

BUNDLES WITH NO INTERMEDIATE COHOMOLOGY ON PRIME ANTICANONICAL THREEFOLDS

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A vector bundle F on a projective variety Y of dimension n has *no intermediate cohomology* if it is *arithmetically Cohen-Macaulay (aCM)* which means:

$$H^k(Y, F(t)) = 0, \quad \text{for all } t \in \mathbb{Z} \text{ and } 0 < k < n.$$

Prime anticanonical threefold of genus g : a projective threefold X embedded by $H_X = -K_X$ which is very ample, with $\text{Pic}(X) \cong \langle H_X \rangle$, of degree $H_X^3 = 2g - 2$.

X belongs to one of the 10 families of non-hyperelliptic Fano threefolds of index 1. We assume X to be *general* (in particular *smooth*).

1. THE PROBLEM

Classify all rank 2 bundles F with no intermediate cohomology on X . Madonna proved that *if F exists*, then it has Chern classes (expressed in multiples of H_X and of a line L_X in X):

$$\begin{aligned} \text{odd case: } & c_1(F) = 1, c_2(F) \in \{1, \dots, g + 3\}. \\ \text{even case: } & c_1(F) = 0, c_2(F) \in \{2, 4\}. \end{aligned}$$

What we do is:

- (1) Construct F .
- (2) Study the moduli space of the aCM bundles on X .

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Remark. *Actually, if $c_1 = 1$, we can only have $c_2(F) = 1$ or $c_2(F) \geq m = \lceil \frac{g+2}{2} \rceil$.*

Notation: $\mathbf{M}_X(2, c_1, c_2)$ is the *Maruyama moduli space*. $\mathbf{M}_X^\mu(2, c_1, c_2)$ is the *Mumford moduli space*. Superscript s means *stable*.

2. CONSTRUCTION OF ACM 2-BUNDLES

Theorem 1. *There exists an aCM vector bundle F_d in $\mathbf{M}_X(2, 1, d)$, for all $m \leq dg + 3$, with $\text{Ext}_X^2(F_d, F_d) = 0$.*

Essentially this theorem is proved in [BF07a].

Theorem 2. *There exists an aCM vector bundle F_d in $\mathbf{M}_X^\mu(2, 0, d)$, for $d \in \{2, 4\}$.*

Sketch of the proof.

- i) Case $d = 2$. The space $\mathbf{M}_X^\mu(2, 0, 2)$ is isomorphic to the Hilbert scheme $\mathcal{H}_2^0(X)$. Each F is aCM.
- ii) Choose two disjoint general conics C, D in X , and consider the semistable extension:

$$0 \rightarrow J_C \rightarrow F^\circ \rightarrow J_C \rightarrow 0.$$

- iii) Note that F° is not locally free. It satisfies $H^1(X, F(t)) = 0$ for all $t \in \mathbb{Z}$.
- iv) Deform F° to a vector bundle F which is aCM.

3. EVEN MODULI SPACE: $g = 7$.

A Fano threefold X of genus 7 has a semiorthogonal decomposition (Kuznetsov):

$$\mathbf{D}^b(X) \cong \langle \mathcal{O}_X, \mathcal{U}_+^*, \Phi(\mathbf{D}^b(\Gamma)) \rangle,$$

where \mathcal{U}_+^* is an exceptional rank 5 bundle, Γ is a curve of genus 7 and Φ is given by a universal vector bundle on $X \times \Gamma$.

Theorem 3. *The space $\mathbf{M}_X^s(2, 0, 4)$ is isomorphic to the Brill-Noether locus $W_{2,4}^1(\Gamma)$, which is a smooth irreducible 5-fold.*

Sketch of the proof. The birational map is:

$$\varphi : F \mapsto \Phi^1(F(1))[-1], \quad \Phi^1 \text{ right adjoint to } \Phi.$$

- i) Every sheaf F in $\mathbf{M}_X^s(2, 0, 4)$ is locally free.
- ii) Every sheaf F in $\mathbf{M}_X^s(2, 0, 4)$ satisfies:

$$H^k(X, F) = H^k(X, F(-1)) = 0, \quad \text{for all } k.$$

- iii) The image $\Phi^1(F(1))$ is a pure sheaf in degree -1 . It is a rank 2 vector bundle of degree 4.
- iv) Compute:

$$H^0(\Gamma, \varphi(F)) \cong \text{Hom}_X(\mathcal{U}_+^*, \Phi(\varphi(F))) \cong \mathbb{C}^2.$$

- v) Use the decomposition of $\mathbf{D}^b(X)$ to prove that $\mathcal{H}^0(\Phi\Phi^1 F(1)) \cong F(1)$. So φ is injective.
- vi) The bundle $\varphi(F)$ is stable: all destabilizing sheaves give back quotients of F (use again the decomposition of $\mathbf{D}^b(X)$).
- vii) Brill-Noether theory for 2-bundles says that $W_{2,4}^1(\Gamma)$ is smooth irreducible of dimension 5.

4. ODD MODULI SPACE: $g = 9$.

A *Fano threefold* X of genus 9 is isomorphic to a linear section of the 6-dimensional Lagrangian Grassmannian Σ . The projective dual linear section is a smooth *plane quartic* Γ . Here $m = 6$. The moduli space $\mathbf{M}_X(2, 1, 6)$ is isomorphic to Γ . It is fine, represented by \mathcal{E} on $X \times \Gamma$. (Iliev-Ranestad, Kuznetsov).

Work in progress. *The space $\mathbf{M}_X(2, 1, 7)$ is isomorphic to $\text{Pic}(\Gamma)$, which is a 3-dimensional Abelian variety.*

The derived category of X has a semiorthogonal decomposition (Kuznetsov):

$$\mathbf{D}^b(X) \cong \langle \mathcal{O}_X, \mathcal{U}^*, \Phi(\mathbf{D}^b(\Gamma)) \rangle,$$

where \mathcal{U}^* is an exceptional rank 3 bundle, Γ is a curve of genus 3 and Φ is given by the universal vector bundle \mathcal{E} .

Sketch of the argument. The isomorphism is:

$$\varphi : F \mapsto \Phi^!F.$$

- i) For any $F \in M_X(2, 1, 7)$ we have $H^1(X, F(-1)) = 0$.
- ii) The image $\Phi^!(F)$ is a line bundle of degree 2.
- iii) Use the decomposition of $\mathbf{D}^b(X)$ to write a resolution of F :

$$0 \rightarrow \mathcal{U}^* \xrightarrow{\zeta} \Phi\Phi^!F \rightarrow F \rightarrow 0.$$

- iv) Prove that φ is injective.

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