BUNDLES WITH NO INTERMEDIATE COHOMOLOGY ON PRIME ANTICANONICAL THREEFOLDS

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A vector bundle F on a projective variety Y of dimension n has *no intermediate cohomology* if it is *arithmetically Cohen-Macaulay (aCM)* which means:

$$\mathrm{H}^k(Y, F(t)) = 0, \qquad \text{for all } t \in \mathbb{Z} \text{ and } 0 < k < n.$$

Prime anticanonical threefold of genus g: a projective threefold X embedded by $H_X = -K_X$ which is very ample, with $\mathrm{Pic}(X) \cong \langle H_X \rangle$, of degree $H_X^3 = 2\,g - 2$.

X belongs to one of the 10 families of non-hyperelliptic Fano threefolds of index 1. We assume X to be *general* (in particular *smooth*).

1. The problem

Classify all rank 2 bundles F with no intermediate cohomology on X. Madonna proved that *if F exists*, then it has Chern classes (expressed in multiples of H_X and of a line L_X in X):

odd case: $c_1(F) = 1$, $c_2(F) \in \{1, \dots, g+3\}$.
even case: $c_1(F) = 0$, $c_2(F) \in \{2, 4\}$.

What we do is:
(1) Construct F.
(2) Study the moduli space of the aCM bundles on X.

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Remark. *Actually, if* $c_1 = 1$*, we can only have* $c_2(F) = 1$ *or* $c_2(F) \geq m = \lceil \frac{g+2}{2} \rceil$.

Notation: $\mathsf{M}_X(2, c_1, c_2)$ is the *Maruyama moduli space*. $\mathsf{M}^{\mu}_X(2, c_1, c_2)$ is the *Mumford moduli space*. Superscript s means *stable*.

2. Construction of aCM 2-bundles

Theorem 1. *There exists an aCM vector bundle* F_d *in* $\mathsf{M}_X(2, 1, d)$*, for all* $m \leq dg + 3$*, with* $\mathrm{Ext}^2_X(F_d, F_d) = 0$.

Essentially this theorem is proved in [BF07a].

Theorem 2. *There exists an aCM vector bundle* F_d *in* $\mathsf{M}^{\mu}_X(2, 0, d)$*, for* $d \in \{2, 4\}$.

Sketch of the proof.

i) Case $d = 2$. The space $\mathsf{M}^{\mu}_X(2, 0, 2)$ is isomorphic to the Hilbert scheme $\mathcal{H}^0_2(X)$. Each F is aCM.
ii) Choose two disjoint general conics C, D in X, and consider the semistable extension:

$$0 \to J_C \to F^{\circ} \to J_C \to 0.$$

iii) Note that F° is not locally free. It satisfies $\mathrm{H}^1(X, F(t)) = 0$ for all $t \in \mathbb{Z}$.
iv) Deform F° to a vector bundle F which is aCM.

3. Even moduli space: $g = 7$.

A *Fano threefold X of genus* 7 has a semiorthogonal decomposition (Kuznetsov):

$$\mathbf{D^b}(X) \cong \langle \mathcal{O}_X, \mathcal{U}_+^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)) \rangle,$$

where $\mathcal{U}_+^*$ is an exceptional rank 5 bundle, Γ is a curve of genus 7 and $\mathbf{\Phi}$ is given by a universal vector bundle on $X \times \Gamma$.

Theorem 3. *The space $\mathsf{M}_X^s(2,0,4)$ is isomorphic to the Brill-Noether locus $W_{2,4}^1(\Gamma)$, which is a smooth irreducible 5-fold.*

Sketch of the proof. The birational map is:

$$\varphi : F \mapsto \mathbf{\Phi}^!(F(1))[-1], \qquad \mathbf{\Phi}^! \text{ right adjoint to } \mathbf{\Phi}.$$

i) Every sheaf F in $\mathsf{M}_X^s(2,0,4)$ is locally free.
ii) Every sheaf F in $\mathsf{M}_X^s(2,0,4)$ satisfies:

$$\mathrm{H}^k(X,F) = \mathrm{H}^k(X,F(-1)) = 0, \qquad \text{for all } k.$$

iii) The image $\mathbf{\Phi}^!(F(1))$ is a pure sheaf in degree -1. It is a rank 2 vector bundle of degree 4.
iv) Compute:

$$\mathrm{H}^0(\Gamma, \varphi(F)) \cong \mathrm{Hom}_X(\mathcal{U}_+^*, \mathbf{\Phi}(\varphi(F))) \cong \mathbb{C}^2.$$

v) Use the decomposition of $\mathbf{D^b}(X)$ to prove that $\mathcal{H}^0(\mathbf{\Phi}\mathbf{\Phi}^! F(1)) \cong F(1)$. So φ is injective.
vi) The bundle $\varphi(F)$ is stable: all destabilizing sheaves give back quotients of F (use again the decomposition of $\mathbf{D^b}(X)$).
vii) Brill-Noether theory for 2-bundles says that $W_{2,4}^1(\Gamma)$ is smooth irreducible of dimension 5.

4. Odd moduli space: $g = 9$.

A *Fano threefold X of genus* 9 is isomorphic to a linear section of the 6-dimensional Lagrangian Grassmannian Σ. The projective dual linear section is a smooth *plane quartic* Γ. Here $m = 6$. The moduli space $\mathsf{M}_X(2,1,6)$ is isomorphic to Γ. It is fine, represented by $\mathcal{E}$ on $X \times \Gamma$. (Iliev-Ranestad, Kuznetsov).

Work in progress. *The space* $\mathsf{M}_X(2,1,7)$ *is isomorphic to* $\mathrm{Pic}(\Gamma)$*, which is a* 3*-dimensional Abelian variety.*

The derived category of X has a semiorthogonal decomposition (Kuznetsov):

$$\mathbf{D^b}(X) \cong \langle \mathcal{O}_X, \mathcal{U}^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)) \rangle,$$

where $\mathcal{U}^*$ is an exceptional rank 3 bundle, Γ is a curve of genus 3 and $\mathbf{\Phi}$ is given by the universal vector bundle $\mathcal{E}$.

Sketch of the argument. The isomorphism is:

$$\varphi : F \mapsto \mathbf{\Phi}^! F.$$

i) For any $F \in M_X(2,1,7)$ we have $\mathrm{H}^1(X, F(-1)) = 0$.
ii) The image $\mathbf{\Phi}^!(F)$ is a line bundle of degree 2.
iii) Use the decomposition of $\mathbf{D^b}(X)$ to write a resolution of F:

$$0 \to \mathcal{U}^* \xrightarrow{\zeta} \mathbf{\Phi}\mathbf{\Phi}^! F \to F \to 0.$$

iv) Prove that φ is injective.

References

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