VECTOR BUNDLES ON FANO THREEFOLDS OF GENUS 7 AND BRILL-NOETHER LOCI

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Fano threefold of genus g: a projective threefold X with $\mathrm{Pic}(X) \cong \langle H_X \rangle$, with $H_X = -K_X$ ample, of degree $H_X^3 = 2\,g - 2$.

Classification due to Fano, Iskovskikh, Takeuchi, Reid, Mukai, et al. We assume X to be *general* (in particular *smooth*) and *non-hyperelliptic* (i.e. H_X is *very ample*).

1. Our Goal

Study the *Maruyama moduli space* $\mathsf{M}_X(2,1,d)$ of Gieseker semistable torsion free sheaves on X with rank 2, and Chern classes $c_1 = H_X$, $c_2 = d\,L_X$, where L_X is a line in X. It is empty for $d < m = \lceil \frac{g+2}{2} \rceil$.

(1) Construct a good component $\mathsf{M}(d)$ of the moduli space $\mathsf{M}_X(2,1,d)$ for $d \geq m$.
(2) Study $\mathsf{M}(d)$ by means of derived categories.

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2. A component of $\mathsf{M}_X(2,1,d)$

Theorem 1. *There exists a vector bundle F_d in $\mathsf{M}_X(2,1,d)$, $\forall\, d \geq m$, s.t.*

$$\operatorname{Ext}^2_X(F_d, F_d) = 0, \tag{1}$$

$$\mathrm{H}^1(X, F_d(-1)) = 0, \tag{2}$$

(3) *F_d splits as $\mathcal{O}_L \oplus \mathcal{O}_L(1)$ on a general line $L \subset X$.*

Proof. By induction on $d \geq m$.

- i) The claim holds for $d = m$. A case by case analysis, due to Gushel, Mukai, Iliev, Ranestad, Markushevich et al.
- ii) Choose L and F_d, and construct F°_{d+1} as the kernel in:

$$0 \to F^\circ_{d+1} \to F_d \to \mathcal{O}_L \to 0.$$

- iii) Note that F°_{d+1} satisfies (1), (2) and (3), but it is not locally free.
- iv) Deform F°_{d+1} to a vector bundle F_{d+1}.

$\square$

3. The case $g = 7$.

A *Fano threefold X of genus* 7 is isomorphic to a linear section of the Spinor tenfold Σ_+ and the orthogonal linear section of Σ_- is a smooth canonical *curve* Γ *of genus* 7.

Here $m = 5$. The moduli space $\mathsf{M}_X(2,1,5)$ is isomorphic to Γ. It is fine, represented by $\mathcal{E}$ on $X \times \Gamma$. (Iliev-Markushevich, Kuznetsov).

Brill-Noether locus $W^s_{r,d}(\Gamma)$: stable vector bundles on Γ with rank r, degree d and at least $s+1$ sections.

Theorem 2. *For any $d \geq 6$, the space $\mathsf{M}_X(d)$ is birational to a generically smooth $(2d-9)$-dimensional component of the Brill-Noether locus $W^{2\,d-11}_{d-5,5\,d-24}(\Gamma)$.*

3.1. **Method: derived category of X.** Consider the adjoint pair:

$$\mathbf{D^b}(\Gamma) \underset{\mathbf{\Phi}^!}{\overset{\mathbf{\Phi}}{\rightleftarrows}} \mathbf{D^b}(X)$$

This is defined by the integral functors given by $\mathcal{E}$:

$\mathbf{\Phi}(-) = \mathbf{R}p_*(q^*(-) \otimes \mathcal{E})$	fully faithful
$\mathbf{\Phi}^!(-) = \mathbf{R}q_*(p^*(-) \otimes \mathcal{E}^*(H_\Gamma))[1]$	right adjoint to $\mathbf{\Phi}$
$\mathbf{\Phi}^*(-) = \mathbf{R}q_*(p^*(-) \otimes \mathcal{E}^*(-H_X))[3]$	left adjoint to $\mathbf{\Phi}$

Semiorthogonal decomposition (Kuznetsov):

$$\mathbf{D^b}(X) \cong \langle \mathcal{O}_X, \mathcal{U}_+^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)) \rangle,$$

where $\mathcal{U}_+^*$ is the restriction to X of the universal rank 5 bundle on Σ_+.

3.2. **Sketch of the proof.** The birational map is defined by:

$$\varphi : F \mapsto \mathbf{\Phi}^! F.$$

i) Take $F \in M_X(d)$ with $\mathrm{H}^1(X, F(-1)) = 0$. Then $\mathbf{\Phi}^!(F)$ is a vector bundle of rank $d-5$ and degree $5\,d-24$.

ii) Use the decomposition of $\mathbf{D^b}(X)$ to write a resolution of F:

$$0 \to \mathrm{Ext}^2_X(F, \mathcal{U}_+)^* \otimes \mathcal{U}_+^* \xrightarrow{\zeta} \mathbf{\Phi}\mathbf{\Phi}^! F \to F \to 0.$$

iii) Compute:

$$\dim_{\mathbb{C}} \mathrm{H}^0(\Gamma, \mathbf{\Phi}^! F) = \dim_{\mathbb{C}} \mathrm{Ext}^2_X(F, \mathcal{U}_+) = 2\,d - 10.$$

iv) Find, by induction on d, a sheaf F such that $\mathbf{\Phi}^!(F)$ is stable. This is an open condition.

v) Prove that φ is injective: the map ζ is canonical since $\mathrm{Hom}_X(\mathcal{U}_+^*, \mathbf{\Phi}\mathbf{\Phi}^! F) \cong \mathrm{Ext}^2_X(F, \mathcal{U}_+)^*$.

vi) Assume now $\mathrm{Ext}^2(F, F) = 0$. Then $\mathbf{\Phi}^! F$ is a smooth point of $W^{2\,d-11}_{d-5, 5\,d-24}(\Gamma)$ since the Petri map is injective.

vii) Conclusion: the component containing $\mathbf{\Phi}^! F$ has dimension $2\,d-9$, hence φ birational.

3.3. **The case $d = 6$: a biregular description.**

Theorem 3. *The space* $\mathsf{M}_X(2,1,6)$ *is isomorphic to* $W^1_{1,6}(\Gamma)$*, which is a smooth irreducible threefold.*

Iliev-Markushevich already proved birationality.

Proof. i) The condition $\mathrm{H}^1(X, F(-1)) = 0$ holds for all $F \in \mathsf{M}_X(2,1,6)$.

ii) Here $\mathbf{\Phi}^!(F)$ is a line bundle with 2 sections.

iii) Brill-Noether theory for line bundles says that $W^1_{1,6}(\Gamma)$ is a smooth irreducible threefold.

□

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