On Weddle Surfaces And Their Moduli

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The Burkhardt quartic hypersurface $B \subset \mathbb{P}^4$ is a hypersurface defined by the vanishing of the unique $Sp(4, \mathbb{Z}/3\mathbb{Z})/\pm Id$ invariant quartic polynomial. Its explicit equation was written down for the first time by H. Burkhardt in 1892 [Bur92]. It was probably known to Coble (or at least one can infer that from his results) that a generic point of $B$ represents a principally polarized abelian surface (ppas for short) with a level 3 structure but it was only recently that G. Van der Geer [vdG87] made this statement clearer. Let us denote $A_2(3)$ this moduli space. In particular Van der Geer ([vdG87], Remark 1) pointed out the fact that the Hessian variety $Hess(B)$ of the Burkhardt quartic is birational to the moduli space parametrizing ppas with a symmetric theta structure and an even theta characteristic, which we will denote by $A_2(3)^+$. The moduli space $A_2(3)^+$ is constructed as a quotient of the Siegel upper half space $\mathbb{H}_2$ by the arithmetic group $\Gamma_2(3,6)$. More generally the Hessian of a given variety is strictly tied to a second variety, called the Steinerian variety, which parametrizes the kernels of the matrices that correspond to the points of the Hessian variety. There exists a very natural map, called the Steinerian map that associates to each point of the Hessian the corresponding projectivized kernel. Moreover, in our case we have $B \cong Stein(B)$ ([Hun96], Chapter 5) and one can view the 10:1 Steinerian map

$$St_+: Hess(B) \longrightarrow B$$

as the forgetful morphism $f: A_2(3)^+ \rightarrow A_2(3)$ which forgets the symmetric line bundle representing the polarization. This means that the following diagram, where the horizontal arrows $Th^+$ and $Q$ are birational isomorphisms, commutes.

$$\begin{array}{ccc}
A_2(3)^+ & \xrightarrow{Th^+} & Hess(B) \subset \mathbb{P}^4 \\
\downarrow f & & \downarrow St_+ \\
A_2(3) & \xrightarrow{Q} & St_+(B) = B
\end{array}$$

Coble also computed in detail a unirationalization

$$\pi : \mathbb{P}^3 \longrightarrow B,$$

given by a system of quartic polynomials that gives rise to a map of degree 6. By analogy with the Steinerian map (1), the degree of this map has lead us to suspect that $\mathbb{P}^3$ could be birational to another moduli space, which we denote by $A_2(3)^-$, that should parametrize ppas with a symmetric theta structure and an odd theta characteristic. Since we want to build $A_2(3)^-$ as a quotient of $\mathbb{H}_2$ by an arithmetic group, the group we are interested in is the odd analogue of $\Gamma_2(3,6)$. Let $Sp(4, \mathbb{Z}/2\mathbb{Z})$ be the symplectic group on a 4-dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$. Because of the characteristic we have $O(4, \mathbb{Z}/2\mathbb{Z}) \cong Sp(4, \mathbb{Z}/2\mathbb{Z})$. Let $O^-(4, \mathbb{Z}/2\mathbb{Z}) \subset Sp(4, \mathbb{Z}/2\mathbb{Z})$ be the stabilizer subgroup of an odd quadratic form.
Proposition 0.0.1 [GH04]
We have an isomorphism
\[ \text{Sp}(4, \mathbb{Z}/2\mathbb{Z}) \cong \Sigma_6 \]
under which \( \text{Sp}(4, \mathbb{Z}/2\mathbb{Z}) \) acts on the set of odd quadratic forms by permutation. Furthermore, let \( \tilde{q} \) be an odd quadratic form, then
\[ O(4, \tilde{q}) \cong O^-(4, \mathbb{Z}/2\mathbb{Z}) \cong \Sigma_5 \subset \Sigma_6. \]

Definition 0.0.2 We will denote by \( \Gamma_2(3)^- \) the group that fits in the following exact sequence
\[ 1 \longrightarrow \Gamma_2(6) \longrightarrow \Gamma_2(3)^- \overset{\text{mod } 2}{\longrightarrow} O^-(4, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 1. \]

Then we have \( \Gamma_2(6) \subset \Gamma_2(3)^- \subset \Gamma_2(3) \) and \( [\Gamma_2(3) : \Gamma_2(3)^-] = 6 \).

Let \( A_2(3)^- \) be the moduli space parametrizing the triples \((A, L, \theta)\), where \( A \) is an abelian surface, \( L \) is a symmetric ample odd line bundle s.t. \( h^0(A, L) = 1 \) and \( \theta \) is a level 3 structure.

Proposition 0.0.3 The quasi-projective variety \( \mathbb{H}_2/\Gamma_2(3)^- \) is the fine moduli space \( A_2(3)^- \) of ppas with a level 3 structure and an odd theta characteristic.

Moreover we prove the following theorem.

Theorem 0.0.4 Let \( A_2(3)^- \) be the moduli space of ppas with a level 3 structure and an odd theta characteristic. The theta-null map \( \text{Th}^- \) given by even theta functions induces a birational isomorphism
\[ \text{Th}^- : A_2(3)^- \longrightarrow \mathbb{P}^3. \]

Furthermore, the \( \mathbb{P}^3 \) of Theorem 0.0.4 is also a web of skew-symmetric matrices whose entries are quadratic polynomials in the four projective coordinates. By taking the kernels of these matrices we find again Coble’s 6 : 1 map \( \pi \) of equation 2 that now appears naturally birational to the forgetful morphism
\[ f^- : A_2(3)^- \longrightarrow A_2(3). \]

Furthermore, the pullback by \( \pi \) of the tangent hyperplane sections of \( B \) are Weddle quartic surfaces. Let us now change slightly of frame to describe a more general construction of such surfaces. Let \( C \) be a genus 2 curve and \( \tau : \xi \mapsto \xi^{-1} \otimes \omega \) the Serre involution on the Picard variety \( \text{Pic}^1(C) \). Chosen an appropriate linearization for the action of \( \tau \) on \( \mathcal{O}_{\text{Pic}^1(C)}(\Theta) \), the Weddle surface \( W \) is the image of \( \text{Pic}^1(C) \) in \( \mathbb{P}^3 = \mathbb{P}H^0(\text{Pic}^1(C), 3\Theta)^* \) (where the plus indicates that we are considering invariant sections). Moreover the surface \( W \) is a birational model of the Kummer surface \( K^1 = \text{Pic}^1(C)/\tau \subset \mathbb{P}H^0(\text{Pic}^1, 2\Theta)^* \). Given a ppa \( A \) with an odd line bundle \( L \) representing the polarization one can as well obtain a Weddle surface by sending \( A \) in the \( \mathbb{P}^3 \) obtained from the eigen-space \( H^0(A, L^3)_+ \) w.r.t. the standard involution
±Id. Then it is clear that one needs to choose a symmetric odd line bundle and a symmetric level 3 theta structure on a ppav to obtain a Weddle surface. More precisely we proved the following result.

**Proposition 0.0.5** Let \((A, H)\) be a ppav, \(L\) a symmetric line bundle representing the polarization and \(n\) an odd positive integer. Then a level \(n\) structure determines a unique symmetric theta structure of level \(n\).

This clarifies the sense in which \(A_2(3)^-\) may be seen as a moduli space of Weddle surfaces.

In the second (independent) part of this note we change our point of view: we fix a smooth genus 2 curve \(C\) and consider the moduli space \(\mathcal{M}_C\) of rank two vector bundles on \(C\) with trivial determinant. It is well known [NR69] that \(\mathcal{M}_C\) is isomorphic to \(\mathbb{P}^3\), seen as the \(2\Theta\)-linear series on the Picard variety Pic\(^1\)(C) and that the semi-stable boundary is the Kummer surface \(K^0 = \text{Jac}(C)/\pm Id \subset |2\Theta|\). The space \(\mathbb{P} \text{Ext}^1(\omega, \omega^{-1}) \cong \mathbb{P}^4 = |\omega^3|^*\) parametrizes extensions classes \((e)\) of \(\omega\) by \(\omega^{-1}\).

\[
0 \longrightarrow \omega^{-1} \longrightarrow E_e \longrightarrow \omega \longrightarrow 0. \quad (e)
\]

There exists also an action of \(\pm Id\) on extension classes. Once chosen appropriate compatible linearizations on Pic\(^1\)(C) and \(C\), we show that the linear system \(\mathbb{P} \text{H}^0(\text{Pic}^1(C), 3\Theta)^*_+\) can be injected in \(\mathbb{P} \text{Ext}^1(\omega, \omega^{-1}) \cong \mathbb{P}^4\) as the hyper-plane whose points represent involution invariant extension classes, i.e. we have \(\mathbb{P} \text{H}^0(\text{Pic}^1(C), 3\Theta)^*_+ \cong |\omega^3|^*_+.\) Our aim is to describe the sub-locus of \(\mathbb{P} \text{Ext}^1(\omega, \omega^{-1})\) which parametrizes strictly semi-stable involution invariant extension classes.

By applying a famous result of Bertram, we have the following lemma.

**Lemma 0.0.6** [Ber92] Let \((e)\) be an extension class in \(|\omega^3|^*_+\) and Sec\((C)\) the secant variety of \(C \subset |\omega^3|^*_+\), then the vector bundle \(E_e\) is not semistable if and only if \(e \in C\) and it is not stable if and only if \(e \in \text{Sec}(C)\).

This means that our locus will be the intersection of \(\text{Sec}(C)\) with the involution-invariant hyperplane given by \(\mathbb{P} \text{H}^0(\text{Pic}^1(C), 3\Theta)^*_+.\) It turns out that the secant variety \(\text{Sec}(C)\) is an hyper-surface in \(\mathbb{P} \text{Ext}^1(\omega, \omega^{-1})\) and that \(\text{deg}(\text{Sec}(C)) = 8\) but the intersection is a very particular one, as the next lemma shows.

**Lemma 0.0.7** The degree of \(\text{Sec}(C) \cap \mathbb{P} \text{H}^0(\text{Pic}^1(C), 3\Theta)^*_+ \subset \mathbb{P} \text{Ext}^1(\omega, \omega^{-1})\) equals 4. The hyperplane \(\mathbb{P} \text{H}^0(\text{Pic}^1(C), 3\Theta)^*_+\) is everywhere tangent to \(\text{Sec}(C)\).

Remark that now we have two quartic surfaces in \(\mathbb{P} \text{H}^0(\text{Pic}^1(C), 3\Theta)^*_+\): our parameter space and the Weddle surface \(W\) associated to Pic\(^1\)(C). With a little technical work we remarked that they both contain the same set of 25 distinct lines and thus are the same surface. This leads us to state the following Theorem.
**Theorem 0.0.8** Let $C$ be a smooth genus 2 curve. The parameter space of strictly semistable involution invariant extension classes of $\omega$ by $\omega^{-1}$ is the Weddle surface $W \subset \mathbb{P}H^0(Pic^1(C), 3\Theta)_+^*$ associated to $Pic^1(C)$.

Let now $\varphi$ be the classifying map

$$\varphi : \mathbb{P} Ext^1(\omega, \omega^{-1}) \rightarrow |2\Theta|$$

$$e \mapsto S\text{-equivalence class of } E_e.$$ 

In a forthcoming paper [Bol07a] we show that the map $\varphi$ also defines a conic bundle over $\mathbb{P}^3 \cong |2\Theta|$. In fact for a general point $p \in \mathbb{P}^3$ the pre-image $\varphi^{-1}(p)$ consists of the intersection of three quadrics, that means $C$ plus a conic. More precisely, let $S \subset \mathbb{P} Ext^1(\omega, \omega^{-1})$ be the cone over the twisted cubic $X \subset |\omega^3|_+ \subset \mathbb{P} Ext^1(\omega, \omega^{-1})$, we have proven the following theorem.

**Theorem 0.0.9** Let $Bl_S\mathbb{P}^4_\omega$ be the blow-up of $\mathbb{P} Ext^1(\omega, \omega^{-1})$ along the cone $S$ and $\mathbb{P}^3_\mathcal{O}$ the blow-up of $\mathbb{P}^3 \cong |2\Theta|$ in the point corresponding to $[\mathcal{O} \oplus \mathcal{O}]$. Let moreover $Bl_\mathcal{O}K^0$ be the Blow-up of the Kummer surface $K^0 = Jac(C)/\pm Id \subset |2\Theta|$ in the origin. Then the rational map $\varphi : \mathbb{P}^4_\omega \rightarrow \mathbb{P}^3_\mathcal{O}$ resolves to a morphism

$$\tilde{\varphi} : Bl_S\mathbb{P}^4_\omega \rightarrow \mathbb{P}^3_\mathcal{O}.$$ 

Furthermore the morphism $\tilde{\varphi}$ is a conic bundle whose degeneration locus is the surface $Bl_\mathcal{O}K^0 \subset \mathbb{P}^3_\mathcal{O}$.

Most of these results are contained in the paper [Bol07b], also available at http://arxiv.org/abs/math/0601251.

**References**


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