposition 8.1, $V_p^{(>)}$ and $V_{1,p}^{(>)}$ are respectively made of the restrictions to p of the vector fields in $\mathscr{T}(\mathbb{M}^4)$ which have the same time orientation as ∂_{x^0} and $\partial_{x_1^0}$ respectively. On the other hand, using (b) again, one finds that $\partial_{x_1^0}$ must belong to $V_p^{(>)}$ or it must belong to $V^{(<)}$. Suppose that the former is valid. In this case $\partial_{x_1^0}$ and $\partial_{x_1^0}$ have the same time orientation. By transitivity one conclude that $V_p^{(>)} = V_{1,p}^{(>)}$. Since $V_p^{(>)} \cup V_p^{(<)} = V_{1,p}^{(>)} \cup V_{1,p}^{(<)}$ and $V_p^{(>)} \cap V_p^{(<)} = V_{1,p}^{(>)} \cap V_{1,p}^{(<)} = \emptyset$ we conclude also that $V_p^{(<)} = V_{1,p}^{(<)}$. If $\partial_{x_1^0} \in V_p^{(<)}$, one concludes that $V_p'^{(>)} = V_{1,p}^{(>)}$ and $V_p'^{(<)} = V_{1,p}^{(<)}$ where $V_p'^{(>,<)}$ are referred to Minkowskian coordinates $-x^0, x^1, x^2, x^3$. Since $V_p'^{(>)} = V_p^{(<)}$ and $V_p'^{(<)} = V_p^{(>)}$, we conclude that $V_p^{(>)} = V_{1,p}^{(<)}$ and $V_p^{(<)} = V_{1,p}^{(>)}$.

Definition 8.4. A pair $(\mathbb{M}^4, \mathcal{O})$ where $\mathcal{O}$ is one of the two equivalence classes of the relation "to have the same time orientation" in the set of smooth timelike vector fields on $\mathbb{M}^4$, is called **time oriented** Minkowski spacetime, and $\mathcal{O}$ is called **time orientation** of $\mathbb{M}^4$.$\diamondsuit$

There is an alternative way to fix a time orientation: form proposition 8.1 and its corollary we also conclude that:

Proposition 8.2. *The assignment of a time orientation $\mathcal{O}$ is equivalent to smoothly select one of the two disjoint halves of V_p, for every $p \in \mathbb{M}^4$: that containing time-like four-vectors which are restriction to p of a vector field in $\mathcal{O}$.*

Definition 8.5. Considering a time oriented Minkowski spacetime $(\mathbb{M}^4, \mathcal{O})$ and an event $p \in \mathbb{M}^4$, the half open cone at p containing four-vectors which are restrictions to p of vector fields in $\mathcal{O}$ is denoted by V_p^+ and is called **future open light cone** at p. Its elements are said **future-directed timelike four-vectors** at p. The lightlike four-vectors of ∂V_p^+ are equally said to be **future-directed lightlike four-vectors** at p. $\diamondsuit$

Remark. Henceforth we suppose to deal with a time oriented Minkowski spacetime and we indicate it by means of $\mathbb{M}^4$ simply.

8.1.3 Curves as stories.

If $I \subset \mathbb{R}$ is an open interval, a C^1 curve (def. 3.5) $\gamma : I \to \mathbb{M}^4$ may represent the story of a particle of matter provided its tangent vector $\dot{\gamma}$ (def. 3.6) is causal and future directed. A particle of light, in particular, has a story represented by a curve with future-directed lightlike tangent vector.

Definition 8.6. A regular curve $\gamma : I \to \mathbb{M}^4$ is said to be **timelike**, **spacelike**, **causal**, **lightlike** if all of its tangent vectors $\dot{\gamma}_p$, $p \in \gamma$, are respectively timelike, spacelike, causal, lightlike.
A causal, timelike, lightlike curve γ with future oriented tangent vectors is said to be **future-directed (resp. causal, timelike, lightlike) curve**.$\diamondsuit$

Definition 8.7. If $\gamma = \gamma(\xi)$, $\xi \in I$ with $I \subset \mathbb{R}$ any interval of $\mathbb{R}$, is any C^1 future-directed causal curve in $\mathbb{M}^4$ and $\xi_0 \in I$, the length function

$$\tau(\xi) := \int_{\xi_0}^{\xi} \sqrt{|g(\dot{\gamma}(l), \dot{\gamma}(l))|}dl \tag{8.7}$$

is called **proper time** of γ. Moreover, for future-directed timelike curves, the tangent vector obtained by re-parameterizing the curve with the proper time is called **four-velocity** of the curve.$\diamondsuit$

Notice that proper time is independent from the used parametrization of γ but it is defined up to the choice of the origin: the point on γ where $\tau = 0$.
From the point of view of physics, when one considers timelike curves, proper time is time measured by an ideal clock at rest with the particle whose story is γ.
For timelike curves proper time can be used as natural parameter to describe the story γ of

the particle because, in that case, directly from (8.7), $d\tau/d\xi > 0$. Notice also that, if $\dot{\gamma}$ is a four-velocity (8.7) implies that:

$$\mathbf{g}(\dot{\gamma}, \dot{\gamma}) = -1 \,, \tag{8.8}$$

that is a four-velocity is *unitary*.
Future-directed causal curves which are straight lines with respect to the affine structure of $\mathbb{M}^4$ (and so they are also complete geodesics for the metric g) are thought to be the stories of particles not subjected to forces. These are particles evolving with inertial motion.

8.1.4 Minkowskian reference frames.

If x^0, x^1, x^2, x^3 are Minkowskian coordinates, the curves tangent to x^0 determine a constant (with respect to Levi-Civita connection associated with g) vector field ∂_{x^0} which is unitary, that is it satisfies $\mathbf{g}(\partial_{x^0}, \partial_{x^0}) = -1$. A unitary constant timelike vector field can always be viewed as the tangent vector to the coordinate x^0 of a Minkowskian coordinate frame. There are anyway *several* Minkowskian coordinate frames for a unitary constant timelike vector field.

Definition 8.8. A constant timelike vector field $\mathscr{F}$ on $\mathbb{M}^4$ is said a **(Minkowskian) reference frame** provided it is future oriented and unitary, i.e. $\mathbf{g}(\mathscr{F}, \mathscr{F}) = -1$. Furthermore the following definitions hold.
(a) Any Minkowskian coordinate frame such that $\partial_{x^0} = \mathscr{F}$, is said to be **co-moving with** (or equivalently **adapted to**) $\mathscr{F}$.
(a) If $\gamma = \gamma(\xi)$, where $\xi \in \mathbb{R}$, is an integral curve of $\mathscr{F}$ (and thus it is both a geodesic and a affine straight line), the length function

$$t_{\mathscr{F}}(\gamma(\xi)) := \int_0^\xi \sqrt{|\mathbf{g}(\mathscr{F}_{\gamma(l)}, \mathscr{F}_{\gamma(l)})|}dl$$

defines a **time coordinate for $\mathscr{F}$ (along γ)**.$\diamondsuit$

For a fixed $\mathscr{F}$, an adapted Minkowskian coordinate frame can be obtained by fixing a three-dimensional affine plane Σ orthogonal to γ in O, and fixing an orthonormal basis in $T_p\mathbb{M}$ such that $\mathbf{e}_0 = \mathscr{F}$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ belong to Σ. Now one proves straightforwardly that the Minkowskian coordinate frame associated with O, $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is co-moving with $\mathscr{F}$ and, along γ, x^0 coincides with $t_{\mathscr{F}}$ up an additive constant. Notice that $t_{\mathscr{F}}$ defines in this way a *global temporal coordinate* associated with $\mathscr{F}$ since one may attribute a time coordinate $t_{\mathscr{F}} := x^0$ to every event of $\mathbb{M}^4$, not only to those of $\mathscr{F}$. By construction, starting from a different integral curve γ' of $\mathscr{F}$, one gets the same global temporal coordinate up to an additive constant. Moreover, if a time coordinate $t_{\mathscr{F}}$ is defined for $\mathscr{F}$, every three-dimensional affine plane orthogonal to $\mathscr{F}$ contains only points with constant value of that time coordinate.

Definition 8.9. Let $\mathscr{F}$ be a reference frame in $\mathbb{M}^4$ and t an associated time coordinate. Any three-dimensional affine plane $\Sigma_t^{(\mathscr{F})}$ orthogonal to $\mathscr{F}$, $t \in \mathbb{R}$ being the value of the time coordinate of the points in the plane, is called **rest space of $\mathscr{F}$ at time t**. $\diamondsuit$

By construction every rest space $\Sigma_t^{(\mathscr{F})}$ of $\mathscr{F}$ is an embedded three-dimensional submanifold of $\mathbb{M}^4$ and the metric induced by $\mathbf{g}$ on $\Sigma_t^{(\mathscr{F})}$ is positive defined.
From the point of view of physics a Minkowski frame is nothing but an inertial reference frame, any time coordinate is time measured by ideal clocks and the metric induced by $\mathbf{g}$ on the rest frames is the mathematical tool corresponding to physical spatial measurements performed by rigid rulers.
The mathematical picture is sufficiently developed to allows one to define the notion of velocity of a particle represented by a C^1 future-directed causal curve γ, with respect to a reference frame $\mathscr{F}$ at time t. The procedure is straightforward. Consider the event $e := \Sigma_t^{(\mathscr{F})} \cap \gamma$ where the curve intersect the rest frame $\Sigma_t^{(\mathscr{F})}$ (the reader should prove that each rest frame intersects γ exactly in a point). $T_e\mathbb{M}^4$ is decomposed to the orthogonal direct sum

$$T_e\mathbb{M}^4 = L(\mathscr{F}_e) \oplus T_e\Sigma_t^{(\mathscr{F})} , \tag{8.9}$$

$L(\mathscr{F}_e)$ being the linear space spanned by the four-vector $\mathscr{F}_e$. As a consequence $\dot{\gamma}_e$ turns out to be uniquely decomposed as

$$\dot{\gamma}_e = \delta t \mathscr{F}_e + \delta X , \tag{8.10}$$

where $\delta X \in T_e\Sigma_t^{(\mathscr{F})}$ and $\delta t \in \mathbb{R}$. The fact that $\dot{\gamma}_e$ is causal prevents δt from vanishing (the reader should prove it), so that it makes sense to give the following definition of velocity.

Definition 8.10. Let γ be a C^1 future-directed causal curve and $\mathscr{F}$ a reference frame in $\mathbb{M}^4$. The **velocity of γ with respect to $\mathscr{F}$ at time** t is the four-vector $\mathbf{v}_{\gamma,t}^{(\mathscr{F})} \in T_{\Sigma_t^{(\mathscr{F})}\cap\gamma}\Sigma_t^{(\mathscr{F})}$, given by

$$\mathbf{v}_{\gamma,t}^{(\mathscr{F})} := \frac{\delta X}{\delta t} .$$

referring to (8.9) and (8.10) with $e := \Sigma_t \cap \gamma$.$\diamondsuit$

Within our framework one has the following physically well-known result.

Proposition 8.3. *Consider a future-directed causal curve γ and fix a reference frame $\mathscr{F}$,*
(a) *The absolute value of $\mathbf{v}_{\gamma,t}^{(\mathscr{F})}$ is bounded up by 1 and that value is attained only at those events along γ where the tangent vector of γ is lightlike.*
(b) *If γ is timelike and $\dot{\gamma}(t)$ is the four-velocity of γ at time t of $\mathscr{F}$:*

$$\dot{\gamma}(t) = \frac{1}{\sqrt{1-{\mathbf{v}_{\gamma,t}^{(\mathscr{F})}}^2}}\,\mathscr{F} + \frac{\mathbf{v}_{\gamma,t}^{(\mathscr{F})}}{\sqrt{1-{\mathbf{v}_{\gamma,t}^{(\mathscr{F})}}^2}} . \tag{8.11}$$

(c) *If γ intersects the rest spaces $\Sigma_{t_1}^{(\mathscr{F})}$ and $\Sigma_{t_2}^{(\mathscr{F})}$ with $t_2 > t_1$ at proper time τ_1 and $\tau_2 > \tau_1$ respectively, the corresponding intervals of time satisfy*

$$\Delta\tau = \int_{t_1}^{t_2} \sqrt{1 - \mathbf{v}_{\gamma,t}^{(\mathscr{F})^2}}\, dt \tag{8.12}$$

and so $\Delta\tau < \Delta t$ unless $\mathbf{v}_{\gamma,t}^{(\mathscr{F})} = 0$ in the considered interval of time and, in that case, $\Delta\tau = \Delta t$.

Comments 8.1.
(1) The absolute value $||\mathbf{v}_{\gamma,t}^{(\mathscr{F})}||$ is referred to the scalar product induced by g in the rest spaces of $\mathscr{F}$. As previously said this is the physical metric tool which corresponds to perform measurements.
(2) The absolute value of the velocity of light, as it corresponds the $||\mathbf{v}_{\gamma,t}^{(\mathscr{F})}||$ evaluated along curves with $\dot{\gamma}_e = \delta t \mathscr{F}_e + \delta X$ such that $\mathbf{g}(\dot{\gamma}_e, \dot{\gamma}_e) = 0$, turns out to have the value 1 with respect to *every* reference frame. This value is usually denoted, changing the units of measure, by c.
(3) One recognizes in (c) the mathematical formulation of the celebrated relativistic phenomenon called *dilatation of time*.

Proof of proposition 8.3 . Using definition 8.10, decomposition (8.10), the orthogonality of δX and $\mathscr{F}$ and the fact that $\mathscr{F}$ is unitary, one has

$$0 \geq \mathbf{g}(\dot{\gamma}, \dot{\gamma}) = -\delta t^2 + g(\delta X, \delta X) = -\delta t^2 + ||\delta X||^2 \,.$$

The the sign = occurs only if $\dot{\gamma}$ is lightlike. This implies the thesis (a) immediately. (b) follows immediately from (8.10) and the definition of $\mathbf{v}_{\gamma,t}^{(\mathscr{F})}$ if imposing $\mathbf{g}(\dot{\gamma}, \dot{\gamma}) = -1$. Finally (c) is a straightforward consequence of (b) since the the factor in front of $\mathscr{F}$ in (8.11) is $dt/d\tau$. That is $dt/d\tau = \frac{1}{\sqrt{1-\mathbf{v}_{\gamma,t}^{(\mathscr{F})^2}}}$. $\Box$

8.2 Lorentz and Poincaré groups.

The following natural question arises: if $\mathscr{F}$ and $\mathscr{F}'$ are two Minkowskian reference frames equipped with co-moving Minkowskian coordinates x^0, x^1, x^2, x^3 and x'^0, x'^1, x'^2, x'^3 respectively, what about the most general realtion between these different systems of coordinates?
From now on we exploit again the convention of summation over repeated indices. Since both coordinate frame are Minkowskian which, in turn, are Cartesian coordinate frames, the relation must be linear:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + T^\mu \,, \tag{8.13}$$

the requirement of non singularity of Jacobian determinant is obviously equivalent to non singularity of the matrix Λ of coefficient $\Lambda^\mu{}_\nu$. Finally, the requirement that in both coordinate system $\mathbf{g}$ must have the form (8.1), i.e.

$$\mathbf{g} = \eta_{\alpha\beta} dx^\alpha \otimes dx^\beta = \eta_{\mu\nu} dx'^\mu \otimes dx'^\nu \tag{8.14}$$

where we have introduced the matrix $\eta = diag(-1, +1, +1, +1)$ of coefficients $\eta_{\alpha\beta}$, leads to the requirement

$$\Lambda^t \eta \Lambda = \eta\,. \tag{8.15}$$

Notice that this relation implies the non singularity of matrices Λ because, taking the determinant of both sides one gets:

$$(\det \Lambda)^2 \det \eta = \det \eta\,,$$

where we exploited $\det(\Lambda^t\eta\Lambda) = \det(\Lambda^t)\det(\eta)\det(\Lambda) = \det(\Lambda)\det(\eta)\det(\Lambda)$. Since $\det \eta = -1$, it must be $\det \Lambda = \pm 1$. Proceeding backwardly one sees that if x'^0, x'^1, x'^2, x'^3 is a Minkowskian coordinate frame and (8.13) hold with Λ satisfying (8.15), then x^0, x^1, x^2, x^3 is a Minkowskian coordinate frame too. Summarizing one has the following straightforward result.

Proposition 8.4. *If x^0, x^1, x^2, x^3 and x'^0, x'^1, x'^2, x'^3 are Minkowskian coordinate frame on $\mathbb{M}^4$, the transformation rule between these coordinates has the form (8.13) where T^μ, $\mu = 0, 1, 2, 3$ are suitable reals and the matrix Λ satisfies (8.15).*
Conversely, if x'^0, x'^1, x'^2, x'^3 is a Minkowskian coordinate frame and (8.13) hold for arbitrary real constants T^μ, $\mu = 0, 1, 2, 3$ and an arbitrary matrix Λ satisfying (8.15), then x^0, x^1, x^2, x^3 is a another Minkowskian coordinate frame.

8.2.1 Lorentz group.

The result proved above allows one to introduce the celebrated Lorentz group. In the following, $M(4, \mathbb{R})$ will denote the algebra of real 4×4 matrices and $GL(4, \mathbb{R})$ (see section 4.1) indicates the group of real 4×4 matrices with non-vanishing determinant. Consider the set of matrices

$$O(1,3) \quad := \{\Lambda \in M(4, \mathbb{R}) \mid \Lambda^t\eta\Lambda = \eta\}\,. \tag{8.16}$$

It is is a subgroup of $GL(4, \mathbb{R})$. To establish it it is sufficient to verify that it is closed with respect to the multiplication of matrices, and this is trivial from (8.16) using the fact that $\eta\eta = I$, and that if $\Lambda \in O(1,3)$ also $\Lambda^{-1} \in O(1,3)$. The proof this last fact is consequence of (a), (b) and (c) in proposition 8.5 whose proofs is completely based on (8.16). We are now in place to give the following defintion.

Definition 8.11. (**Lorentz Group**.) The **Lorentz group** is the group of matrices, with group strucuture induced by that of $GL(4, \mathbb{R})$,

$$O(1,3) \quad := \{\Lambda \in M(4, \mathbb{R}) \mid \Lambda^t\eta\Lambda = \eta\}\,.$$

♢

The next technical proposition will allow us to introduce some physically relevant subgroups of $O(1,3)$ later.

Proposition 8.5. *The Lorentz group enjoys the following properties.*
(a) $\eta, -I, -\eta, \in O(1,3)$.
(b) $\Lambda \in O(1,3)$ *if and only if* $\Lambda^t \in O(1,3)$.
(c) *If* $\Lambda \in O(1,3)$ *then* $\Lambda^{-1} = \eta \Lambda^t \eta$.
(d) *If* $\Lambda \in O(1,3)$ *then* $\det \Lambda = \pm 1$. *In particular, if* $\Lambda, \Lambda' \in O(1,3)$ *and* $\det \Lambda = \det \Lambda' = 1$ *then* $\det(\Lambda\Lambda') = 1$ *and* $\det \Lambda^{-1} = 1$.
(e) *If* $\Lambda \in O(1,3)$ *then* $\Lambda^0{}_0 \geq 1$ *or* $\Lambda^0{}_0 \leq -1$. *In particular, if* $\Lambda, \Lambda' \in O(1,3)$ *and* $\Lambda^0{}_0 \geq 1$ *and* $\Lambda'^0{}_0 \geq 1$ *then* $(\Lambda\Lambda')^0{}_0 \geq 1$ *and* $(\Lambda^{-1})^0{}_0 \geq 1$.

Proof. The proof of (a) is immediate from (8.16) also using $\eta\eta = I$. To prove (b) we start from $\Lambda^t \eta \Lambda = \eta$. Since Λ is not singular, Λ^{-1} exists and one has $\Lambda^t \eta \Lambda \Lambda^{-1} = \eta \Lambda^{-1}$, namely $\Lambda^t \eta = \eta \Lambda^{-1}$. Therefore, applying η on the right $\Lambda^t = \eta \Lambda^{-1} \eta$. Finally applying $\Lambda\eta$ on the left one gets

$$\Lambda \eta \Lambda^t = \eta \,,$$

so $\Lambda^t \in O(1,3)$ if $\Lambda \in O(1,3)$.
To show (c) we notice that $\eta \Lambda^t \eta \in O(1,3)$ because this set is closed with respect to composition of matrices and $\eta, \Lambda^t \in O(1,3)$ for (a) and (b). Finally: $\Lambda(\eta\Lambda^t\eta) = (\lambda\eta\Lambda^t)\eta = \eta\eta = I$. Since every $\Lambda \in SO(1,3)$ is non singular as noticed below (8.15) we can conclude that $\eta\Lambda^t\eta = \Lambda^{-1}$ but also that $\Lambda^{-1} \in O(1,3)$ if $\Lambda \in O(1,3)$.
The first part of (d) has been proved previously. The remaining part is straightforward: $\det(\Lambda\Lambda') = (\det\Lambda)\cdot(\det\Lambda') = 1 \cdot 1 = 1$ and $\det(\Lambda^{-1}) = (\det\Lambda)^{-1} = (1)^{-1} = 1$.
Let us conclude the proof by demonstrating (e) whose proof is quite involved. The constraint $A^t \eta A = \eta$ gives :

$$(A^0{}_0)^2 = 1 + \sum_{\alpha=1}^{3} A^0{}_\alpha A^0{}_\alpha \,, \tag{8.17}$$

so that $A^0{}_0 \geq 1$ or $A^0{}_0 \leq -1$ if $A \in O(1,3)$. This proves the first statement of (e) if $\Lambda = A$. Let us pass to the second part. Suppose that $\Lambda, \Lambda' \in O(1,3)$ and $\Lambda^0{}_0 \geq 1$ and $\Lambda'^0{}_0 \geq 1$, we want to show that $(\Lambda\Lambda')^0{}_0 \geq 1$. Actually it is sufficient to show that $(\Lambda\Lambda')^0{}_0 > 0$ because of the first statement. We shall prove it.
We start from the identity

$$(\Lambda\Lambda')^0{}_0 = \Lambda^0{}_0 \Lambda'^0{}_0 + \sum_{\alpha=1}^{3} \Lambda^0{}_\alpha \Lambda'^\alpha{}_0 \,.$$

It can be re-written down as

$$(\Lambda\Lambda')^0{}_0 = \Lambda^0{}_0 (\Lambda'^t)^0{}_0 + \sum_{\alpha=1}^{3} \Lambda^0{}_\alpha (\Lambda'^t)^0{}_\alpha$$

Using Cauchy-Schwarz' inequality:

$$\left| \sum_{\alpha=1}^{3} \Lambda^0{}_\alpha (\Lambda'^t)^0{}_\alpha \right|^2 \leq \left(\sum_{\alpha=1}^{3} \Lambda^0{}_\alpha \Lambda^0{}_\alpha \right) \left(\sum_{\beta=1}^{3} (\Lambda'^t)^0{}_\beta (\Lambda'^t)^0{}_\beta \right) \,,$$

so that

$$\left|(\Lambda\Lambda')^0{}_0 - \Lambda^0{}_0(\Lambda'^t)^0{}_0\right|^2 \leq \left(\sum_{\alpha=1}^{3}\Lambda^0{}_\alpha\Lambda^0{}_\alpha\right)\left(\sum_{\beta=1}^{3}(\Lambda'^t)^0{}_\beta(\Lambda'^t)^0{}_\beta\right) . \tag{8.18}$$

(8.17) implies, for $A \in O(1,3)$,:

$$\sum_\alpha A^0{}_\alpha A^0{}_\alpha < (A^0{}_0)^2 .$$

Exploiting that inequality in (8.18) for $A = \Lambda, \Lambda'^t$ (using the fact that $O(1,3)$ is closed with respect to transposition of matrices), we obtains:

$$\left|(\Lambda\Lambda')^0{}_0 - \Lambda^0{}_0(\Lambda'^t)^0{}_0\right|^2 < (\Lambda^0{}_0)^2((\Lambda'^t)^0{}_0)^2 . \tag{8.19}$$

Since $\Lambda^0{}_0 \geq 0$ and $(\Lambda'^t)^0{}_0 = \Lambda'^0{}_0 \geq 0$ by hypotheses, we have

$$\left|(\Lambda\Lambda')^0{}_0 - \Lambda^0{}_0\Lambda'^0{}_0\right| < \Lambda^0{}_0\Lambda'^0{}_0 \tag{8.20}$$

that is

$$\Lambda^0{}_0\Lambda'^0{}_0 - \Lambda^0{}_0\Lambda'^0{}_0 < (\Lambda\Lambda')^0{}_0 < \Lambda^0{}_0\Lambda'^0{}_0 + \Lambda^0{}_0\Lambda'^0{}_0$$

and thus

$$0 < (\Lambda\Lambda')^0{}_0 < 2\Lambda^0{}_0\Lambda'^0{}_0 .$$

In particular $(\Lambda\Lambda')^0{}_0 > 0$ which is that we wanted to prove.
To prove the last statement, notice that, if ${\Lambda^0}_0 \geq 1$, from (c), ${(\Lambda^{-1})^0}_0 = {(\eta\Lambda^t\eta)^0}_0 = {\Lambda^0}_0 \geq 1$. □

8.2.2 Poincaré group.

Considering the complete transformation (8.13) we can introduce the celebrated Poincaré group, also called inhomogeneous Lorentz group. To do it, we notice that the set

$$IO(1,3) := O(1,3) \times \mathbb{R}^4 , \tag{8.21}$$

is a group when equipped with the composition rule

$$(\Lambda, T) \circ (\Lambda', T') := (\Lambda\Lambda', T + \Lambda T') . \tag{8.22}$$

The proof of it is immediate and it is left to the reader.

Definition 8.12. The **Poincaré group** or **inhomogeneous Lorentz group** is the set of matrices

$$IO(1,3) := O(1,3) \times \mathbb{R}^4 ,$$

equipped with the composition rule

$$(\Lambda, T) \circ (\Lambda', T') := (\Lambda\Lambda', T + \Lambda T') .$$

$\diamondsuit$

We make only a few remarks.

Comments 8.2.
(1) The composition rule (8.22) is nothing but that obtained by composing the two transformations of Minkowskian coordinates:

$$x_1^\mu = \Lambda^\mu{}_\nu x_2^\nu + T^\mu \quad \text{and } x_2^\nu = \Lambda'^\nu{}_\tau x_3^\tau + T'^\nu$$

obtaining

$$x_1^\mu = (\Lambda\Lambda')^\mu{}_\tau x_3^\tau + T^\mu + (\Lambda T')^\mu .$$

It is also evident that $IO(1,3)$ it is nothing but the semi-direct product (see subsection 4.1.2) of $O(1,3)$ and the group of spacetime displacements $\mathbb{R}^4$, $IO(1,3) \times_\psi \mathbb{R}^4$. In this case, for every $\Lambda \in O(1,3)$, the group isomorphism $\psi_\Lambda : \mathbb{R}^4 \to \mathbb{R}^4$ is nothing but $\psi_T := \Lambda T$.
(2) From a kinematic point of view it makes sense to define the velocity of $\mathscr{F}$ with respect to $\mathscr{F}'$ when they are connected by Poincaré transformation (Λ, T). The velocity turns out to be constant in space and time. Indeed, let $\mathscr{F}$ and $\mathscr{F}'$ be Minkowskian reference frames with associated co-moving Minkowskian coordinate frames x^0, x^1, x^2, x^3 and x'^0, x'^1, x'^2, x'^3 respectively and suppose that (8.13) hold. Let γ represent the story of a point *at rest* with respect to $\mathscr{F}$, that is, γ admits parametrization $x^i(\xi) = x_0^i$ constant for $i = 1,2,3$, $x^0 = x^0(\xi)$. The velocity of γ with respect to $\mathscr{F}'$ does not depend on x_0^i and it is constant in $\mathscr{F}'$-time so that, indicating it by $\mathbf{v}_{\mathscr{F}}^{(\mathscr{F}')}$, their components in coordinates x'^1, x'^2, x'^3 turns out to be:

$$\mathbf{v}_{\mathscr{F}}^{(\mathscr{F}')i} = \frac{\Lambda^i{}_0}{\Lambda^0{}_0} . \tag{8.23}$$

Indeed we have $x'^i(x^0) = \Lambda^i_0 x^0 + \sum_{j=1}^3 \Lambda^i_j x^j$ where the coordinates x^j are taken constant. Therefore $\frac{dx'^i}{dx^0} = \Lambda^i{}_0$. Similarly: $x'^0(x^0) = \Lambda^0{}_0 x^0 + \sum_{j=1}^3 \Lambda^0{}_j x^j$ where, once again, the coordinates x^j are taken constant, so that: $\frac{dx'^0}{dx^0} = \Lambda^i{}_0$. Putting all together we achieve (8.23).
(3) With the same proof as for (8.23), noticing that x^0 is the proper time of the story of any point at rest with $\mathscr{F}$, we immediately obtains that the components of the four-velocity of such a point computed in the Minkowskian coordinates of $\mathscr{F}'$ are

$$\gamma_{\mathscr{F}}^0 = \Lambda^0{}_0 , \quad \gamma_{\mathscr{F}}^i := \Lambda^i{}_0 , \quad i = 1,2,3 . \tag{8.24}$$

Chapter 9

Decomposition of Lorentz group.

The final goal of this chapter is to discuss the interplay of standard physical decomposition of Lorentz group in boost and spatial rotation and the polar decomposition theorem proved in chapter 7. We start by focussing again on the features of Lorentz and Poincaré groups, referring to the Lie group structure in particular.

Remark. We assume henceforth that the reader is familiar with the basic notions of matrix Lie groups [KNS].

9.1 Lie group structure, distinguished subgroups, and connected components of the Lorentz group.

We have a first elementary but important result concerning Lorentz group and its Lie group structure.

Proposition 9.1. *The Lorentz group $O(1,3)$ is a Lie group which is a Lie subgroup of $GL(4,\mathbb{R})$. Similarly, the Poincaré group $IO(1,3)$ is a Lie group which can be viewed as a Lie subgroup of $GL(5,\mathbb{R})$.*

Proof. From the general theory of Lie groups [KNS] we know that to show that $O(1,3)$ is a Lie subgroup of the Lie group of $GL(2,\mathbb{R})$ it is sufficient to prove that the former is a topologically-closed algebraic subgroup of the latter. The fact that $O(1,3)$ is a closed subset of $GL(2,\mathbb{R})$, where the latter is equipped with the topology (and the differentiable structure) induced by $\mathbb{R}^{16}$, is obvious from $\Lambda^t\eta\Lambda = \eta$ since the product of matrices and the transposition of a matrix are continuous operations. Concerning the second statement, it is possible to provide $IO(1,3)$ with the structure of Lie group which is also subgroup of $GL(5,\mathbb{R})$ and that includes $O(1,3)$ and $\mathbb{R}^4$

as Lie subgroups, as follows. One start with the injective map

$$IO(1,3) \ni (\Lambda, T) \mapsto \left[\begin{array}{c|c} 1 & 0 \\ \hline T & \Lambda \end{array}\right] \in GL(5,\mathbb{R})\,, \qquad (9.1)$$

and he verifies that the map is in fact an injective group homomorphism. The matrix group in the right hand side of (9.1) define, in fact a closed subset of $\mathbb{R}^{25}$ and thus of $GL(5,\mathbb{R})$. As a consequence this matrix group is a Lie group which is a Lie subgroup of $GL(5,\mathbb{R})$. $\Box$

Consider two Minkowskian frames $\mathscr{F}$ and $\mathscr{F}'$ and let x^0, x^1, x^2, x^3, x'^0, x'^1, x'^2, x'^3 be two respectively co-moving Minkowski coordinate frames. We know that $\partial_{x^0} = \mathscr{F}$ and $\partial_{x'^0} = \mathscr{F}'$, finally we know that $\mathscr{F}$ and $\mathscr{F}'$ must have the same time-orientation they being future directed. We conclude that it must be $g(\partial_{x^0}, \partial_{x'^0}) < 0$. Only transformations (8.13) satisfying that constraint may make sense physically speaking. From (8.13) one has $\partial_{x^0} = \Lambda^\mu{}_0 \partial_{x'^\mu}$, so that the requirement $g(\partial_{x^0}, \partial_{x'^0}) < 0$ is equivalent to $\Lambda_0{}^0 > 0$ which is, in turn, equivalent to $\Lambda_0{}^0 > 1$ because of the first statement in (e) in proposition 8.5. One expect that the Poincaré or Lorentz transformations fulfilling this constraint form a subgroup. Indeed this is the case and the group is called the *orthochronous subgroup*: it embodies all physically sensible transformations of coordinates between inertial frames in special relativity. The following proposition states that results introducing also two other relevant subgroups. The proof of the following proposition is immediate from (e) and (d) of proposition 8.5.

Proposition 9.2. *The subsets of $IO(1,3)$ and $O(1,3)$ defined by*

$$IO(1,3)\!\uparrow := \{(\Lambda, T) \in IO(1,3) \mid \Lambda^0{}_0 \geq 1\}\,, \quad O(1,3)\!\uparrow := \{\Lambda \in O(1,3) \mid \Lambda^0{}_0 \geq 1\}\,, \qquad (9.2)$$

and called respectively **orthochronous Poincaré group** *and* **orthochronous Lorentz group***, are (Lie) subgroups of $IO(1,3)$ and $O(1,3)$ respectively.*
The subsets of $IO(1,3)$ and $O(1,3)$ defined by

$$ISO(1,3) := \{(\Lambda, T) \in IO(1,3) \mid \det \Lambda = 1\}\,, \quad SO(1,3) := \{\Lambda \in O(1,3) \mid \det\Lambda = 1\}\,, \ (9.3)$$

and called respectively **proper Poincaré group** *and* **proper Lorentz group***, are (Lie) subgroups of $IO(1,3)$ and $O(1,3)$ respectively.*

We remark that the condition $\Lambda^0{}_0 \geq 1$ can be replaced with the equivalent constraint $\Lambda^0{}_0 > 0$, whereas the condition $\det\Lambda = 1$ can be replaced with the equivalent constraint $\det\Lambda > 0$. Since the intersection of a pair of (Lie) groups is a (Lie) group, we can give the following final definition.

Definition 9.1. The (Lie) subgroups of $IO(1,3)$ and $O(1,3)$ defined by

$$ISO(1,3)\!\uparrow := IO(1,3)\!\uparrow \cap ISO(1,3) \quad SO(1,3)\!\uparrow := O(1,3)\!\uparrow \cap SO(1,3) \qquad (9.4)$$

are called respectively **orthochronous proper Poincaré group** and **orthochronous proper Lorentz group**. $\Diamond$

To conclude the short landscape of properties of Lorentz group initiated in the previous chapter, we state and partially prove (the complete proof needs a result we shall achieve later) the following proposition about connected components of Lorentz group. These are obtained by starting from $SO(1,3)\uparrow$ and transforming it under the left action of the elements of discrete subgroup of $O(1,3)$: $\{I, \mathscr{P}, \mathscr{T}, \mathscr{P}\mathscr{T}\}$ where $\mathscr{T} := \eta$ and $\mathscr{P} := -\eta$ (so that $\mathscr{P}\mathscr{T} = \mathscr{T}\mathscr{P} = -I$, $\mathscr{P}\mathscr{P} = \mathscr{T}\mathscr{T} = I$). In this context $\mathscr{T}$ is called *time reversal* operator – since it changes the time orientation of causal four-vectors – and $\mathscr{P}$ is also said to be the *(parity) inversion* operator – since it corresponds to the spatial inversion in the rest space.

Proposition 9.3. *$SO(1,3)$ admits four connected components which are respectively, with obvious notation, $SO(1,3)\uparrow$, $\mathscr{P}SO(1,3)\uparrow$, $\mathscr{T}SO(1,3)\uparrow$, $\mathscr{P}\mathscr{T}SO(1,3)\uparrow$. Only the first is a subgroup.*

Proof. By construction: (1) if $\Lambda \in \mathscr{P}SO(1,3)\uparrow$, both $\det\Lambda = -1$ and $\Lambda^0{}_0 \geq 1$, (2) if $\Lambda \in \mathscr{T}SO(1,3)\uparrow$, both $\Lambda^0{}_0 \leq -1$ and $\det\Lambda = 1$, (3) if $\Lambda \in \mathscr{P}\mathscr{T}SO(1,3)\uparrow$, both $\Lambda^0{}_0 \leq -1$ and $\det\Lambda = -1$.
Thus the last statement is an immediate consequence of the fact that, as I satisfies $\det I = 1$ and $(I)^0_0 = 1$, it cannot belong to the three sets by construction $\mathscr{P}SO(1,3)\uparrow$, $\mathscr{T}SO(1,3)\uparrow$, $\mathscr{P}\mathscr{T}SO(1,3)\uparrow$. Assume that $SO(1,3)\uparrow$ is connected. We shall prove it later. Since the maps $O(1,3) \ni \Lambda \mapsto \mathscr{T}\Lambda$ and $O(1,3) \ni \Lambda \mapsto \mathscr{P}\Lambda$ are continuous, they transform connected sets to connected sets. As a consequence $\mathscr{P}SO(1,3)\uparrow$, $\mathscr{T}SO(1,3)\uparrow$, $\mathscr{P}\mathscr{T}SO(1,3)\uparrow$ are connected sets. To conclude, it is sufficient to prove that the considered sets are pair-wisely disconnected. To this end it is sufficient to exhibit continuous real-valued function defined on $O(1,3)$ which cannot vanish but they change their sign passing from a set to the other[1]. By construction two functions are sufficient $O(1,3) \ni \Lambda \mapsto \det\Lambda$ and $O(1,3) \ni \Lambda \mapsto \Lambda^0{}_0$. $\Box$

9.2 Spatial rotations and boosts.

The final aim of this chapter is to state and prove the decomposition theorem for the group $SO(1,3)\uparrow$. To this end, two ingredients have to be introduced: spatial rotations and boosts, which are distinguished types of Lorentz transforms in $SO(1,3)\uparrow$.

9.2.1 Spatial rotations.

Consider a reference frame $\mathscr{F}$, what is the relation between two Minkowskian coordinate frames x^0, x^1, x^2, x^3 and x'^0, x'^1, x'^2, x'^3 both co-moving with $\mathscr{F}$? These transformations are called **internal** to $\mathscr{F}$. The answer is quite simple. The class of all internal Poincaré transformations is completely obtained by imposing the further constraint $\partial_{x^0} = \partial_{x'^0} (= \mathscr{F})$ on the equations

[1]Indeed, assuming that X is a topological space and $f : X \to \mathbb{R}$ a continuous function, if X is connected $f(X)$ is so. Therefore, if $a, b \in f(X)$ with $a < 0$ and $b > 0$, then $f(X)$ is a connected subset of $\mathbb{R}$ including a, b. The connected subsets of $\mathbb{R}$ are the intervals, so that $f(X)$ has to contain all reals between $a < 0$ and $b > 0$, in particular $0 \in f(X)$. If f cannot vanish, X cannot be connected.

(8.13) and assuming that $\Lambda \in O(1,3)\uparrow$.
Since $\partial_{x^0} = \Lambda^\mu{}_0 \partial_{x'^\mu}$, the constraint is equivalent to impose $\Lambda^0{}_0 = 1$ and $\Lambda^i{}_0 = 0$ for $i = 1, 2, 3$. Lorentz condition $\Lambda^t \eta \Lambda = \eta$ implies in particular that:

$$(\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3} \Lambda^0{}_i \Lambda^0{}_i \,,$$

and thus, since $\Lambda^0{}_0 = 1$, we find that $\Lambda^0{}_i = 0$. Summing up, internal Lorentz transformations must have the form

$$\Omega_R = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right] , \tag{9.5}$$

By direct inspection one finds that, in this case, $\Omega^t \eta \Omega = \eta$ reduces to

$$R^t R = I \,, \tag{9.6}$$

This is nothing but the equation determining the orthogonal group $O(3)$. Conversely, starting from any matrix $R \in O(3)$ and thus satisfying (9.6), and defining Ω_R as in (9.5), it is immediate to verify that $\Omega_R \in O(1,3)\uparrow$ and

$$x'^\mu = (\Omega_R)^\mu{}_\nu (x^\nu + T^\nu) \,,$$

with $T \in \mathbb{R}^4$ fixed arbitrarily, is an internal Poincaré transformation. It is immediate to show also that Ω_R with form (9.5) belongs to $SO(1,3)\uparrow$ if and only if $R \in SO(3)$, the group of special rotations made of rotations of $O(3)$ with unitary determinant.

Comments 9.1. It is worthwhile noticing, from a kinematic point of view that, the velocity of $\mathscr{F}$ with respect to $\mathscr{F}'$ seen in (3) in comments 8.2 is invariant under changes of co-moving Minkowskian coordinates when the transformations of coordinates are internal to $\mathscr{F}$ and $\mathscr{F}'$. We leave the simple proof of this fact to the reader.

Definition 9.2. The Lorentz transformations Ω_R defined in (9.5) with $R \in O(3)$ are called **spatial rotations**. If $R \in SO(3)$, Ω_R is called **spatial proper rotations**. $\diamondsuit$

Since the translational part is trivial, from now on we will focus on the Lorentz part of Poincaré group only.

9.2.2 Lie algebra analysis.

Focusing on the Lorentz group, we wish to extract the *non internal* part of a Lorentz transformation, i.e. what remains after one has taken spatial rotations into account. This goal will

be achieved after a preliminary analysis of the Lie algebra of $SO(1,3)\uparrow$ and the corresponding exponentiated operators. As $SO(1,3)\uparrow$ is a matrix Lie group which is Lie subgroup of $GL(4,\mathbb{R})$, its Lie algebra can be obtained as a Lie algebra of matrices in $M(4,\mathbb{R})$ with the commutator $[\cdot,\cdot]$ given by the usual matrix commutator. The topology and the differential structure on $SO(1,3)\uparrow$ are those induced by $\mathbb{R}^{16}$.
As the maps $O(1,3) \ni \Lambda \mapsto \det\Lambda$ $O(1,3) \ni \Lambda \mapsto \Lambda^0{}_0$ are continuous and $detI = 1$ and $(I)^0{}_0 > 0$, every $\Lambda \in O(1,3)$ sufficiently close to I must belong to $SO(1,3)\uparrow$ and *viceversa*. Hence the Lie algebra of $O(1,3)$, $o(1,3)$, coincides with that of its Lie subgroup $SO(1,3)\uparrow$ because it is completely determined by the behavior of the group in an arbitrarily small neighborhood of the identity.

Proposition 9.4. *The Lie algebra of $SO(1,3)\uparrow$ admits a vector basis made of the following 6 matrices called* **boost generators** K_1, K_2, K_3 *and* **spatial rotation generators** S_1, S_2, S_3*:*

$$K_1 = \left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] , \quad K_2 = \left[\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] , \quad K_3 = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] . \tag{9.7}$$

$$S_i = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & T_i & \\ 0 & & & \end{array}\right] \quad \textit{with} \;\; T_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} , \; T_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} , \; T_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9.8}$$

These generators enjoy the following commutation relations, which, as a matter of facts, determines the structure tensor of $o(1,3)$:

$$[S_i, K_j] = \sum_{k=1}^{3} \epsilon_{ijk} K_k \, , \quad [S_i, S_j] = \sum_{k=1}^{3} \epsilon_{ijk} S_k \quad [K_i, K_j] = -\sum_{k=1}^{3} \epsilon_{ijk} S_k \, . \tag{9.9}$$

Above ϵ_{ijk} is the usual completely antisymmetric Ricci indicator with $\epsilon_{123} = 1$.

Proof. If a matrix $N \in M(4,\mathbb{R})$ is in $o(1,3)$, the generated one-parameter subgroup $\{e^{uN}\}_{u\in\mathbb{R}}$ in $GL(4,\mathbb{R})$ satisfies $(e^{uN})^t \eta e^{uN} = \eta$ for u in a neighborhood of 0, that is $e^{uN^t}\eta e^{uN} = \eta$ for the same values of u. Taking the derivative at $t = 0$ one gets the necessary condition

$$N^t\eta + \eta N = 0 \, . \tag{9.10}$$

These equations are also sufficient. Indeed, from standard properties of the exponential map of matrices, one has

$$\frac{d}{du}\left(e^{uN^t}\eta e^{uN} - \eta\right) = e^{uN^t}\left(N^t\eta + \eta N\right) e^{uN} \, .$$

Thus, the validity of (9.10) implies that $e^{uN^t}\eta e^{uN} - \eta = \text{constant}$. For $u = 0$ one recognizes that the constant is 0 and so $(e^{uN})^t\eta e^{uN} = \eta$ is valid (for every $u \in \mathbb{R}$). Eq (9.10) supplies 10 linearly independent conditions so that it determines a subspace of $M(4,\mathbb{R})$ with dimension 6. The 6 matrices $S_i, K_j \in M(4.\mathbb{R})$ are linearly independent and satisfy (9.10), so they are a basis for $o(1,3)$. The relations (9.9) can be checked by direct inspection. □

From now on $\mathbf{K}, \mathbf{S}, \mathbf{T}$ respectively denote the formal vector with components K_1, K_2, K_3, the formal vector with components S_1, S_2, S_3 and the formal vector with components T_1, T_2, T_3. $\mathbb{S}^2$ will indicate the sphere of three-dimensional unit vectors.
Generators S_1, S_2, S_3 individuates proper spatial rotations as stated in the following proposition.

Proposition 9.5. *The following facts about Proper rotations holds.*
(a) *Every proper spatial rotations has the form $\Omega_R = e^{\theta \mathbf{n}\cdot\mathbf{S}}$ – or equivalently $R = e^{\theta\mathbf{n}\cdot\mathbf{T}}$ for all $R \in SO(3)$ – with suitable $\theta \in \mathbb{R}$ and $\mathbf{n} \in \mathbb{S}^2$ depending on R.*
(b) *Every matrix $e^{\theta\mathbf{n}\cdot\mathbf{S}}$ with $\theta \in \mathbb{R}$ and $\mathbf{n} \in \mathbb{S}^2$ is a proper rotation Ω_R, and is associated with $R = e^{\theta\mathbf{n}\cdot\mathbf{T}} \in SO(3)$.*
(c) *The following equivalent identities hold true, for every $U \in SO(3)$, $\mathbb{N} \in \mathbb{S}^2$, $\theta \in \mathbb{R}$:*

$$\Omega_U e^{\theta\mathbf{n}\cdot S}\Omega_U{}^t = e^{\theta(U\mathbf{n})\cdot S}\,, \quad U e^{\theta\mathbf{n}\cdot T}U^t = e^{\theta(U\mathbf{n})\cdot T}\,. \tag{9.11}$$

The latter holds, more generally, also if $U \in SL(3,\mathbb{C})$.
(d) *The Lie group of the spatial proper rotations $SO(3)$ is connected, but not simply connected, its fundamental group being $\pi_1(SO(1)) = \mathbb{Z}_2$.*

Proof. (a) and (c). Since, from the given definitions,

$$e^{i\theta\mathbf{n}\cdot S} = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & e^{\mathbf{n}\cdot\mathbf{T}} & \\ 0 & & & \end{array}\right], \tag{9.12}$$

it is obvious that $\Omega_R = e^{\theta\mathbf{n}\cdot\mathbf{S}}$ are completely equivalent $R = e^{\theta\mathbf{n}\cdot\mathbf{T}}$, so we deal with the latter.
If $R \in SO(3)$, the induced operator in $\mathbb{R}+i\mathbb{R}$ is unitary and thus it admits a base of eigenvectors with eigenvalues λ_i with $|\lambda_i| = 1$, $i = 1,2,3$. As the characteristic polynomial of R is real, an eigenvalue must be real, the remaining pair of eigenvalues being either real or complex and conjugates. Since $\det R = \lambda_1\lambda_2\lambda_3 = 1$, 1 is one of the eigenvalues. If another eigenvalue coincides with 1 all three eigenvalues must do it and $R = I$. In this case every non-vanishing real vector is an eigenvector of R. Otherwise, the eigenspace of $\lambda = 1$ must be one-dimensional and thus, as R is real, it must contain a real eigenvector. We conclude that, in every case, R has a real normalized eigenvector $\mathbf{n}$ with eigenvalue 1. Consider an orthonormal basis $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3 := \mathbf{n}$, related with the initial one by means of $R' \in SO(3)$, and represent R in the new basis. Imposing the requirement that $\mathbf{n}_3$ is an eigenvector with eigenvalue 1 as well as that the represented transformation belong to $SO(3)$, one can easily prove that, in such a base, R

is represented by the matrix $e^{\theta T_3}$ for some $\theta \in [0, 2\pi]$. In other words, coming back to the initial basis, $R = R' e^{\theta T_3} R'^t$ for some $R' \in SO(3)$. Now notice that $(T_i)_{jk} = -\epsilon_{ijk}$. This fact entails that $\sum_{i,j,k} U_{pi} U_{qj} U_{rk} \epsilon_{ijk} = \epsilon_{pqr}$ for all $U \in SL(3, \mathbb{C})$. That identity can be re-written as $\mathbf{n} \cdot U \mathbf{T} U^t = (U\mathbf{n}) \cdot \mathbf{T}$ for every $U \in SL(3, \mathbb{C})$. By consequence, if $U \in SO(3)$ in particular, it also holds $U e^{\theta \mathbf{n} \cdot \mathbf{T}} U^t = e^{\theta (U\mathbf{n}) \cdot \mathbf{T}}$. (This proves the latter in (9.11), the former is a trivial consequence of the given definitions). Therefore, the identity found above for any $R \in SO(3)$, $R = R' e^{\theta T_3} R'^t$ with $R' \in SO(3)$, can equivalently be written as $R = e^{\theta \mathbf{n} \cdot \mathbf{T}}$ for some versor $\mathbf{n} = R' \mathbf{e}_3$.
(b) Finally, every matrix $e^{\theta \mathbf{n} \cdot \mathbf{T}}$ belongs to $SO(3)$ because $(e^{\theta \mathbf{n} \cdot \mathbf{T}})^t = e^{\theta \mathbf{n} \cdot \mathbf{T}^t} = e^{-\theta \mathbf{n} \cdot \mathbf{T}} = (e^{\theta \mathbf{n} \cdot \mathbf{T}})^{-1}$ and $\det e^{\theta \mathbf{n} \cdot \mathbf{T}} = e^{\theta \mathbf{n} \cdot \mathrm{tr}\, \mathbf{T}} = e^0 = 1$.
(e) We sketch here the idea of the proof only. First consider the subgroup made of marices $e^{\theta \mathbf{e}_3 \cdot \mathbf{T}}$ with $\theta \in \mathbb{R}$. By the explicit form of these matrices, i.e.

$$T_3 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{9.13}$$

one see that, all these matrices are biunivocally determined by couples $(\theta, \mathbf{n}) \in [0, \pi] \times \{-\mathbf{e}_3, \mathbf{e}_3\}$ with the only exception given by the pairs $(\pi, -\mathbf{e}_3)$ and $(\pi, \mathbf{e}_3)$ which individuate the same rotation. This result generalizes to the subgroup $\{e^{\theta \mathbf{n} \cdot \mathbf{T}}\}_{\theta \in \mathbb{R}}$ for every fixed $\mathbf{n} \in \mathbb{S}^2$, in view of (c). From now on we restrict θ to range in $[0, \pi]$. By direct inspection, reducing to the case $\mathbf{n} = \mathbf{e}_3$ using (c) again, one sees that, $e^{\theta \mathbf{n} \cdot \mathbf{T}} = e^{\theta' \mathbf{n}' \cdot \mathbf{T}}$ is possible only if $\mathbf{n} = \mathbf{n}'$ and $\theta = \theta'$, with the only exception $\theta = \theta' = \pi$, where also $\mathbf{n} = -\mathbf{n}'$ is allowed as a consequence of the analysis above. The proof of this fact is an immediate consequence of the fact that $\mathbf{n}'$ is an eigenvector (with eigenvalue 1) of $e^{\theta' \mathbf{n}' \cdot \mathbf{T}}$, so that, if $e^{\theta \mathbf{e}_3 \cdot \mathbf{T}} = e^{\theta' \mathbf{n}' \cdot \mathbf{T}}$ $\mathbf{n}'$ must be an eigenvector of the matrix (9.13) (with eingenvalue 1). As a conclusion we obtain that $SO(3)$ in biunivocally defined by the points of a set B constructed as follows. B is the ball in $\mathbb{R}^3$ with radius π, where each two points on the surface of the ball which belong to the same diameter are identified (in other words, the pairs $(\pi, \mathbf{n})$ and $(\pi, -\mathbf{n})$ individuate the same element of $SO(3)$ as we said above). A closer scrutiny proves that this one-to-one correspondence is actually a homeomorphism when B is endowed with the natural topology indiced by $\mathbb{R}^3$ and the said identifications. The group of continuous closed paths in B is $\mathbb{Z}_2$ as one may symply prove. $\Box$

9.2.3 Boosts.

From the analysis performed above, we conclude that what remains of a Lorentz transformation in $SO(1,3)\uparrow$, one one has taken spatial rotations into account, are transformations obtained by exponentiating the generators $\mathbf{K}$.

Definition 9.3. The elements of $SO(1,3)\uparrow$ with the form $\Lambda = e^{\chi \mathbf{m} \cdot \mathbf{K}}$, with $\chi \in \mathbb{R}$ and $\mathbf{m} \in \mathbb{S}^2$, are called **boosts** or **pure transformations**.$\diamondsuit$

Let us investigate the basic properties of boosts. These are given by the following proposition.

Proposition 9.6. *The boost enjoy the following properties.*
(a) *All matrices $\Lambda = e^{\chi \mathbf{m}\cdot\mathbf{K}}$ with arbitrary $\chi \in \mathbb{R}$ and $\mathbf{m} \in \mathbb{S}^2$ belong to $SO(1,3)\uparrow$ and thus are boosts.*
(b) *For every pair $\mathbf{m} \in \mathbb{S}^1$, $\chi \in \mathbb{R}$ and every $R \in SO(3)$ one has*

$$\Omega_R e^{\chi \mathbf{m}\cdot\mathbf{K}} \Omega_R{}^t = e^{\chi (R\mathbf{m})\cdot\mathbf{K}} . \tag{9.14}$$

(c) *For $\mathbf{n} \in \mathbb{S}^2$, the explicit form of $e^{\chi \mathbf{n}\cdot\mathbf{K}}$ reads:*

$$e^{\chi \mathbf{n}\cdot\mathbf{K}} = \left[\begin{array}{c|c} \cosh\chi & (\sinh\chi)\mathbf{n}^t \\ \hline (\sinh\chi)\mathbf{n} & I - (1-\cosh\chi)\mathbf{n}\mathbf{n}^t \end{array}\right] , \tag{9.15}$$

(d) *Every boost is symmetric and (strictly) positive defined.*
(e) *For every fixed $\mathbf{m} \in \mathbb{S}^2$, $\{e^{\chi \mathbf{m}\cdot\mathbf{K}}\}_{\chi\in\mathbb{R}}$ is a subgroup of $SO(1,3)$. However, the set of all the boost transformations is not a subgroup of $SO(1,3)$ and it is homeomorphic to $\mathbb{R}^3$ when equipped with the topology induced by $SO(1,3)\uparrow$.*

Proof. (a) It has been proved in the proof of proposition 9.4, taking into account that $N := \chi \mathbf{m}\cdot\mathbf{K} \in o(1,3)$.
(b) Fix $\mathbf{n} \in \mathbb{S}^2$ and $j = 1,2,3$. Now, for $i = 1,2,3$ define the functions

$$f_j(\theta) := e^{\theta \mathbf{n}\cdot S} K_j e^{\theta \mathbf{n}\cdot S} , \quad g_j(\theta) := \sum_{k=1}^{3} \left(e^{\theta \mathbf{n}\cdot T}\right)_{jk} K_k .$$

Taking the first derivative in θ and using both (9.9) and the explicit form of the matrices T_h, one finds that the smooth functions f_k and the smooth functions g_k satisfies the same system of differential equation of order 1 written in normal form: for $j = 1,2,3$,

$$\frac{df_j}{d\theta} = \sum_{k=1}^{3} n_k \epsilon_{kjp} f_p(\theta) , \quad \frac{dg_j}{d\theta} = \sum_{k=1}^{3} n_k \epsilon_{kjp} g_p(\theta) .$$

Since $f_j(0) = g_j(0)$ for $j = 1,2,3$, we conclude that these functions coincide for every $\theta \in \mathbb{R}$:

$$e^{\theta \mathbf{n}\cdot S} K_j e^{\theta \mathbf{n}\cdot S} = \sum_{k=1}^{3} \left(e^{\theta \mathbf{n}\cdot T}\right)_{jk} K_k .$$

Now, by means of exponentiation we get (9.14) exploiting (c) of proposition 9.5.
(c) First consider the case $\mathbf{n} = \mathbf{e}_3$. In this case directly from Taylor's expansion formula

$$e^{\chi \mathbf{e}_3\cdot\mathbf{K}} = \left[\begin{array}{c|ccc} \cosh\chi & 0 & 0 & \sinh\chi \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\chi & 0 & 0 & \cosh\chi \end{array}\right] = \left[\begin{array}{c|c} \cosh\chi & (\sinh\chi)\mathbf{e}_3^t \\ \hline (\sinh\chi)\mathbf{e}_3 & I - (1-\cosh\chi)\mathbf{e}_3\mathbf{e}_3^t \end{array}\right] . \tag{9.16}$$

If $\mathbf{n} \in \mathbb{S}^2$ there is $R \in SO(3)$ such that $\mathbf{n} = R\mathbf{e}_3$. Using this R in (b) with $\mathbf{m} = \mathbf{e}_3$ one gets (9.15) with $\mathbf{n}$.
(d) Symmetry is evident from (9.15). Using (b), it is sufficient to prove positivity if $\mathbf{m} = \mathbf{e}_3$. In this case strictly positivity can be checked by direct inspection using (9.16).
(e) The first statement is trivial, since $\{e^{\chi\mathbf{m}\cdot\mathbf{K}}\}_{\chi\in\mathbb{R}}$ is a one-parameter subgroup of $SO(1,3)\uparrow$. By direct inspection using (9.15) for $\mathbf{n} = \mathbf{e}_1$ and $\mathbf{n} = \mathbf{e}_3$ one verifies that the product of these boosts cannot be represented as in (9.15). Concerning the last statement, one proves that $e^{\mathbf{u}\cdot\mathbf{K}} = e^{\mathbf{u}'\cdot\mathbf{K}}$ implies $\mathbf{u} = \mathbf{u}'$, reducing to study the simpler case $e^{\chi\mathbf{e}_3\cdot\mathbf{K}} = e^{\mathbf{u}'\cdot\mathbf{K}}$, making use of (b). This case can be examined by taking advantage of (9.16) and (9.15), yielding to $\chi\mathbf{e}_3 = \mathbf{u}'$ straightforwardly. Summarizing, $\mathbb{R}^3 \ni \mathbf{u} \mapsto e^{\mathbf{u}\cdot\mathbf{K}}$ is bijective onto the set of all boost transformations. Directly from (9.16) one see that the map $\mathbb{R}^3 \ni \mathbf{v} \to e^{\mathbf{v}\cdot\mathbf{K}} \in \mathbb{R}^{16}$ is a smooth embedding and individuates a submanifold of $\mathbb{R}^{16}$. A way to verify it is to identify the three components of $\mathbf{v} = (\sinh\chi)\mathbf{n}$ with the coordinates x^{ij}, when $i = 1$ and $j = 2, 3, 4$, in the space $\mathbb{R}^{16}$ of the real 4×4 matrices, and prove that the hypotheses of the theorem of regular values are valid for the equations $f^{ij} := x^{ij} - x^{ij}(x^{12}, x^{13}, x^{14}) = 0$, which determines the other 12 components of the boost $e^{\mathbf{v}\cdot\mathbf{K}} \in \mathbb{R}^{16}$, referring to the its explicit experssion as given in (9.16). In particular the Jacobian sub-matrix with elements $\{\partial f^{rs}/\partial x^{ij}\}_{r,s=1,\ldots,16, i=1, j=2,3,4}$ has range 3 and thus $(x^{12}, x^{13}, x^{14}) = \mathbf{v}$ are (global) admissible coordinates on the space of boosts, and the map $\mathbb{R}^3 \ni \mathbf{v} \to e^{\mathbf{v}\cdot\mathbf{K}} \in \mathbb{R}^{16}$, restricted to its image in the co-domain, define a diffeomorphism from $\mathbb{R}^3$ onto the space of the boost. Thus in particular, the set of the boosts is homeomorphic to $\mathbb{R}^3$ when endowed with the topology induced by $\mathbb{R}^{16}$. Since the topology of $SO(1,3)\uparrow$ is that induced by $\mathbb{R}^{16}$ itself, we conclude that the set of the boosts is homeomorphic to $\mathbb{R}^3$ when endowed with the topology induced by $SO(1,3)\uparrow$. $\Box$

Recalling (3) in comments 8.2, the kinematic meaning of parameters χ and $\mathbf{m}$ in boosts is clear from the following last proposition.

Proposition 9.7. *Let $\mathscr{F}$ and $\mathscr{F}'$ be Minkowskian reference with associated co-moving Minkowskian coordinate frames x^0, x^1, x^2, x^3 and x'^0, x'^1, x'^2, x'^3 respectively and suppose that (8.13) hold with*

$$\Lambda = e^{\chi\mathbf{m}\cdot\mathbf{K}} .$$

Let γ represent the story of a point at rest with respect to $\mathscr{F}$, that is, γ admits parametrization $x^i(\xi) = x^i_0$ constant for $i = 1, 2, 3$, $x^0 = x^0(\xi)$. The velocity of γ with respect to $\mathscr{F}'$ does not depend on x^i_0 and it is constant in $\mathscr{F}'$-time so that, indicating it by $\mathbf{v}^{(\mathscr{F}')}_{\mathscr{F}}$, it holds

$$\mathbf{v}^{(\mathscr{F}')}_{\mathscr{F}} = (\tanh\chi)\mathbf{m} .$$

Proof. The proof follows immediately from (9.15) and definition 8.10. $\Box$

Comments 9.2. Boosts along $\mathbf{n} := \mathbf{e}_3$ are know in the literature as "special Lorentz transformations" along z (an analogous name is given replacing $\mathbf{e}_3$ with $\mathbf{e}_1$ and $\mathbf{e}_2$).

9.2.4 Decomposition theorem for $SO(1,3)\!\uparrow$.

In the previous subsection we have determined two classes of transformations in $SO(1,3)\!\uparrow$: spatial pure rotations and boosts. It is natural to wonder if those transformations encompass, up to products of some of them, the whole group $SO(1,3)\!\uparrow$. The answer is positive and is stated in a theorem of decomposition of $SO(1,3)\!\uparrow$ we go to state. The proof of the theorem will be given in the next section using a nonstandard approach based on the so-called polar decomposition theorem.

Theorem 9.1. *Take $\Lambda \in SO(1,3)\!\uparrow$, the following holds.*
(a) *There is exactly one boost P and exactly a spatial pure rotation U (so that $P = e^{\chi \mathbf{m}\cdot\mathbf{K}}$, $U = e^{\theta \mathbf{m}\cdot\mathbf{S}}$ for some $\chi, \theta \in \mathbb{R}$ and $\mathbf{m}, \mathbf{n} \in \mathbb{S}^2$) such that*

$$\Lambda = UP\,.$$

(b) *There is exactly one boost P' and exactly a spatial pure rotation U' (so that $P' = e^{\chi' \mathbf{m}'\cdot\mathbf{K}}$, $U' = e^{\theta' \mathbf{m}'\cdot\mathbf{S}}$ for some $\chi', \theta' \in \mathbb{R}$ and $\mathbf{m}', \mathbf{n}' \in \mathbb{S}^2$) such that*

$$\Lambda = P'U'\,.$$

(c) *It holds $U' = U$ and $P' = UPU^t$.*

9.3 Proof of decomposition theorem by polar decomposition of $SO(1,3)\!\uparrow$.

If $\Lambda \in SO(1,3)\!\uparrow$ it can be seen as a linear operator in the finite-dimensional real vector space $\mathbb{R}^4$. Therefore one may consider the polar decomposition $\Lambda = PU = U'P'$ given in theorem 7.2. $U = U'$ are now orthogonal operators of $O(4)$ and P, P' are a symmetric positive operators. *A priori* those decompositions could be physically meaningless because U and P, P' could not to belong to $SO(1,3)\!\uparrow$: the notions of symmetry, positiveness, orthogonal group $O(4)$ are refereed to the positive scalar product of $\mathbb{R}^4$ instead of the indefinite Lorentz scalar product. Nevertheless we shall show that the polar decompositions of $\Lambda \in SO(1,3)\!\uparrow$ are in fact physically meaningful. Indeed, they coincides with the known decompositions of Λ in spatial-rotation and boost parts as in theorem 9.1.
Let us focus attention on the real vector space $V = \mathbb{R}^4$ endowed with the usual positive scalar product. In that case $\mathscr{L}(V|V) = M(4,\mathbb{R})$, orthogonal operators are the matrices of $O(4)$ and the adjoint $A^\dagger$ of $A \in \mathscr{L}(V|V)$ (see chapter 7) coincides with the transposed matrix A^t, therefore symmetric operators are symmetric matrices. We have the following theorem which proves theorem 9.1 as an immediate consequence.

Theorem 9.2. *If $UP = P'U = \Lambda$ (with $P' = UPU^t$) are polar decompositions of $\Lambda \in SO(1,3)\!\uparrow$:*

(a) $P, P', U \in SO(1,3)\uparrow$, *more precisely* P, P' *are boosts and* U *a spatial proper rotation;*
(b) *there are no other decompositions of* Λ *as a product of a Lorentz boost and a spatial proper rotation different from the two polar decompositions above.*

Proof. If $P \in M(4,\mathbb{R})$ we exploit the representation:

$$P = \left[\begin{array}{c|c} g & B^t \\ \hline & \\ C & A \\ & \end{array}\right], \tag{9.17}$$

where $g \in \mathbb{R}$, $B, C \in \mathbb{R}^3$ and $A \in M(3,\mathbb{R})$.
(a) We start by showing that $P, U \in O(1,3)$. As $P = P^t$, $\Lambda^t\eta\Lambda = \Lambda$ entails $PU^t\eta UP = \eta$. As $U^t = U^{-1}$ and $\eta^{-1} = \eta$, the obtained identity is equivalent to $P^{-1}U^t\eta UP^{-1} = \eta$ which, together with $PU^t\eta UP = \eta$, implies $P\eta P = P^{-1}\eta P^{-1}$, namely $\eta P^2\eta = P^{-2}$, where we have used $\eta = \eta^{-1}$ once again. Both sides are symmetric (notice that $\eta = \eta^t$) and positive by construction, by theorem 7.1 they admit unique square roots which must coincide. The square root of P^{-2} is P^{-1} while the square root of $\eta P^2\eta$ is $\eta P\eta$ since $\eta P\eta$ is symmetric positive and $\eta P\eta\eta P\eta = \eta PP\eta = \eta P^2\eta$. We conclude that $P^{-1} = \eta P\eta$ and thus $\eta = P\eta P$ because $\eta = \eta^{-1}$. Since $P = P^t$ we have found that $P \in O(1,3)$ and thus $U = \Lambda P^{-1} \in O(1,3)$. Let us prove that $P, U \in SO(1,3)\uparrow$. $\eta = P^t\eta P$ entails det $P = \pm 1$, on the other hand $P = P^t$ is positive and thus det $P \geq 0$ and $P^0{}_0 \geq 0$. As a consequence det $P = 1$ and $P^0{}_0 \geq 0$. We have found that $P \in SO(1,3)\uparrow$. Let us determine the form of P using (9.17). $P = P^t$, $P \geq 0$ and $P\eta P = \eta$ give rise to the following equations: $C = B$, $0 < g = \sqrt{1+B^2}$, $AB = gB$, $A = A^*$, $A \geq 0$ and $A^2 = I + BB^t$. Since $I + BB^t$ is positive, the solution of the last equation $A = \sqrt{A^2} = I + BB^t/(1+g) \geq 0$ is the unique solution by theorem 7.1. We have found that a matrix $P \in O(1,3)$ with $P \geq 0$, $P = P^*$ must have the form

$$P = \left[\begin{array}{c|c} \cosh\chi & (\sinh\chi)\mathbf{n}^t \\ \hline & \\ (\sinh\chi)\mathbf{n} & I - (1-\cosh\chi)\mathbf{n}\mathbf{n}^t \\ & \end{array}\right] = e^{\chi\mathbf{n}\cdot\mathbf{K}}\,, \tag{9.18}$$

where we have used the parameterization $B = (\sinh\chi)\mathbf{n}$, $\mathbf{n}$ being any versor in $\mathbb{R}^3$ and $\chi \in \mathbb{R}$. By (c) in proposition we have found that P is a boost. (The same proofs apply to P'.)
Let us pass to consider U. Since $\Lambda, P \in SO(1,3)\uparrow$, from $\Lambda P^{-1} = U$, we conclude that $U \in SO(1,3)\uparrow$. $U\eta = \eta(U^t)^{-1}$ (i.e. $U \in O(1,3)$) and $U^t = U^{-1}$ (i.e. $U \in O(4)$) entail that $U\eta = \eta U$ and thus the eigenspaces of η, E_λ (with eigenvalue λ), are invariant under the action of U. In those spaces U acts as an element of $O(\dim(E_\lambda))$ and the whole matrix U has a block-diagonal form. $E_{\lambda=-1}$ is generated by $\mathbf{e}_0$ and thus U reduces to $\pm I$ therein. The sign must be $+$ because of the requirement $U^0{}_0 > 0$. The eigenspace $E_{\lambda=1}$ is generated by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and therein U reduces to an element of $R \in O(3)$. Actually the requirement det $U = 1$ (together with $U^0{}_0 = 1$) implies that $R \in SO(3)$ and thus one has that $U = \Omega_R$ is a spatial pure rotation

as well.
(b) Suppose that $\Omega B = \Lambda \in SO(1,3)\uparrow$ where B is a pure boost and Ω is a spatial proper rotation. $\Omega \in O(4)$ by construction, on the other hand, from (d) of proposition 9.6: $B^t = B > 0$ Thus $\Omega B = \Lambda$ is a polar decomposition of Λ. Uniqueness of polar decomposition (theorem 7.2) implies that $\Omega = U$ and $B = P$. The other case is analogous.$\Box$

We want emphasize here a topological consequence of this theorem with the following proposition.

Proposition 9.8. *$SO(1,3)\uparrow$ is homeomorphic to $SO(3) \times \mathbb{R}^3$, as a consequence it is connected, arch-connected and $\pi_1(SO(1,3)\uparrow) = \mathbb{Z}_2$, so that it is not simply connected.*

Proof. Using the fact that $SO(1,3)\uparrow$ is bijectively identified with the product $SO(3)\times\mathbb{R}^3$, where the $\mathbb{R}^3$ is the space of the boosts as in (c) of proposition 9.6, it arises that $SO(1,3)\uparrow$ is homeomorphic to the topological space $SO(3) \times \mathbb{R}^3$. Indeed, the map $\mathbb{R}^{16} \ni \Lambda \mapsto (U,P) \in \mathbb{R}^{16} \times \mathbb{R}^{16}$ is continuous since the functions used to construct the the factors U and P of the polar decomposition of a nonsingular matrix are continuous in the $\mathbb{R}^{16}$ topology; on the other hand the multiplication of matrices $\mathbb{R}^{16} \times \mathbb{R}^{16} \ni (U,P) \mapsto UP \in \mathbb{R}^{16}$ is trivially continuos in the same topology; and finally the topologies used in the restricted spaces of the matrices here employed, i.e. $\Lambda \in SO(1,3)\uparrow$, $U \in SO(3)$ and $P \in \mathbb{R}^3$ (space of the boosts), are actually those induced by $\mathbb{R}^{16}$. As soon as both the factors of $SO(3) \times \mathbb{R}^3$ are connected, $SO(1,3)\uparrow$ is connected too. It is also arch connected, it being a differentiable manifold and thus admitting a topological base made of (smooth-)arch-connected open sets. Moreover, the fundamental group of $SO(1,3)\uparrow$ is, as it happens for product manifolds [Sernesi], the product of the fundamental groups of the factors. As $\pi_1(SO(3)) = \mathbb{Z}_2$ by (d) in proposition 9.5, whereas $\pi_1(\mathbb{R}^3) = \{1\}$, one finds again $\pi_1(SO(1,3)\uparrow) = \mathbb{Z}_2$. $\Box$

Chapter 10

Towards the theory of spinors: the interplay of $SL(2,\mathbb{C})$ and $SO(1,3)\!\uparrow$.

In this chapter we introduce some elementary results about the interplay of $SL(2,\mathbb{C})$ and $SO(1,3)\!\uparrow$. The final goal is to prepare the background to develop the theory of spinors and spinorial representations.

10.1 Elementary properties of $SL(2,\mathbb{C})$ and $SU(2)$.

As well known, $SL(2,\mathbb{C})$ denotes the Lie subgroup of $GL(2,\mathbb{C})$ made of all the $2x2$ complex matrices with unital determinant [KNS, Ruhl, Streater-Wightman, Wightman]. $SL(2,\mathbb{C})$ can be viewd as a real Lie group referring to real coordinates. We remind the reader that $U(2)$ is the group of complex 2×2 unitary matrices. $SU(2) := U(2) \cap SL(2,\mathbb{C})$. Evidently, both are (Lie) subgroup of $SL(2,\mathbb{C})$.

10.1.1 Almost all on Pauli matrices.

The (real) Lie algebra of $SL(2,\mathbb{C})$, $sl(2,\mathbb{C})$ and that of $SU(2)$, $su(2)$ are 6-dimensional and 3-dimensiona; respectively. Performing an analysis similar to that we done about Lorentz group in the previous chapter, one sees that

$$sl(2,\mathbb{C}) = \{M \in M(2,\mathbb{C}) \mid trM = 0\}\,, \quad su(2) = \{M \in M(2,\mathbb{C}) \mid M = M^\dagger\,,\ trM = 0\}\,. \tag{10.1}$$

Above † denotes the Hermitean conjugate: $M^\dagger := \overline{M}^t$ and the bar indicates the complex conjugation. By direct inspection one verifies that $sl(2,\mathbb{C})$ admits a basis made of the following six matrices

$$-i\sigma_1/2,\ -i\sigma_2/2,\ -i\sigma_3/2,\ \sigma_1/2,\ \sigma_2/2,\ \sigma_3/2\,, \tag{10.2}$$

and the first three define a basis of $su(2)$ as well. We have introduced the well-known *Pauli's matrices*

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \,, \; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \,, \; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \,. \tag{10.3}$$

By direct inspection one sees that these matrices fulfill the following identities

$$tr\sigma_i = 0 \,, \;\; \sigma_i = (\sigma_i)^\dagger \,, \qquad \sigma_i\sigma_j = \delta_{ij}I + i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k \,, \quad \text{for } i,j = 1,2,3, \tag{10.4}$$

where, as usual ϵ_{ijk} is the completely antisymmetric Ricci indicator with $\epsilon_{123} = 1$.

Comments 10.1. As a very important consequence of the definition of Pauli matrices, one immediately obtains the following commutation relations of the generators of $sl(2,\mathbb{C})$.

$$\left[-i\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = \sum_{k=1}^{3}\epsilon_{ijk}\frac{\sigma_k}{2} \,, \quad \left[-i\frac{\sigma_i}{2}, -i\frac{\sigma_j}{2}\right] = \sum_{k=1}^{3}\epsilon_{ijk} - i\frac{\sigma_k}{2} \,, \quad \left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = -\sum_{k=1}^{3}\epsilon_{ijk}\frac{\sigma_k}{2} \,. \tag{10.5}$$

It is then evident that the unique real vector space isomorphism from $sl(2,\mathbb{C})$ to the lie algebra of $SO(1,3)\!\uparrow$, $so(1,3)\!\uparrow$ that identifies S_i with $-i\sigma_i/2$ and $\sigma_i/2$ with K_i, $i = 1,2,3$, preserves the Lie commutator in view of (9.9). In other words $sl(2,\mathbb{C})$ and $so(1,3)\!\uparrow$ are isomorphic as real Lie algebras. The analog arises by comparison of the Lie algebra of $su(2)$ and that of $so(3)$. As is known [KNS] this implies that there exists a local Lie-group isomorphism form a neighborhood of the identity of $SL(2,\mathbb{C})$ to a neighborhood of $SO(1,3)\!\uparrow$, and form a neighborhood of the identity of $SU(2)$ to a neighborhood of $SO(3)$ respectively, whose differential maps reduce to the found Lie-algebra isomorphisms. In the following we shall study those Lie-group isomorphisms proving that, actually, they extend to surjective global Lie-group homomorphisms, which are the starting point for the theory of (relativistic) spinors.

In the applications to relativity, it is convenient to define some other "Pauli matricies". First of all, define

$$\sigma_0 := I \,, \tag{10.6}$$

so that σ_μ makes sense if $\mu = 0,1,2,3$. In the following, as usual, $\eta_{\mu\nu}$ denotes the components of the Minkowskian metric tensor in canonical form (i.e. the elements of the matrix $diag(-1,1,1,1,))$, and $\eta^{\mu\nu}$ are the components of the inverse Minkowskian metric tensor. Extending the procedure of raising indices, it is customary to define the **Pauli matrices with raised indices**:

$$\sigma^\mu := \eta^{\mu\nu}\sigma_\nu \,. \tag{10.7}$$

Above $\nu,\mu := 0,1,2,3$ and it is assumed the convention of summation of the repeated indices. Finally, essentially for technical reasons, it is also customary to define the **primed Pauli matricies**

$$\sigma'_0 := I \,, \; \sigma'_i := -\sigma_i \,, \quad i = 1,2,3 \,. \tag{10.8}$$

and those with raised indices

$$\sigma'^{\mu} := \eta^{\mu\nu}\sigma'_\nu \,. \tag{10.9}$$

It is worth noticing that $\{\sigma_\mu\}_{\mu=0,1,2,3}$ is a vector basis of the real space $\mathcal{H}(2,\mathbb{C})$ of Hermitean complex 2×2 matrices and $\{\sigma_\mu, i\sigma_\mu\}_{\mu=0,1,2,3}$ is a vector basis of the algebra of complex 2×2 matrices $M(2,\mathbb{C})$ viewed as complex vector space.

Exercises 10.1.

1. Prove that $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ is a basis for the *real* vector space of the 2×2 complex Hermitean matrices.

2. Prove that $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}I$ and $\sigma'_i\sigma'_j + \sigma'_j\sigma'_i = 2\delta_{ij}I$, $i = 1,2,3$.

3. Prove that:

$$tr(\sigma'_\mu\sigma_\nu) = -\eta_{\mu\nu}\,,\ \ tr(\sigma'^{\mu}\sigma^{\nu}) = -\eta^{\mu\nu}\,,\ \ tr(\sigma'^{\mu}\sigma_\nu) = -\delta^\mu_\nu\,. \tag{10.10}$$

and

$$tr(\sigma_\mu\sigma'_\nu) = -\eta_{\mu\nu}\,,\ \ tr(\sigma^{\mu}\sigma'^{\nu}) = -\eta^{\mu\nu}\,,\ \ tr(\sigma^{\mu}\sigma'_\nu) = -\delta^\mu_\nu\,. \tag{10.11}$$

(Hint. Use the result in exercise 2 and the cyclic property of the trace.)

4. Defining $\epsilon := i\sigma_2$, prove that:

$$\epsilon\sigma^t_\mu\epsilon = \epsilon\overline{\sigma_\mu}\epsilon = -\sigma'_\mu\,. \tag{10.12}$$

5. Prove that $\{\sigma_\mu\}_{\mu=0,1,2,3}$ is a vector basis of $\mathcal{H}(2,\mathbb{C})$ and that $\{\sigma_\mu, i\sigma_\mu\}_{\mu=0,1,2,3}$ is a vector basis of the algebra $M(2,\mathbb{C})$ viewed as complex vector space.

10.1.2 Properties of exponentials of Pauli matrices and consequences for $SL(2,\mathbb{C})$ and $SU(2)$.

We have a technical, but very important result, stated in the following proposition.

Proposition 10.1. *The following results holds.*
(a) *If $\sigma := (\sigma_1, \sigma_2, \sigma_3)$ and the $so(3)$ generators T_j being defined as in proposition 9.4,*

$$e^{-i\theta\mathbf{m}\cdot\sigma/2}\mathbf{n}\cdot\sigma e^{i\theta\mathbf{m}\cdot\sigma/2} = \left(e^{\theta\mathbf{m}\cdot\mathbf{T}}\mathbf{n}\right)\cdot\sigma\,, \quad \textit{for all } \mathbf{n},\mathbf{m}\in\mathbb{S}^2, \theta\in\mathbb{R}. \tag{10.13}$$

(b) *The following pair of decompositions are valid:*

$$e^{\chi\mathbf{n}\cdot\sigma/2} = \cosh\frac{\chi}{2}\,I + \sinh\frac{\chi}{2}\,\mathbf{n}\cdot\sigma\,, \quad \textit{for all } \mathbf{n}\in\mathbb{S}^2, \chi\in\mathbb{R}, \tag{10.14}$$

$$e^{-i\theta\mathbf{m}\cdot\sigma/2} = \cos\frac{\theta}{2}\,I - i\sin\frac{\theta}{2}\,\mathbf{m}\cdot\sigma\,, \quad \textit{for all } \mathbf{m}\in\mathbb{S}^2, \theta\in\mathbb{R}, \tag{10.15}$$

(c) *A matrix $A\in M(2,\mathbb{C})$ has the form $e^{-i\theta\mathbf{n}\cdot\sigma/2}$, for some $\theta\in\mathbb{R}, \mathbf{n}\in\mathbb{S}^2$, if and only if $A\in SU(2)$. The Lie group $SU(2)$ is homeomorphic to $\mathbb{S}^3$.*
(d) *A matrix $A\in M(2,\mathbb{C})$ has the form $e^{\chi\mathbf{n}\cdot\sigma/2}$, for some $\chi\in\mathbb{R}, \mathbf{n}\in\mathbb{S}^2$, if and only if A is a*

Hermitean positive element of $SL(2,\mathbb{C})$. The set of all Hermitean positive element of $SL(2,\mathbb{C})$ equipped with the topology induced by $SL(2,\mathbb{C})$ is homeomorphic to $\mathbb{R}^3$ and it is not a subgroup of $SL(2,\mathbb{C})$.

Proof. (a) and (b) Fix $\mathbf{m} \in \mathbb{S}^2$ and consider the smooth functions

$$f_i(\theta) = e^{-i\theta \mathbf{m}\cdot\sigma/2}\sigma_i e^{i\theta\mathbf{m}\cdot\sigma/2} .$$

By direct inspection, making use of the first set commutation rules in (10.5), one obtains

$$\frac{df_i}{d\theta} = \sum_{k,p=1}^{3} m^k \epsilon_{kip} f_p(\theta) .$$

Reminding that $(T_i)_{rs} = -\epsilon_{irs}$ and considering the other set of smooth functions

$$g_i(\theta) = \left(e^{\theta\mathbf{m}\cdot\mathbf{T}}\mathbf{e}_i\right)\cdot\sigma ,$$

one veryfies that, again,

$$\frac{dg_i}{d\theta} = \sum_{k,p=1}^{3} m^k \epsilon_{kip} g_p(\theta) .$$

Since both the set of smooth functions satisfy the same system of first-order differential equations, in normal form, with the same initial condition $f_i(0) = g_i(0) = \sigma_i$, we conclude that $f_i(\theta) = g_i(\theta)$ for all $\theta \in \mathbb{R}$ by uniqueness. This results yields (10.13) immediately. The established identity implies, by exponentiation

$$e^{-i\theta\mathbf{m}\cdot\sigma/2}e^{s\mathbf{n}\cdot\sigma}e^{i\theta\mathbf{m}\cdot\sigma/2} = \exp\{s\left(e^{\theta\mathbf{m}\cdot\mathbf{T}}\mathbf{n}\right)\cdot\sigma\}$$

for either $s \in \mathbb{R}$ or $s \in i\mathbb{R}$. Using these improved results, and rotating $\mathbf{n}$ into $\mathbf{e}_3$ with a suitable element $e^{-i\theta\mathbf{m}\cdot\sigma/2}$ of $SO(3)$, one realizes immediately that (10.14) and (10.15) are equivalent to

$$\begin{aligned} e^{\chi\sigma_3/2} &= \cosh\frac{\chi}{2}\, I + \sinh\frac{\chi}{2}\,\sigma_3 , \quad \text{for all } \chi \in \mathbb{R}, \\ e^{-i\theta\sigma_3/2} &= \cos\frac{\theta}{2}\, I - i\sin\frac{\theta}{2}\sigma_3 , \quad \text{for all } \theta \in \mathbb{R}. \end{aligned}$$

In turn, these identities can be proved by direct inspection, using the (diagonal) explicit form of σ_3.
(c) If $A := e^{-i\theta\mathbf{n}\cdot\sigma/2}$, then, making use of the well-known identity, $det(e^F) = e^{trF}$, we have $detA = e^{tr(-i\theta\mathbf{n}\cdot\sigma/2)} = e^{-i\theta\mathbf{n}\cdot tr(\sigma)/2} = e^0 = 1$, so that $A \in SL(2,\mathbb{C})$. We also have

$$A^\dagger = (e^{-i\theta\mathbf{n}\cdot\sigma/2})^\dagger = e^{(-i\theta\mathbf{n}\cdot\sigma/2)^\dagger} = e^{i\theta\mathbf{n}\cdot\sigma^\dagger/2} = e^{i\theta\mathbf{n}\cdot\sigma^\dagger/2} = A^{-1} .$$

This proves that every matrix of the form $e^{-i\theta\mathbf{n}\cdot\sigma/2}$ belongs to $SU(2)$. Now we go to establish also the converse. Consider a matrix $V \in SU(2)$. We know, by the spectral theorem that it is decomposable as $V = e^{-i\lambda}P_- + e^{i\lambda}P_+$, where $P_\pm$ are orthonormal projectors onto one-dimensional subspaces and $\lambda \in \mathbb{R}$. Consider the Hermitean operator $H := -\lambda P_- + \lambda P_+$. By construction $e^{iH} = V$. On the other hand, every Hermiten 2×2 matrix H can be written as $H = t^0 I + \sum_{i=1}^3 t^i\sigma_i$ for some reals t^0, t^1, t^2, t^3, that is $H = t^0 I - \theta\mathbf{n}\cdot\sigma/2$ for some $\theta \in \mathbb{R}$ and $\mathbf{n} \in \mathbb{S}^2$. We have found that $V = e^{t^0}e^{-i\theta\mathbf{n}\cdot\sigma/2}$. The requirement $detV = 1$ impose $t^0 = 0$, so that we have found that $V = e^{-i\theta\mathbf{n}\cdot\sigma/2}$ if and only if $V \in SU(2)$ as requested. Consider now the generic $V \in SU(2)$. In view of (b), it can be written as the matrix:

$$V(\mathbf{n},\theta) = \cos\frac{\theta}{2}\, I - i\sin\frac{\theta}{2}\,\mathbf{n}\cdot\sigma\,, \quad \text{for any } \mathbf{n} \in \mathbb{S}^2, \theta \in \mathbb{R}.$$

Making explicit the form of V with the help of the explicit expression of Pauli matrices, it turns out to be is that the assignment of such a $V(\mathbf{n},\theta)$ is equivalent to the assignment a point on the surface in $\mathbb{R}^4$:

$$X^0 := \cos\frac{\theta}{2}\,,\; X^1 = n^1\sin\frac{\theta}{2}\,,\; X^2 = n^2\sin\frac{\theta}{2}\,, X^3 = n^3\sin\frac{\theta}{2}\,, \quad \text{for any } \mathbf{n} \in \mathbb{S}^2, \theta \in \mathbb{R}.$$

This surface it is nothing but the 3-sphere $\mathbb{S}^3$, $(X)^2+(X^1)^2+(X^2)^2+(X^3)^2 = 1$. This one-to-one correspondence between matrices $V(\mathbf{n},\theta)$ and the points of 3-sphere $\mathbb{S}^3$ can be proved to be a homeomorphism from $\mathbb{S}^3$ and the $SU(2)$, the idea of the proof being the same as that given for the space of the boost in the proof of (e) in proposition 9.6.

(d) With the same procedure as in the case (c), one finds that every matrix $A := e^{\chi\mathbf{n}\cdot\sigma/2}$, with $\chi \in \mathbb{R}$ belongs to $SL(2,\mathbb{C})$ and fulfills $A = A^\dagger$. Let us prove that it is positive, too. Positivity means that $(x|Ax) \geq 0$ for every $x \in \mathbb{C}^2$, where $(|)$ is the standard Hermitean scalar product in $\mathbb{C}^2$. Notice that, if V is unitary (and thus preserves the scalar product) $(x|Ax) \geq 0$ for every $x \in \mathbb{C}^2$ if and only if $(x|V^\dagger AVx) \geq 0$ for every $x \in \mathbb{C}^2$. Using (a), one can fix the matrix $V = e^{-i\theta\mathbf{m}\cdot\sigma}$, in order that $V^\dagger\mathbf{n}\cdot\sigma V = \sigma_3$. Therefore $V^\dagger e^{\chi\mathbf{n}\cdot\sigma/2}V = e^{\chi\sigma_3/2}$. Since σ_3 is diagonal $e^{\chi\sigma_3/2}$ is diagonal as well and positivity of $e^{\chi\sigma_3/2}$ can be checked by direct inspection, and it result to be verified trivially. We have established that every matrix of the form $e^{\chi\mathbf{n}\cdot\sigma/2}$ is a positive Hermitean element of $SL(2,\mathbb{C})$. Now we go to prove the converse. Consider a Hermitean positive matrix $P \in SL(2,C)$. We know, by the spectral theorem of Hermitean operators that it is decomposable as $P = \lambda_1P_1+\lambda_2P_2$, where $P_{1,2}$ are orthonormal projectors onto one-dimensional subspaces and $\lambda_1, 2 > 0$ from positivity and the requirement $1 = detV = \lambda_1\lambda_2$. Consider the Hermitean operator $H := \ln\lambda_1 P_1 + \ln\lambda_2 P_2$. By construction $e^H = P$. As before, every Hermiten 2×2 matrix H can be written as $H = t^0I + \sum_{i=1}^3 t^i\sigma_i$ for some reals t^0, t^1, t^2, t^3, that is $H = t^0 I + \chi\mathbf{n}\cdot\sigma/2$ for some $\chi \in \mathbb{R}$ and $\mathbf{n} \in \mathbb{S}^2$. We have found that $P = e^{t^0}e^{\chi\mathbf{n}\cdot\sigma/2}$. The requirement $detP = 1$ impose $t^0 = 0$, so that we have found that $P = e^{\chi\mathbf{n}\cdot\sigma/2}$ if and only if $P \in SL(2,\mathbb{C})$ is Hermitean and positive as requested. Consider now the generic positive Hermitean element $P \in SL(2,\mathbb{C})$. In view of (b), it can be written as the matrix:

$$P(\mathbf{n},\chi) = \cosh\frac{\chi}{2}\, I + \sinh\frac{\chi}{2}\,\mathbf{n}\cdot\sigma\,, \quad \text{for any } \mathbf{n} \in \mathbb{S}^2, \chi \in \mathbb{R}.$$

Making explicit the form of P with the help of the explicit expression of Pauli matrices, it turns out to be is that the assignment of such a P is equivalent to the assignment a point on the surface in $\mathbb{R}^4$:

$$X^0 := \cosh\frac{\chi}{2}\ ,\ X^1 = n^1 \sinh\frac{\chi}{2}\ ,\ X^2 = n^2 \sinh\frac{\chi}{2}\ ,\ X^3 = n^3 \sinh\frac{\chi}{2}\ ,\quad \text{for any } \mathbf{n} \in \mathbb{S}^2, \chi \in \mathbb{R}.$$

This surface it is nothing but the hyperboloid $-(X)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 = -1$ which is trivially homeomorphic to $\mathbb{R}^3$. This one-to-one correspondence between matrices $P(\mathbf{n}, \chi)$ and the points of the hyperboloid can be proved to be a homeomorphism from $\mathbb{R}^3$ and the set of all Hermitean positive element of $SL(2, \mathbb{C})$ equipped with the topology induced by $SL(2, \mathbb{C})$, the idea of the proof being the same as that given for the space of the boost in the proof of (e) in proposition 9.6. □

10.1.3 Polar decomposition theorem and topology of $SL(2, \mathbb{C})$.

We are in place to state and prove a decomposition theorem for $SL(2, \mathbb{C})$ which will play an important role in the following section.

Theorem 10.1. *Take $L \in SL(2, \mathbb{C})$, the following holds.*
(a) *There is exactly one positive Hermitean matrix $H \in SL(2, \mathbb{C})$ (so that $H = e^{\chi \mathbf{n} \cdot \sigma / 2}$ for some $\chi \in \mathbb{R}$ and $\mathbf{n} \in \mathbb{S}^2$) and exactly a unitary matrix $V \in U(2)$ (so that $V = e^{-i\theta \mathbf{m} \cdot \sigma}$ for some $\theta \in \mathbb{R}$ and $\mathbf{m} \in \mathbb{S}^2$) such that*

$$L = VH\ .$$

(b) *There is exactly one positive Hermitean matrix $H' \in SL(2, \mathbb{C})$ (so that $H' = e^{\chi' \mathbf{n}' \cdot \sigma / 2}$ for some $\chi' \in \mathbb{R}$ and $\mathbf{n}' \in \mathbb{S}^2$) and exactly a unitary matrix $V' \in U(2)$ (so that $V' = e^{-i\theta' \mathbf{m}' \cdot \sigma}$ for some $\theta' \in \mathbb{R}$ and $\mathbf{m}' \in \mathbb{S}^2$) such that*

$$L = H'V'\ .$$

(c) *It holds $V' = V$ and $H' = VHV^t$.*

Proof. This is nothing but the specialization of polar decomposition theorem (theorem 7.2) applied to complex 2×2 matrices. The explicit expressions for H, V, H', V' arise from (b) and (c) in proposition 10.1. □

We want emphasize here a topological consequence of this theorem. Using the fact that $SL(2, \mathbb{C})$ is bijectively identified with the product $SU(2) \times \mathbb{R}^3$ i.e. $\mathbb{S}^3 \times \mathbb{R}^3$, where the $\mathbb{R}^3$ is the topological space of the Hermitean positive elements of $SL(2, \mathbb{C})$ as in (c) of proposition 10.1, it arises (the proof is the same as that in proposition 9.8) that $SL(2, \mathbb{C})$ is homeomorphic to the topological space $\mathbb{S}^3 \times \mathbb{R}^3$. As soon as both the factors are connected, $SL(2, \mathbb{C})$ is connected too. It is also arch connected, it being a differentiable manifold and thus admitting a topological base made of (smooth-)arch-connected open sets. Moreover, the fundamental group of $SL(2, \mathbb{C})$ is, as it happens for product manifolds [Sernesi], the product of the fundamental groups of the factors. As $\pi_1(SL(2, \mathbb{C})) = \pi_1(\mathbb{R}^3) = \{1\}$, one finds again $\pi_1(SL(2, \mathbb{C})) = \{1\}$. We have found that the following proposition holds.

Proposition 10.2. *$SL(2,\mathbb{C})$ is homeomorphic to $\mathbb{S}^3 \times \mathbb{R}^3$, as a consequence it is connected, arch-connected and $\pi_1(SL(2,\mathbb{C})) = \{1\}$, so that it is simply connected.*

Comments 10.2. We can draw further conclusions from the proposition above, making use of general theorems on Lie groups [KNS]. Since $SL(2,\mathbb{C})$ is simply connected, it is the only simply-connected Lie group (up to Lie-group isomorphisms) which admits $sl(2,\mathbb{C})$ as Lie algebra. As a consequence, it has to coincide with the universal covering of $SO(1,3)\uparrow$ (since $so(1,3)\uparrow$ is isomorphic to $sl(2,\mathbb{C})$, as noticed at the beginning of this section, and since $SO(1,3)\uparrow$ is not simply connected due to proposition 9.8) and thus there must be a Lie-group homomorphism from $SL(2,\mathbb{C})$ onto $SO(1,3)\uparrow$ which is a local Lie-group isomorphism about the units and whose differential computed in the tangent space on the group unit coincides with the Lie-algebra isomorphism which identifies $sl(2,\mathbb{C})$ and $so(1,3)\uparrow$.

10.2 The interplay of $SL(2,\mathbb{C})$ and $SO(1,3)\uparrow$.

In this subsection we establish the main result which enable one to introduce the notion of spinor in relation with vectors. It is the covering Lie-group homomorphism from $SL(2,\mathbb{C})$ onto $SO(1,3)\uparrow$ as discussed in the comment 10.2.

10.2.1 Construction of the covering homomorphism $\Pi : SL(2,\mathbb{C}) \to SO(1,3)\uparrow$.

First of all we notice that the real vector space $\mathcal{H}(4,\mathbb{C})$ of 2×2 complex Hermitean matrices admits a basis made of the extended Pauli matrices σ_μ with $\mu = 0,1,2,3$. So that, if t^μ denotes the μ-th component of $t \in \mathbb{R}^4$, referred to the canonical basis, there is a linear bijective map

$$\mathbb{R}^4 \ni t \to H(t) := t^\mu \sigma_\mu \in \mathcal{H}(4,\mathbb{C})\,. \tag{10.16}$$

Above, we have adopted the convention of summation over repeated indices and that convention will be always assumed henceforth in reference to Greek indices. As a second step we notice the following remarkable identity, which arises by direct inspection from the given definitions,

$$detH(t) = -\eta_{\mu\nu} t^\mu t^\nu\,, \quad \text{for every } t \in \mathbb{R}^4. \tag{10.17}$$

$$tr(\sigma'_\mu \sigma_\nu) = -\eta_{\mu\nu}\,,\ \ tr(\sigma'^\mu \sigma^\nu) = -\eta^{\mu\nu}\,,\ \ tr(\sigma'^\mu \sigma_\nu) = -\delta^\mu_\nu\,. \tag{10.18}$$

As a further step, we observe that, if $H(t) \in \mathcal{H}(4,\mathbb{C})$, then $LH(t)L^\dagger \in \mathcal{H}(4,\mathbb{C})$ because

$$(LH(t)L^\dagger)^\dagger = (L^\dagger)^\dagger H(t)^\dagger L^\dagger = LH(t)L^\dagger\,,$$

and thus $LH(t)L^\dagger$ has the form $H(t')$ for some $t' \in \mathbb{R}$. However, if $L \in SL(2,\mathbb{C})$, it also holds $L^\dagger \in SL(2,\mathbb{C})$ (because $detL^\dagger = \overline{detL^t} = \overline{detL} = 1$) and thus the determinant of $H(t)$ coincides with that of $H(t') = LH(t)L^\dagger$ by Binet's rule:

$$detH(t') = det(LH(t)L^\dagger) = (detL)(detH(t))detL^\dagger = 1(detH(t))1 = detH(t)\,.$$

Now (10.17) implies that $LH(t)L^\dagger = H(t')$ with $\eta_{\mu\nu}t^\mu t^\nu = \eta_{\mu\nu}t'^\mu t'^\nu$. This result has a quite immediate fundamental consequence.

Proposition 10.3. *There is a group homomorphism* $\Pi : SL(2,\mathbb{C}) \ni L \mapsto \Lambda_L \in O(1,3)$ *which is uniquely determined by*

$$LH(t)L^\dagger = H\left(\Lambda_L t\right) \ , \quad \textit{for all } t \in \mathbb{R}^4. \tag{10.19}$$

Proof. As notice above, one has $LH(t)L^\dagger = H(t')$. Since the correspondence (10.16) is bijective, for a fixed $L \in SL(2,\mathbb{C})$, a map $f_L : \mathbb{R}^4 \ni t \mapsto t'$ is well defined (there is a unique t' for each fixed t). So, we can write $LH(t)L^\dagger = H(f_L(t))$. On the other hand f_L is linear due to the linearity and bijectivity of $t \mapsto H(t)$, in particular. Indeed, one has $LH(at+bs)L^\dagger = aLH(t)L^\dagger + bLH(s)L^\dagger$, i.e $H(f_L(at+bs)) = aH(f_L(t)) + bH(f_L(s)) = H(af_L(t) + bf_L(s))$, so that $f_L(at+bs) = af_L(t)+bf_L(s)$, because $t \mapsto H(t)$ is invertible. It remains to prove that $f_L \in O(1,3)$. We know that, as discussed above, $\eta_{\mu\nu}t^\mu t^\nu = \eta_{\mu\nu}(f_Lt)^\mu(f_Lt)^\nu$ for every $t \in \mathbb{R}^4$, namely

$$\eta(t|t) = \eta\left(f_L t\,|f_L t\right) \ , \quad \text{for every } t \in \mathbb{R}^4. \tag{10.20}$$

However, as a general fact, if $s, t \in \mathbb{R}^4$, symmetry of the pseudo scalar product η entails:

$$\eta(s|t) = \frac{1}{4}\left(\eta(s+t|s+t) - \eta(s-t|s-t)\right)$$

so that, (10.20) implies

$$\eta(s|t) = \eta\left(f_L s\,|f_L t\right) \ , \quad \text{for every } s,t \in \mathbb{R}^4.$$

Let us indicate f_L by Λ_L and prove that $\Pi : L \mapsto \Lambda_L$ is a group homomorphism. Trivially, $L_I = I$ because $IH(t)I^\dagger = H\left(It\right)$ for every $t \in \mathbb{R}^4$. Moreover

$$H(\Lambda_{LL'}t) = (LL')H(t)(LL')^\dagger = L(L'H(t)L'^\dagger)L^\dagger = L(H(\Lambda_{L'}t))L^\dagger = H(\Lambda_L\Lambda_{L'}t), \quad \text{for all } t \in \mathbb{R}^4,$$

and thus $\Lambda_{LL'} = \Lambda_L\Lambda_{L'}$. The proof is concluded. □

10.2.2 Properties of Π.

The obtained result can be made stronger in several steps. First of all, we prove that Π it is a Lie-group homomorphism.

Proposition 10.4. *The group homomorphism* $\Pi : SL(2,\mathbb{C}) \ni L \mapsto \Lambda_L \in O(1,3)$ *determined by (10.19) enjoys the following properties.*
(a) *It is a Lie-group homomorphism.*
(b) *the following explicit formula holds:*

$$(\Lambda_L)^\nu{}_\mu = -tr\left(\sigma'^\nu L\sigma_\mu L^\dagger\right) \ , \quad \textit{for every } L \in SL(2,\mathbb{C}), \tag{10.21}$$

(c) *The kernel of* Π *includes* $\{\pm I\}$ *since*

$$\Pi(L) = \Pi(L') \quad \textit{if } L' = -L. \tag{10.22}$$

Proof. (a) and (b) Since the differentiable structures of $SL(2,\mathbb{C})$ and $O(1,3)$ are those induced by $\mathbb{R}^{16}$ and the the operations of taking the trace, taking the Hermitean conjugate and multiplying matrices are trivially differentiable, (10.21) implies that the group homomorphism Π is differentiable and thus is a Lie-group homomorphism. To prove (10.21). Choosing $t = \delta^\alpha_\mu e_\alpha$, where $\{e_\lambda\}_{\lambda=0,1,2,3}$ is the canonical base of $\mathbb{R}^4$, (10.19) produces:

$$L\sigma_\mu L^\dagger = Lt^\alpha\sigma_\alpha L^\dagger = (\Lambda_L)^\gamma{}_\beta\, t^\beta\sigma_\gamma = (\Lambda_L)^\gamma{}_\beta\delta^\beta_\mu\sigma_\gamma = (\Lambda_L)^\gamma{}_\mu\,\sigma_\gamma\,.$$

As a consequence

$$\sigma'^\nu L\sigma_\mu L^\dagger = (\Lambda_L)^\gamma{}_\mu\,\sigma'^\nu\sigma_\gamma\,.$$

To conclude, it is enough applying the identity $tr(\sigma'^\nu\sigma_\gamma) = -\delta^\nu_\gamma$. (see (10.10) in exercises 10.1)
(c) Using the very definition of Π, it is evident that $\Pi(L) = \Pi(-L)$. $\Box$

Now we study the interplay of Π and exponentials $\exp\{a\mathbf{n}\cdot\sigma\}$ as those appearing in the decomposition theorem 10.1.

Proposition 10.5. *The Lie-group homomorphism* $\Pi : SL(2,\mathbb{C}) \ni L \mapsto \Lambda_L \in O(1,3)$ *determined by (10.19) enjoys the following features, using notation as in subsection 9.2.2.*
(a) *For every* $\theta \in \mathbb{R}$ *and* $\mathbf{n} \in \mathbb{S}^2$, *one has*

$$\Pi\left(e^{-i\theta\mathbf{n}\cdot\sigma/2}\right) = e^{\theta\mathbf{n}\cdot\mathbf{S}}\,, \tag{10.23}$$

so that $\Pi(SU(2)) = SO(3)$. *Moreover, if* $V, V' \in SU(2)$, $\Pi(V) = Pi(V')$ *if and only if* $V = \pm V'$.
(b) *For every* $\chi \in \mathbb{R}$ *and* $\mathbf{n} \in \mathbb{S}^2$, *one has*

$$\Pi\left(e^{\chi\mathbf{n}\cdot\sigma/2}\right) = e^{\chi\mathbf{n}\cdot\mathbf{K}}\,, \tag{10.24}$$

so that Π *maps the set of* 2×2 *complex Hermitean positive elements of* $SL(2,\mathbb{C})$ *onto the set of Lorentz boosts. Moreover this map is injective.*

Proof. (a) The identity (10.13) implies immediately that, with obvious notation

$$e^{-i\theta\mathbf{n}\cdot\sigma/2}t^\mu\sigma_\mu\left(e^{-i\theta\mathbf{n}\cdot\sigma/2}\right)^\dagger = e^{-i\theta\mathbf{n}\cdot\sigma/2}(t^0I + \mathbf{t}\cdot\sigma)\sigma_\mu e^{i\theta\mathbf{n}\cdot\sigma/2} = t^0I + \left(e^{\theta\mathbf{n}\cdot\mathbf{T}}\mathbf{t}\right)\cdot\sigma = \left(e^{\theta\mathbf{n}\cdot\mathbf{S}}t\right)^\mu\sigma_\mu\,.$$

This results, in view of the definition of Π, and since t is arbitrary, implies (10.23). Remembering that every element of $SU(2)$ has the form $e^{\theta\mathbf{n}\cdot theoremSL(2,C)SO(1,3)\mathbf{S}}$ (see proposition 9.5), we have also obtained that $\Pi(SU(2)) = SO(3)$. To conclude we have to prove that $\Pi(V) = \Pi(V')$ implies $V' = -V$ when $V, V' \in SU(2)$. To this end, we notice that, from the definition of Π,

$\Pi(V) = \Pi(V')$ is equivalent to $V_1\sigma_i V_1^\dagger = \sigma_i$ for $i = 1, 2, 3$, where $V_1 := V'V^\dagger$. Since $V_1^\dagger = V_1^{-1}$ because $V_1 \in SU(2)$, it also hold $V_1\sigma_k = \sigma_k V_1$ for $i = 1, 2, 3$. Thus V_1 commutes with all complex combinations of I and the Pauli matrices. But these combinations amount to all of the elements of $M(2,\mathbb{C})$. As a consequence it must be $V_1 = \lambda I$ for some $\lambda \in \mathbb{C}$. The requirements $det V_1 = 1$ and $V_1^\dagger = V^{-1}$ imply $\lambda = \pm 1$, so that $V = \pm V'$.
(b) The identity to be proved is now

$$e^{\chi \mathbf{n}\cdot\sigma/2} t^\mu \sigma_\mu \left(e^{\chi \mathbf{n}\cdot\sigma/2}\right)^\dagger = \left(e^{\chi \mathbf{n}\cdot \mathbf{K}} t\right)^\mu \sigma_\mu \,,$$

that is, since $\sigma_\mu^\dagger = \sigma_\mu$,

$$e^{\chi \mathbf{n}\cdot\sigma/2} t^\mu \sigma_\mu e^{\chi \mathbf{n}\cdot\sigma/2} = \left(e^{\chi \mathbf{n}\cdot \mathbf{K}} t\right)^\mu \sigma_\mu \,.$$

It can be, equivalently re-written:

$$e^{-i\theta\mathbf{m}\cdot\sigma/2} e^{\chi \mathbf{n}\cdot\sigma/2} e^{i\theta\mathbf{m}\cdot\sigma/2} e^{-i\theta\mathbf{m}\cdot\sigma/2} t^\mu \sigma_\mu e^{i\theta\mathbf{m}\cdot\sigma/2} e^{-i\theta\mathbf{m}\cdot\sigma/2} e^{\chi \mathbf{n}\cdot\sigma/2} e^{-i\theta\mathbf{m}\cdot\sigma/2} = \left(e^{\chi \mathbf{n}\cdot \mathbf{K}} t\right)^\mu \sigma_\mu \,.$$

That is, fixing $\mathbf{m}$ and θ suitably, and then making use of (10.13),

$$e^{\chi\sigma_3/2} e^{-i\theta\mathbf{m}\cdot\sigma/2} t^\mu \sigma_\mu e^{i\theta\mathbf{m}\cdot\sigma/2} e^{\chi\sigma_3/2} = \left(e^{\chi \mathbf{n}\cdot \mathbf{K}} t\right)^\mu \sigma_\mu \,.$$

In turn, employing (a) in this proposition, and (b) of Proposition 9.6 this turns out to be equivalent to:

$$e^{\chi\sigma_3/2} s^\mu \sigma_\mu e^{\chi\sigma_3/2} = \left(e^{\chi K_3} s\right)^\mu \sigma_\mu \,, \tag{10.25}$$

where $s = e^{\theta \mathbf{m}\cdot\mathbf{S}} t$ is generic. (10.25) can be proved to hold by direct inspection, expanding the exponentials via (10.14), the last formula in (10.4), and (9.16). Since all the boost of Lorentz group have the form $e^{\chi \mathbf{n}\cdot\sigma/2}$, it remains to prove that Π restricted to the Hermitean positive matrices of $SL(2,\mathbb{C})$ is injective. To this end suppose that $\Pi(e^{\chi \mathbf{n}\cdot\sigma/2}) = \Pi(e^{\chi' \mathbf{n}'\cdot\sigma/2})$. This is equivalent to, for every $t \in \mathbb{R}^4$:

$$e^{\chi \mathbf{n}\cdot\sigma/2} t^\mu \sigma_\mu e^{\chi \mathbf{n}\cdot\sigma/2} = e^{\chi' \mathbf{n}'\cdot\sigma/2} t^\mu \sigma_\mu e^{\chi' \mathbf{n}'\cdot\sigma/2} \,.$$

This is equivalent to write, for every $\mu = 0, 1, 2, 3$:

$$e^{\chi \mathbf{n}\cdot\sigma/2} \sigma_\mu e^{\chi \mathbf{n}\cdot\sigma/2} = e^{\chi' \mathbf{n}'\cdot\sigma/2} \sigma_\mu e^{\chi' \mathbf{n}'\cdot\sigma/2} \,.$$

Taking $\mu = 0$, and using $\sigma_0 = I$, these identities produces:

$$e^{\chi \mathbf{n}\cdot\sigma} = e^{\chi' \mathbf{n}'\cdot\sigma} \,.$$

By direct inspection, making use of (10.14), one finds that this is possible only if

$$\cosh\frac{\chi}{2}\,\sigma_0 + \sinh\frac{\chi}{2}\sum_{j=1}^{3} n^j\sigma_j = \cosh\frac{\chi'}{2}\,\sigma_0 + \sinh\frac{\chi'}{2}\sum_{j=1}^{3} n'^j\sigma_j$$

and thus $\chi n^j = \chi' n'^j$, because the σ_μ are a basis of the space of Hermitean matrices and using well-known properties of hyperbolic functions. As a consequence, it finally holds: $\chi\mathbf{n}\cdot\sigma = \chi'\mathbf{n}'\cdot\sigma$, We have found that $\Pi(e^{\chi\mathbf{n}\cdot\sigma/2}) = \Pi(e^{\chi'\mathbf{n}'\cdot\sigma/2})$ implies $e^{\chi\mathbf{n}\cdot\sigma/2} = e^{\chi'\mathbf{n}'\cdot\sigma/2}$ as wanted. □

We are in place to state and prove the conclusive theorem.

Theorem 10.2. *The Lie-group homomorphism $\Pi : SL(2,\mathbb{C}) \ni L \mapsto \Lambda_L \in O(1,3)$ determined by (10.19) satisfies the following facts hold.*
(a) $\Pi(SL(2,\mathbb{C})) = SO(1,3)\uparrow$.
(b) $\Pi(SU(2)) = SO(3)$.
(c) *The kernel of Π is $\{\pm I\}$, that is $\Pi(L') = \Pi(L")$ if and only if $L' = \pm L"$.*
(d) *Π individuates a local Lie-group isomorphism (about the unit) between $SL(2,\mathbb{C})$ and $SO(1,3)\uparrow$, so that $d\Pi|_I : sl(2,\mathbb{C}) \to so(1,3)\uparrow$ is a Lie-algebra isomorphism.*
(e) *$\Pi\restriction_{SU(2)}$ individuates a local Lie-group isomorphism (about the unit) between $SU(2)$ and $SO(3)$, so that $d\Pi\restriction_{SU(2)}|_I : su(2) \to so(3)$ is a Lie-algebra isomorphism.*
(f) *If $L = HV$ is the polar decomposition of $L \in SL(2,\mathbb{C})$, with V unitary and H positive Hermitean, $\Pi(H)\Pi(V)$ is the polar decomposition of $\Pi(L) \in SO(1,3)\uparrow$, where $\Pi(V)$ is a spatial rotation and $\Pi(H)$ a boost.*

Proof. We start from (f). The proof of this statement is almost evident: if $L \in SL(2,\mathbb{C})$ the matrices of its polar decomposition in $SL(2,\mathbb{C})$ have the form $H = e^{\chi\mathbf{n}\cdot\sigma/2}$ and $V = e^{-i\theta\mathbf{m}\cdot\sigma/2}$ due to theorem 10.1. Thus, the action of Π produces, in view of proposition 10.5, $\Pi(V) = e^{\theta\mathbf{m}\cdot\mathbf{S}}$ and $\Pi(H) = e^{\chi\mathbf{n}\cdot\mathbf{K}}$, so that $\Pi(L) = \Pi(H)\Pi(V) = e^{\chi\mathbf{n}\cdot\mathbf{K}}e^{\theta\mathbf{m}\cdot\mathbf{S}}$. The latter is, trivially, a polar decomposition in $SO(1,3)\uparrow$ of the product of these factors and thus the *unique* one (theorems 9.1 and 9.2). But the product of these factors is $\Pi(L)$ by construction. So $\Pi(H)\Pi(V)$ is the polar decomposition of $\Pi(L)$ in $SO(1,3)\uparrow$ as wanted.
(a) Let $\Lambda \in SO(1,3)\uparrow$, so that $\Lambda = e^{\theta\mathbf{m}\cdot\mathbf{S}}e^{\chi\mathbf{n}\cdot\mathbf{K}}$ via polar decomposition. Due to (f), we have immediately: $\Pi(e^{-i\theta\mathbf{m}\cdot\sigma/2}e^{\chi\mathbf{n}\cdot\sigma/2}) = \Lambda$, and thus $\Pi(SL(2,\mathbb{C})) \supset SO(1,3)\uparrow$. On the other hand, if $L \in SL(2,\mathbb{C})$, as established in the proof of (f), $\Pi(L) = \Pi(H)\Pi(V) = e^{\chi\mathbf{n}\cdot\mathbf{K}}e^{\theta\mathbf{m}\cdot\mathbf{S}}$ so that $\Pi(L) \in SO(1,3)\uparrow$ because is the product of two elements of that group. Therefore $\Pi(SL(2,\mathbb{C})) \subset SO(1,3)\uparrow$ and thus $\Pi(SL(2,\mathbb{C})) = SO(1,3)\uparrow$. (b) It follows from (a) in proposition 10.5 and (a) and (b) of proposition 9.5.
(c) Suppose that $\Pi(L') = \Pi(L")$. This is equivalent to say that $\Pi(L) = I$, where $L = L'L"^{-1}$. By polar decomposition $L = HV$ and, for (f), $\Pi(H)\Pi(V)$ must be the polar decomposition of $I \in SO(1,3)\uparrow$. A polar decomposition of I is, trivially obtained as the product of $I \in SO(3)$ and the trivial boost I. By uniqueness this is the only polar decomposition of I. (f) entails that $\Pi(H) = I$ so and $\Pi(V) = I$. In view of proposition 10.5, $H = I$ and $V = \pm I$, so that $L = \pm I$ and $L' = \pm L"$.
(d) and (e). As is well-known from the general theory of Lie groups, given a basis $e_1, \ldots, e_n$ in the tangent space at the unit element (i.e the Lie algebra of the group), the set of parameters $(t_1, \ldots, t_n)$ of the one-parameter subgroups $t_k \mapsto \exp\{t_k e_k\}$ generated by the elements of the basis, for a sufficiently small range of the parameters about 0, i.e. $|t_k| < \delta$, individ-

uates a coordinate patch, compatible with the differentiable structure, in a neighborhood of the unit element 1 and centered on that point, i.e. 1 corresponds to $(0,\ldots,0)$. Fix the basis $-i\sigma_1/2,\ -i\sigma_2/2,\ -i\sigma_3/2,\ \sigma_1/2,\ \sigma_2/2,\ \sigma_3/2$ in $sl(2,\mathbb{C})$, associated with the coordinates system $(s^1,s^2,s^3,t^1,t^2,t^3)$ about $I \in SL(2,\mathbb{C})$, and the basis S_1,S_2,S_3,K_1,K_2,K_3 in $so(1,3)\!\uparrow$, associated with the coordinates system $(x^1,x^2,x^3,y^1,y^2,y^3)$ about $I \in SO(1,3)\!\uparrow$. By proposition 10.5, $\Pi(e^{-is_k\sigma_k/2}) = e^{s_kS_k}$ and $\Pi(e^{t_j\sigma_j/2}) = e^{t_jK_j}$, so that, in the said coordinates, the action of Π, is nothing but:

$$\Pi(s^1,s^2,s^3,t^1,t^2,t^3) = (s^1,s^2,s^3,t^1,t^2,t^3)\ ,$$

that is, the identity map. In other words, Π is a diffeomorphism, and thus a Lie-group isomorphism in the constructed coordinate patches about the units of the two groups.
By construction the differential $d\Pi|_I sl(2,\mathbb{C}) \to so(1,3)\!\uparrow$ maps each element of the first basis $-i\sigma_1/2,\ -i\sigma_2/2,\ -i\sigma_3/2,\ \sigma_1/2,\ \sigma_2/2,\ \sigma_3/2$ into the corresponding element of the other basis S_1,S_2,S_3,K_1,K_2,K_3. As noticed early in the comment 10.1, this map is also an isomorphism of Lie algebras since it preserves the commutations rules. The case concerning $SU(2)$ and $SO(3)$ has the same proof, restricting to the bases $-i\sigma_1/2,\ -i\sigma_2/2,\ -i\sigma_3/2$ and S_1,S_2,S_3. $\Box$

Chapter 11

Spinors.

Spinors are important in relativity – because, in a sense, they generalize the notion of four-vector – in quantum mechanics and in quantum field theory. They are the mathematical tool used to describe the spin (or the helicity) of particles and to formulate the quantum relativistic equation of the electron, the so called Dirac equation, but also of other semi-integer particles as neutrinos and, more generally, fermions. Nowadays, their relevance includes several other fields of pure mathematics, especially in relations with the Clifford algebras, but also noncommutative geometry. However, our approach will be very elementary.

11.1 The space of Weyl spinors.

In this section we use again the definition of conjugate space $\overline{W}$ and conjugate dual space $\overline{W^*}$ (which is naturally isomorphic to $\overline{W}^*$) as defined in definition 2.1, making use of their properties as stated in theorem 2.3. In general we make use of various definitions, notions and results contained in chapters 2 and 3. However we will confine ourselves to the case of a two-dimensional vector space W on the complex field $\mathbb{C}$. Another ingredient will be a preferred non degenerate anti symmetric tensor $\epsilon \in W^* \otimes W^*$ called the *metric spinor*. Non degenerate means as usual that, viewing ϵ as a linear map $\epsilon : W \to W^*$, it turns out to be bijective. The metric spinor will be used to raise and lower indices, similarly to the metric tensor.

Definition 11.1. A (Weyl) **spinor space** is a two-dimensional vector space W on the complex field $\mathbb{C}$, equipped with a preferred non degenerate anti symmetric tensor $\epsilon \in W^* \otimes W^*$, called the **metric spinor**. The elements of W are called **spinors** (or, equivalently, **Weyl spinors**), those of $\overline{W}$ are called **conjugate spinors**, those of W^* are called **dual spinors**, and those of $\overline{W^*}$ are called **dual conjugate spinors**. The elements belonging to a tensor products of $W, W^*, \overline{W}, \overline{W^*}$ are generically called **spinorial tensors**. $\diamondsuit$

Notation 11.1.

a. Referring to a basis in W and the associated in the spaces W^*, $\overline{W}$ and $\overline{W^*}$, the components of the spinors, i.e. vectors in W, are denoted by ξ^A (where χ may be replaced with another Greek letter). The components of dual spinors, that is vectors of W^*, are denoted by ξ_A. The components of conjugate spinors, that is vectors of $\overline{W}$, are denoted by $\xi^{A'}$. The components of dual conjugate spinors, that is vectors of $\overline{W^*}$, are denoted either by $\xi_{A'}$. Notation for spinorial tensor is similar.

b. We make sometimes use of the *abstract index notation*, so, for instance $\Xi^A{}_{B'}$ denotes a spinorial tensor in $W \otimes \overline{W^*}$.

c. Sometimes will be more convenient to use an intrinsic notation, in that case a spinor or a spinorial tensor will be indicated with a Greek letter without indices, e.g. $\xi \in W$ or $\Xi \in W \otimes W^*$.

d. If Ξ is a spinorial tensor, and $\mathcal{B} := \{e_A\}_{A=1,2} \subset W$ is a basis, ${}_{\mathcal{B}}\Xi$ denotes the *matrix* whose elements are the components of Ξ with respect to $\mathcal{B}$, or with respect the relevant basis, canonically associated with $\mathcal{B}$ in the tensor space containing Ξ. For instance, if $\Xi \in W \otimes \overline{W}$, so that $\{e_A \otimes \overline{e}_{B'}\}_{A,B'=1,2} \subset W \otimes \overline{W}$ is the basis in $W \otimes \overline{W}$ canonically associated with $\mathcal{B}$, ${}_{\mathcal{B}}\Xi$ is the matrix of elements $\Xi^{AB'}$, where $\Xi = \Xi^{AB'} e_A \otimes \overline{e}_{B'}$.

11.1.1 The metric spinor to lower and raise indices.

Let us come to the use of the metric spinor ϵ_{AB}. Exactly as in the case of the metric tensor (see section 5.2), since ϵ is non degenerate, the components of

$$\epsilon = \epsilon_{AB} e^{*A} \otimes e^{*B} , \quad \epsilon_{AB} = \epsilon(e_A, e_B) = -\epsilon_{BA} ,$$

individuate a nonsingular antisymmetric matrix referring to a fixed basis of W, $\{e_A\}_{A=1,2}$ and the associated canonical bases in $W^* \otimes W^*$. The matrix whose components are ϵ^{AB}, and satisfy

$$\epsilon_{AB}\, \epsilon^{BC} = -\delta_A^C , \tag{11.1}$$

defines a second spinorial tensor, the **inverse metric spinor**:

$$\epsilon^{AB} e_A \otimes e_B = -\epsilon^{BA} e_A \otimes e_B .$$

The definition turns out to be independent from the used basis, exactly as for the metric tensor. Notice the sign $-$ in (11.1). Following the same procedure as that for the metric tensor, one sees that the metric spinor and the inverse metric spinor individuate a natural isomorphism ad its inverse respectively, from W to W^*. In components, or referring to the abstract index notation, the isomorphism corresponds to the procedure of lowering indices, and its inverse correspond to the procedure of raising indices. These are defined, respectively, as

$$W \ni \xi^A \mapsto \xi_A := \xi^B \epsilon_{BA} \in W^* \quad \text{and} \quad W^* \ni \eta_A \mapsto \eta^A := \epsilon^{AB} \eta_B \in W . \tag{11.2}$$

Notice that these two procedures work by summing over *different* indices. *The order cannot be interchanged here*, because ϵ_{AB} and ϵ^{AB} are antisymmetric, differently from the corresponding metric tensors. With the given choices, it results

$$\xi^A = \epsilon^{AC}(\xi^B \epsilon_{BC}) .$$

The procedures extend to spinorial tensors constructed with several factors V and V^*, exactly as for the metric tensor. In particular one finds, in view of (11.1) again:

$$\epsilon_{AB} = \epsilon^{CD}\epsilon_{CA}\epsilon_{DB} \quad \text{and} \quad \epsilon^{AB} = \epsilon_{CD}\epsilon^{CA}\epsilon^{DB} \; .$$

11.1.2 The metric spinor in the conjugate spaces of spinors.

The space $\overline{W}$ is naturally anti-isomorphic to W, and the same happens for W^* and $\overline{W^*}$, as established in theorem 2.3. As a consequence a preferred metric spinor is defined on $\overline{W}$, induced by ϵ. If $F : V \to \overline{V}$ is the natural anti isomorphism described in theorem 2.3,

$$\overline{\epsilon}(\xi', \eta') := \overline{\epsilon(F^{-1}(\xi'), F^{-1}(\eta'))} \; , \quad \text{for all } \xi', \eta' \in \overline{W}, \tag{11.3}$$

defines an anti symmetric bilinear map from $\overline{W} \times \overline{W}$ to $\mathbb{C}$. The corresponding tensor of $\overline{W^*} \otimes \overline{W^*}$, is indicated by $\epsilon_{A'B'}$ or, indifferently, $\overline{\epsilon}_{A'B'}$ in the abstract index notation. Fixing a basis $\{e_A\}_{A=1,2}$ and working in the canonically associated basis $\{\overline{e^{*A'}} \otimes \overline{e^{*B'}}\}_{A',B'=1,2}$ in $\overline{W^*} \otimes \overline{W^*}$, in view of the last statement of theorem 2.3, one finds

$$\overline{\epsilon}(\overline{e}_A, \overline{e}_B) = \epsilon_{AB} \; ,$$

that is, the components of the spinorial tensor $\overline{\epsilon} \in \overline{W^*} \otimes \overline{W^*}$ are the same as those of the metric spinor. As a consequence $\overline{\epsilon}$ is non degenerate and thus it defines a metric spinor on $\overline{W}$. There is, as a consequence, an inverse metric spinor $\overline{\epsilon}^{A'B'}$ also for $\overline{\epsilon}_{A'B'}$, and the procedure of raising and lowering indices can be performed in the conjugated spaces, too.
Finally, one can consider spinorial tensors defined in tensor spaces with four types of factors: V, V^*, $\overline{V}$ and $\overline{V^*}$, and the procedure of raising and lowering indices can be performed in those spaces employing the relevant metric spinors.

Exercises 11.1.

1. Prove that the inverse metric spinor $\overline{\epsilon}^{A'B'}$ defined as above, may be obtained equivalently from ϵ^{AB} and the natural anti isomorphism $G : V^* \to \overline{V^*}$ (theorem 2.3), following the analog of the procedure to define $\overline{\epsilon}_{A'B'}$ from ϵ_{AB} and the natural anti isomorphism $F : V \to \overline{V}$, as done in (11.3).

11.1.3 Orthonormal bases and the role of $SL(2, \mathbb{C})$.

Consider the spinor space W with metric tensor ϵ. The components of the non degenerate anti symmetric metric tensor ϵ, for a fixed basis $\{e_A\}_{A=1,2} \subset W$, are

$$\epsilon_{11} = \epsilon_{22} = 0 \; , \quad \text{and} \quad \epsilon_{12} = -\epsilon_{21} = \epsilon(e_1, e_2) \; .$$

Notice that $\epsilon(e_1, e_2) \neq 0$, otherwise ϵ would be degenerate. Therefore, we can rescale e_1 and/or e_2, in order to achieve

$$\epsilon_{11} = \epsilon_{22} = 0 \; , \quad \text{and} \quad \epsilon_{12} = -\epsilon_{21} = 1 \; .$$

As we shall see later, these bases are important, especially in relation with standard tensors in Minkowski spacetime, so we state a formal definition.

Definition 11.2. Referring to a spinor space W with metric spinor ϵ, a basis $\{e_A\}_{A=1,2} \subset W$ is said (ϵ-)**orthonormal** if the components of ϵ, referred to the associated canonical basis $\{e^{*A} \otimes e^{*B}\}_{A,B=1,2} \subset W$, read

$$\epsilon_{11} = \epsilon_{22} = 0\,, \quad \text{and} \quad \epsilon_{12} = -\epsilon_{21} = 1\,. \tag{11.4}$$

♢

If $\{e_A\}_{A=1,2} \subset W$ is ϵ-orthonormal, the components of ϵ^{AB} with respect to the canonically associated basis in $V^* \otimes V^*$ are again:

$$\epsilon^{11} = \epsilon^{22} = 0\,, \quad \text{and} \quad \epsilon^{12} = -\epsilon^{21} = 1\,, \tag{11.5}$$

This happens thanks to the sign $-$ in the right-hand side of (11.1). Conversely, if, referring to some basis $\{e_A\}_{A=1,2} \subset W$ and to the canonically associated ones, one finds that (11.5) is satisfied, he/she is sure that $\{e_A\}_{A=1,2} \subset W$ is ϵ-orthonormal, because it arises from (11.1) that (11.5) implies (11.4). We conclude that (11.4) and (11.5) are equivalent.
This result has an important implication concerning the role of $SL(2,\mathbb{C})$ in spinor theory. It plays the same role as the Lorentz group plays in vector theory, in relation to pseudo orthonormal frames. If $\{e_A\}_{A=1,2} \subset W$ is ϵ-orthonormal, so that (11.5) is true, consider another basis, $\{\tilde{e}_A\}_{A=1,2} \subset W$, with $e_A = L^B{}_A \tilde{e}_B$ for some matrix $L \in GL(2,\mathbb{C})$, whose components are the coefficients $L^B{}_A$. As soon as (11.4) and (11.5) are equivalent, $\{f_A\}_{A=1,2}$ is orthonormal if and only if the coefficients

$$\tilde{\epsilon}^{AB} = L^A{}_C L^B{}_D \epsilon^{CD} \tag{11.6}$$

satisfy the four requirements

$$\tilde{\epsilon}^{11} = 0\,, \quad \tilde{\epsilon}^{22} = 0\,, \quad \tilde{\epsilon}^{12} = -\tilde{\epsilon}^{21}\,, \quad \tilde{\epsilon}^{12} = 1\,.$$

Taking (11.5) into account in (11.6), the only nontrivial condition among the requirements written above is the last one, which is equivalent to say that

$$1 = L^1{}_1 L^2{}_2 - L^1{}_2 L^2{}_1\,.$$

We recognize the determinant of L in the right-hand side. In other words, if $\{e_A\}_{A=1,2} \subset W$ is ϵ-orthonormal, another basis $\{\tilde{e}_A\}_{A=1,2} \subset W$, with $e_A = L^B{}_A \tilde{e}_B$ for some matrix $L \in GL(2,\mathbb{C})$ (whose components are the coefficients $L^B{}_A$), is ϵ-orthonormal if and only if

$$\det L = 1\,.$$

Let us summarize the obtained results within a proposition that includes some further immediate results.

Proposition 11.1. *Consider the spinor space W with metric tensor ϵ. Let $\{e_A\}_{A=1,2} \subset W$ be a basis. The following facts hold.*
(1) *Referring to the bases canonically associated with $\{e_A\}_{A=1,2} \subset W$, the following conditions are equivalent:*
(i) $\{e_A\}_{A=1,2}$ is ϵ-orthonormal;
(ii) $\epsilon^{11} = \epsilon^{22} = 0$, $\epsilon^{12} = -\epsilon^{21} = 1$;
(iii) $\overline{\epsilon}_{1'1'} = \overline{\epsilon}_{2'2'} = 0$, $\overline{\epsilon}_{1'2'} = -\overline{\epsilon}_{2'1'} = 1$;
(iv) $\overline{\epsilon}^{1'1'} = \overline{\epsilon}^{2'2'} = 0$, $\overline{\epsilon}^{1'2'} = -\overline{\epsilon}^{2'1'} = 1$.
(2) *Assume that $\{e_A\}_{A=1,2}$ is ϵ-orthonormal and let $\{\tilde{e}_A\}_{A=1,2} \subset W$ be another basis, with*

$$e_A = L^B{}_A \tilde{e}_B\,, \quad \textit{for some } L \in GL(2,\mathbb{C}) \textit{ with components } L^B{}_A.$$

$\{\tilde{e}_A\}_{A=1,2}$ is ϵ-orthonormal if and only if $L \in SL(2,\mathbb{C})$.

11.2 Four-Vectors constructed with spinors in Special Relativity.

In this subsection we show how it is possible to build up Minkowskian four-vectors starting from spinors. This is the starting point to construct, in theoretical physics, the differential equations describing the motion of relativistic particles with spin.
First of all we focus on the Hermitean spinorial tensors in the space $W \otimes \overline{W}$. To this end fix a basis $\mathcal{B} := \{e_A\}_{A=1,2} \subset W$ and consider the basis $\{e_A \otimes \overline{e}_{B'}\}_{A,B'=1,2}$, canonically associated with the given one, in the space $W \otimes \overline{W}$. An element $\Xi^\dagger \in W \otimes \overline{W}$ is the **Hermitean conjugate** of a given $\Xi \in W \otimes \overline{W}$ **with respect to** $\mathcal{B}$, if the matrix of the components of $\Xi^\dagger$, referred to the base in $W \otimes \overline{W}$ canonically associated with $\mathcal{B}$, is the adjoint (i.e. Hermitean conjugate) of that of Ξ. In other words, if $\Xi = \Xi^{AB'} e_A \otimes \overline{e}_{B'}$ and $\Xi^\dagger = \Xi^{\dagger AB'} e_A \otimes \overline{e}_{B'}$, it has to be

$$\Xi^{\dagger AB'} := \overline{\Xi^{BA'}}\,.$$

Equivalently

$$_{\mathcal{B}}\Xi^\dagger := \overline{_{\mathcal{B}}\Xi^t}\,.$$

Proposition 11.2. *Given two basis $\mathcal{B} := \{e_A\}_{A=1,2} \subset W$ and $\widetilde{\mathcal{B}} := \{\widetilde{e}_A\}_{A=1,2} \subset W$, $\Xi^\dagger \in W \otimes \overline{W}$ is the hermitean conjugate of $\Xi \in W \otimes \overline{W}$ with respect to $\mathcal{B}$ if and only if it is the hermitean conjugate of Ξ with respect to $\widetilde{\mathcal{B}}$. Thus, the notion of Hermitean conjugate is* intrinsic.

Proof. Assume that $\Xi = \Xi^{AB'} e_A \otimes \overline{e}_{B'} = \widetilde{\Xi}^{AB'} \widetilde{e}_A \otimes \overline{\widetilde{e}}_{B'}$. If $e_B = L^A{}_B \widetilde{e}_A$ and L denotes the matrix whose elements are the coefficients $L^B{}_A$ one has

$$\widetilde{\Xi}^{AB'} = L^A{}_C \overline{L^{B'}{}_{C'}} \Xi^{CD'}\,.$$

Using a matricial notation, the found identity reads:

$$_{\widetilde{\mathcal{B}}}\Xi = L_{\mathcal{B}}\Xi L^\dagger\,.$$

Taking the Hermitean conjugate we achieve:

$$\widetilde{_{\mathcal{B}}\Xi^{\dagger}} = (L_{\mathcal{B}}\Xi L^{\dagger})^{\dagger} = (L_{\mathcal{B}}\Xi^{\dagger} L^{\dagger}) .$$

We have so found that

$$\tilde{\Xi}^{\dagger AB'} = L^{A}{}_{C}\overline{L^{B'}{}_{C'}}\Xi^{\dagger CD'} .$$

That is just what we wanted to achieve. □

Definition 11.3. An element $\Xi \in W \otimes \overline{W}$ is said to be **real** if $\Xi = \Xi^{\dagger}$. $(W \otimes \overline{W})_{\mathbb{R}}$ denotes the space of real elements of $W \otimes \overline{W}$. ♢

The space $(W \otimes \overline{W})_{\mathbb{R}}$ is a real four dimensional vector space. Indeed, we known (see the previous chapter) that the matrices σ^{μ}, $\mu = 0, 1, 2, 3$ defined as in (10.7), form a basis of the real space of complex 2×2 Hermitean matrix $\mathcal{H}(2, \mathbb{C})$. As a consequence, if $\mathcal{B} = \{e_A\}_{A=1,2}$ is an ϵ-orthonormal basis of W, $\Xi \in (W \otimes \overline{W})_{\mathbb{R}}$ if and only if there are four reals, $_{\mathcal{B}}t^0, _{\mathcal{B}}t^1, _{\mathcal{B}}t^2, _{\mathcal{B}}t^3$, bijectively defined by the matrix $_{\mathcal{B}}\Xi$ of the components of Ξ referred to the basis $\{e_A \otimes \overline{e}_{B'}\}_{A,B'=1,2}$, such that

$$_{\mathcal{B}}\Xi = {_{\mathcal{B}}t^{\mu}}\sigma_{\mu} . \tag{11.7}$$

The inverse relation can be obtained making use of the matrices ${\sigma'}^{\mu}$ (10.9) and the elementary result, presented in 3 in exercises 10.1:

$$_{\mathcal{B}}t^{\mu} = -tr({_{\mathcal{B}}\Xi}\, {\sigma'}^{\mu}) . \tag{11.8}$$

To go on, the idea is to intepret the four real numbers $_{\mathcal{B}}t^{\mu}$ as the components of a four-vector in the Minkowski spacetime $\mathbb{M}^4$, with respect to some **g**-orthonormal frame $\{f_{\mu}\}_{\mu=0,1,2,3}$ of the space of the translations T^4 (**g** is the metric with segnature $(-1, +1, +1, +1)$).
The remarkable fact is the following. The found assignment of a vector t^{μ} in correspondence with a real spinorial tensor $\Xi \in W \otimes \overline{W}$ could seem to depend on the fixed basis $\mathcal{B} \subset W$. Actually this is not the case, in view of the covering homomorphism $\Pi : SL(2, \mathbb{C}) \to SO(1, 3)\uparrow$ as discussed in the previous chapter. The following theorem clarifies the relationship between real spinorial tensors of $(W \otimes \overline{W})_{\mathbb{R}}$ and four-vectors.

Theorem 11.1. *Consider a spinor space W, with metric spinor ϵ, and Minkowski spacetime $\mathbb{M}^4$ with metric **g** with signature $(-1, +1, +1, +1)$ and, as usual, define $\eta := diag(-1, 1, 1, 1)$. Finally, let $\Pi : SL(2, \mathbb{C}) \ni L \mapsto \Lambda_L \in SO(1, 3)\uparrow$ be the covering homomorphism discussed in theorem 10.2.*
*Fix an ϵ-orthonormal basis $\mathcal{B} := \{e_A\}_{A=1,2}$ in W and a **g**-orthonormal basis $\mathcal{P} := \{f_{\mu}\}_{\mu=0,1,2,3}$ in the space of four-vectors T^4 of $\mathbb{M}^4$. The following holds.*
(a) *There is a real-vector-space isomorphism*

$$h_{\mathcal{B},\mathcal{P}} : (W \otimes \overline{W})_{\mathbb{R}} \ni \Xi \to t_{\Xi} \in T^4$$

between the space of real spinorial tensors of $W \otimes \overline{W}$ and T^4 which is defined by (11.8), with inverse given by (11.7), where ${}_{\mathcal{B}}t^\mu$ are the components of t_Ξ referred to the base $\{f_\mu\}_{\mu=0,1,2,3}$.
(b) *The definition of $h_{\mathcal{B},\mathcal{P}}$ is independent from the fixed bases, modulo the action of Π. In other words, if $\widetilde{\mathcal{B}} = \{\widetilde{e}_A\}_{A=1,2}$ is another ϵ-orthonormal basis of W so that $e_B = L^A{}_B \widetilde{e}_A$, where the coefficients $L^A{}_B$ individuates $L \in SL(2,\mathbb{C})$, and $\widetilde{\mathcal{P}} := \{\widetilde{f}_\nu\}_{\nu=0,1,2,3} \subset T^4$ is the $\mathbf{g}$-orthonormal basis with $f_\nu = (\Lambda_L)^\mu{}_\nu \widetilde{f}_\mu$, then*

$$h_{\mathcal{B},\mathcal{P}} = h_{\widetilde{\mathcal{B}},\widetilde{\mathcal{P}}} \,.$$

(c) *With the given notations, for every pair $\Xi, \Sigma \in (W \otimes \overline{W})_{\mathbb{R}}$, it holds*

$$\Xi^{AB'}\Sigma^{CD'}\epsilon_{AC}\epsilon_{B'D'} = -\eta_{\mu\nu}\, t^\mu_\Xi \, t^\nu_\Sigma \,. \tag{11.9}$$

So that, in particular, the spinorial tensor,

$$\Gamma := -\epsilon \otimes \overline{\epsilon} \;\; \in V \otimes V^* \otimes \overline{V} \otimes \overline{V^*} \,, \tag{11.10}$$

individuates a Lorentzian metric in $(W \otimes \overline{W})_{\mathbb{R}}$ with signature $(-1,+1,+1,+1)$.

Proof. (a) It has been proved immediately before the statement of the theorem.
(b) Adopting notation as in (11.8) and (11.7), the thesis is equivalent to

$$L_{\mathcal{B}}\, \Xi\, L^\dagger = (\Lambda_L)^\mu{}_\nu \, {}_{\mathcal{B}}t^\nu \,.$$

This is nothing but the result proved in proposition 10.3.
(c) With the given definitions and making use of 4 in exercises 10.1, we have:

$$\Xi^{AB'}\Sigma^{CD'}\epsilon_{AC}\epsilon_{B'D'} = t^\mu_\Xi t^\nu_\Sigma \sigma_\mu^{AB'}\sigma_\nu^{CD'}\epsilon_{AC}\epsilon_{B'D'} = t^\mu_\Xi t^\nu_\Sigma tr\left(\epsilon^t\sigma^t_\mu\epsilon\sigma_n u\right) = -t^\mu_\Xi t^\nu_\Sigma tr\left(\epsilon\sigma^t_\mu\epsilon\sigma_n u\right)$$

$$= t^\mu_\Xi t^\nu_\Sigma tr(\sigma'_\mu\sigma_n u) = -\eta_{\mu\nu} t^\mu_\Xi t^\nu_\Sigma \,,$$

where, in the last passage we employed 3 in exercises 10.1. $\Box$

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