

VECTOR BUNDLES ON FANO THREEFOLDS OF GENUS 7 AND BRILL-NOETHER LOCI

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Fano threefold of genus g : a projective threefold X with $\text{Pic}(X) \cong \langle H_X \rangle$, with $H_X = -K_X$ ample, of degree $H_X^3 = 2g - 2$.

Classification due to Fano, Iskovskikh, Takeuchi, Reid, Mukai, et al. We assume X to be *general* (in particular *smooth*) and *non-hyperelliptic* (i.e. H_X is *very ample*).

1. OUR GOAL

Study the *Maruyama moduli space* $\mathbf{M}_X(2, 1, d)$ of Gieseker semistable torsion free sheaves on X with rank 2, and Chern classes $c_1 = H_X$, $c_2 = dL_X$, where L_X is a line in X . It is empty for $d < m = \lceil \frac{g+2}{2} \rceil$.

- (1) Construct a good component $\mathbf{M}(d)$ of the moduli space $\mathbf{M}_X(2, 1, d)$ for $d \geq m$.
- (2) Study $\mathbf{M}(d)$ by means of derived categories.

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2. A COMPONENT OF $\mathbf{M}_X(2, 1, d)$

Theorem 1. *There exists a vector bundle F_d in $\mathbf{M}_X(2, 1, d)$, $\forall d \geq m$, s.t.*

- (1) $\text{Ext}_X^2(F_d, F_d) = 0$,
- (2) $H^1(X, F_d(-1)) = 0$,
- (3) F_d splits as $\mathcal{O}_L \oplus \mathcal{O}_L(1)$ on a general line $L \subset X$.

Proof. By induction on $d \geq m$.

- i) The claim holds for $d = m$. A case by case analysis, due to Gushel, Mukai, Iliev, Ranestad, Markushevich et al.
- ii) Choose L and F_d , and construct F_{d+1}° as the kernel in:

$$0 \rightarrow F_{d+1}^\circ \rightarrow F_d \rightarrow \mathcal{O}_L \rightarrow 0.$$

- iii) Note that F_{d+1}° satisfies (1), (2) and (3), but it is not locally free.
- iv) Deform F_{d+1}° to a vector bundle F_{d+1} .

□

3. THE CASE $g = 7$.

A *Fano threefold X of genus 7* is isomorphic to a linear section of the Spinor tenfold Σ_+ and the orthogonal linear section of Σ_- is a smooth canonical *curve Γ of genus 7*.

Here $m = 5$. The moduli space $\mathbf{M}_X(2, 1, 5)$ is isomorphic to Γ . It is fine, represented by \mathcal{E} on $X \times \Gamma$. (Iliev-Markushevich, Kuznetsov).

Brill-Noether locus $W_{r,d}^s(\Gamma)$: stable vector bundles on Γ with rank r , degree d and at least $s + 1$ sections.

Theorem 2. *For any $d \geq 6$, the space $\mathbf{M}_X(d)$ is birational to a generically smooth $(2d - 9)$ -dimensional component of the Brill-Noether locus $W_{d-5,5d-24}^{2d-11}(\Gamma)$.*

3.1. Method: derived category of X . Consider the adjoint pair:

$$\mathbf{D}^b(\Gamma) \begin{array}{c} \xrightarrow{\Phi} \\ \xleftarrow{\Phi^!} \end{array} \mathbf{D}^b(X)$$

This is defined by the integral functors given by \mathcal{E} :

$$\begin{aligned} \Phi(-) &= \mathbf{R}p_*(q^*(-) \otimes \mathcal{E}) && \text{fully faithful} \\ \Phi^!(-) &= \mathbf{R}q_*(p^*(-) \otimes \mathcal{E}^*(H_\Gamma))[1] && \text{right adjoint to } \Phi \\ \Phi^*(-) &= \mathbf{R}q_*(p^*(-) \otimes \mathcal{E}^*(-H_X))[3] && \text{left adjoint to } \Phi \end{aligned}$$

Semiorthogonal decomposition (Kuznetsov):

$$\mathbf{D}^b(X) \cong \langle \mathcal{O}_X, \mathcal{U}_+^*, \Phi(\mathbf{D}^b(\Gamma)) \rangle,$$

where \mathcal{U}_+^* is the restriction to X of the universal rank 5 bundle on Σ_+ .

3.2. Sketch of the proof. The birational map is defined by:

$$\varphi : F \mapsto \Phi^!F.$$

- i) Take $F \in M_X(d)$ with $H^1(X, F(-1)) = 0$. Then $\Phi^!(F)$ is a vector bundle of rank $d - 5$ and degree $5d - 24$.
- ii) Use the decomposition of $\mathbf{D}^b(X)$ to write a resolution of F :

$$0 \rightarrow \text{Ext}_X^2(F, \mathcal{U}_+)^* \otimes \mathcal{U}_+^* \xrightarrow{\zeta} \Phi\Phi^!F \rightarrow F \rightarrow 0.$$

- iii) Compute:

$$\dim_{\mathbb{C}} H^0(\Gamma, \Phi^!F) = \dim_{\mathbb{C}} \text{Ext}_X^2(F, \mathcal{U}_+) = 2d - 10.$$

- iv) Find, by induction on d , a sheaf F such that $\Phi^!(F)$ is stable. This is an open condition.
- v) Prove that φ is injective: the map ζ is canonical since $\text{Hom}_X(\mathcal{U}_+^*, \Phi\Phi^!F) \cong \text{Ext}_X^2(F, \mathcal{U}_+)^*$.
- vi) Assume now $\text{Ext}^2(F, F) = 0$. Then $\Phi^!F$ is a smooth point of $W_{d-5, 5d-24}^{2d-11}(\Gamma)$ since the Petri map is injective.
- vii) Conclusion: the component containing $\Phi^!F$ has dimension $2d - 9$, hence φ birational.

3.3. The case $d = 6$: a biregular description.

Theorem 3. *The space $M_X(2, 1, 6)$ is isomorphic to $W_{1,6}^1(\Gamma)$, which is a smooth irreducible threefold.*

Iliev-Markushevich already proved birationality.

- Proof.* i) The condition $H^1(X, F(-1)) = 0$ holds for all $F \in M_X(2, 1, 6)$.
- ii) Here $\Phi^1(F)$ is a line bundle with 2 sections.
- iii) Brill-Noether theory for line bundles says that $W_{1,6}^1(\Gamma)$ is a smooth irreducible threefold. □

REFERENCES

- [BF07] MARIA CHIARA BRAMBILLA AND DANIELE FAENZI, *Vector bundles on Fano threefolds of genus 7 and Brill-Noether loci*, Available at the authors' web-pages, 2007.
- [IM04a] ATANAS ILIEV AND DIMITRI MARKUSHEVICH, *Elliptic curves and rank-2 vector bundles on the prime Fano threefold of genus 7*, *Adv. Geom.* **4** (2004), no. 3, 287–318.
- [IM04b] ———, *Parametrization of $\text{Sing}(\Theta)$ for a Fano 3-fold of Genus 7 by Moduli of Vector Bundles*, Available at <http://www.arxiv.org/abs/math.AG/0403122>, 2004.
- [IM05] ATANAS ILIEV AND LAURENT MANIVEL, *Pfaffian Lines and Vector Bundles on Fano Threefolds of Genus 8*, *J. Alg. Geom.*, to appear. Available at <http://www.arxiv.org/abs/math.AG/0504595>, 2005.
- [IP99] VASILII A. ISKOVSKIKH AND YURI. G. PROKHOROV, *Fano varieties*, *Algebraic geometry, V*, *Encyclopaedia Math. Sci.*, vol. 47, Springer, Berlin, 1999, pp. 1–247.
- [Kuz05] ALEXANDER G. KUZNETSOV, *Derived categories of the Fano threefolds V_{12}* , *Mat. Zametki* **78** (2005), no. 4, 579–594, English translation in *Math. Notes* **78**, no. 3-4, 537–550 (2005).
- [Kuz06] ———, *Hyperplane sections and derived categories*, *Izv. Ross. Akad. Nauk Ser. Mat.* **70** (2006), no. 3, 23–128, Available at <http://www.arxiv.org/abs/math.AG/0503700>.