On Weddle Surfaces And Their Moduli

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The Burkhardt quartic hypersurface $\mathcal{B} \subset \mathbb{P}^4$ is a hypersurface defined by the vanishing of the unique $Sp(4, \mathbb{Z}/3\mathbb{Z})/\pm Id$ invariant quartic polynomial. Its explicit equation was written down for the first time by H. Burkhardt in 1892 [Bur92]. It was probably known to Coble (or at least one can infer that from his results) that a generic point of $\mathcal{B}$ represents a principally polarized abelian surface (ppas for short) with a level 3 structure but it was only recently that G. Van der Geer [vdG87] made this statement clearer. Let us denote $\mathcal{A}_2(3)$ this moduli space. In particular Van der Geer ([vdG87], Remark 1) pointed out the fact that the Hessian variety $Hess(\mathcal{B})$ of the Burkhardt quartic is birational to the moduli space parametrizing ppas with a symmetric theta structure and an even theta characteristic, which we will denote by $\mathcal{A}_2(3)^+$. The moduli space $\mathcal{A}_2(3)^+$ is constructed as a quotient of the Siegel upper half space $\mathbb{H}_2$ by the arithmetic group $\Gamma_2(3,6)$. More generally the Hessian of a given variety is strictly tied to a second variety, called the *Steinerian* variety, which parametrizes the kernels of the matrices that correspond to the points of the Hessian variety. There exists a very natural map, called the *Steinerian* map that associates to each point of the Hessian the corresponding projectivized kernel. Moreover, in our case we have $\mathcal{B} \cong Stein(\mathcal{B})$ ([Hun96], Chapter 5) and one can view the 10:1 Steinerian map

$$St_+ : Hess(\mathcal{B}) \longrightarrow \mathcal{B} \tag{1}$$

as the forgetful morphism $f : \mathcal{A}_2(3)^+ \rightarrow \mathcal{A}_2(3)$ which forgets the symmetric line bundle representing the polarization. This means that the following diagram, where the horizontal arrows Th^+ and Q are birational isomorphisms, commutes.

$$\begin{array}{ccc} \mathcal{A}_2(3)^+ & \stackrel{Th^+}{\longrightarrow} & Hess(\mathcal{B}) \subset \mathbb{P}^4 \\ f \downarrow & & \downarrow St_+ \\ \mathcal{A}_2(3) & \stackrel{Q}{\longrightarrow} & St_+(\mathcal{B}) = \mathcal{B} \end{array}$$

Coble also computed in detail a unirationalization

$$\pi : \mathbb{P}^3 \longrightarrow \mathcal{B}, \tag{2}$$

given by a system of quartic polynomials that gives rise to a map of degree 6. By analogy with the Steinerian map (1), the degree of this map has lead us to suspect that $\mathbb{P}^3$ could be birational to another moduli space, which we denote by $\mathcal{A}_2(3)^-$, that should parametrize ppas with a symmetric theta structure and an odd theta characteristic. Since we want to build $\mathcal{A}_2(3)^-$ as a quotient of $\mathbb{H}_2$ by an arithmetic group, the group we are interested in is the *odd analogue* of $\Gamma_2(3,6)$. Let $Sp(4, \mathbb{Z}/2\mathbb{Z})$ be the symplectic group on a 4-dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$. Because of the characteristic we have $O(4, \mathbb{Z}/2\mathbb{Z}) \cong Sp(4, \mathbb{Z}/2\mathbb{Z})$. Let $O^-(4, \mathbb{Z}/2\mathbb{Z}) \subset Sp(4, \mathbb{Z}/2\mathbb{Z})$ be the stabilizer subgroup of an odd quadratic form.

Proposition 0.0.1 *[GH04]*
We have an isomorphism

$$Sp(4,\mathbb{Z}/2\mathbb{Z}) \cong \Sigma_6$$

under which $Sp(4,\mathbb{Z}/2\mathbb{Z})$ *acts on the set of odd quadratic forms by permutation. Furthermore, let* $\tilde{q}$ *be an odd quadratic form, then*

$$O(4,\tilde{q}) \cong O^-(4,\mathbb{Z}/2\mathbb{Z}) \cong \Sigma_5 \subset \Sigma_6.$$

Definition 0.0.2 *We will denote by* $\Gamma_2(3)^-$ *the group that fits in the following exact sequence*

$$1 \longrightarrow \Gamma_2(6) \longrightarrow \Gamma_2(3)^- \stackrel{mod\ 2}{\longrightarrow} O^-(4,\mathbb{Z}/2\mathbb{Z}) \longrightarrow 1.$$

Then we have $\Gamma_2(6) \subset \Gamma_2(3)^- \subset \Gamma_2(3)$ and $[\Gamma_2(3) : \Gamma_2(3)^-] = 6$.
Let $\mathcal{A}_2(3)^-$ be the moduli space parametrizing the triples (A, L, θ), where A is an abelian surface, L is a symmetric ample odd line bundle s.t. $h^0(A,L) = 1$ and θ is a level 3 structure.

Proposition 0.0.3 *The quasi-projective variety* $\mathbb{H}_2/\Gamma_2(3)^-$ *is the fine moduli space* $\mathcal{A}_2(3)^-$ *of ppas with a level 3 structure and an odd theta characteristic.*

Moreover we prove the following theorem.

Theorem 0.0.4 *Let* $\mathcal{A}_2(3)^-$ *be the moduli space of ppas with a level 3 structure and an odd theta characteristic. The theta-null map* Th^- *given by even theta functions induces a birational isomorphism*

$$Th^- : \mathcal{A}_2(3)^- \longrightarrow \mathbb{P}^3.$$

Furthermore, the $\mathbb{P}^3$ of Theorem 0.0.4 is also a web of skew-symmetric matrices whose entries are quadratic polynomials in the four projective coordinates. By taking the kernels of these matrices we find again Coble's $6:1$ map π of equation 2 that now appears naturally birational to the forgetful morphism

$$f^- : \mathcal{A}_2(3)^- \longrightarrow \mathcal{A}_2(3).$$

Furthermore, the pullback by π of the tangent hyperplane sections of $\mathcal{B}$ are Weddle quartic surfaces. Let us now change slightly of frame to describe a more general construction of such surfaces. Let C be a genus 2 curve and $\tau : \xi \mapsto \xi^{-1} \otimes \omega$ the Serre involution on the Picard variety $Pic^1(C)$. Chosen an appropriate linearization for the action of τ on $\mathcal{O}_{Pic^1(C)}(\Theta)$, the Weddle surface W is the image of $Pic^1(C)$ in $\mathbb{P}^3 = \mathbb{P}H^0(Pic^1(C), 3\Theta)^*_+$ (where the plus indicates that we are considering invariant sections). Moreover the surface W is a birational model of the Kummer surface $K^1 = Pic^1(C)/\tau \subset \mathbb{P}H^0(Pic^1, 2\Theta)^*$. Given a ppas A with an odd line bundle L representing the polarization one can as well obtain a Weddle surface by sending A in the $\mathbb{P}^3$ obtained from the eigen-space $H^0(A, L^3)_+$ w.r.t. the standard involution

$\pm Id$. Then it is clear that one needs to choose a symmetric odd line bundle *and* a symmetric level 3 theta structure on a ppas to obtain a Weddle surface. More precisely we proved the following result.

Proposition 0.0.5 *Let (A, H) be a ppav, L a symmetric line bundle representing the polarization and n an odd positive integer. Then a level n structure determines a unique symmetric theta structure of level n.*

This clarifies the sense in which $\mathcal{A}_2(3)^-$ may be seen as a *moduli space of Weddle surfaces.*

In the second (independent) part of this note we change our point of view: we fix a smooth genus 2 curve C and consider the moduli space $\mathcal{M}_C$ of rank two vector bundles on C with trivial determinant. It is well known [NR69] that $\mathcal{M}_C$ is isomorphic to $\mathbb{P}^3$, seen as the 2Θ-linear series on the Picard variety $Pic'(C)$ and that the semi-stable boundary is the Kummer surface $K^0 = Jac(C)/\pm Id \subset |2\Theta|$. The space $\mathbb{P}Ext^1(\omega, \omega^{-1}) \cong \mathbb{P}^4 = |\omega^3|^*$ parametrizes extensions classes (e) of ω by ω^{-1}.

$$0 \longrightarrow \omega^{-1} \longrightarrow E_e \longrightarrow \omega \longrightarrow 0. \qquad (e)$$

There exists also an action of $\pm Id$ on extension classes. Once chosen appropriate compatible linearizations on $Pic^1(C)$ and C, we show that the linear system $\mathbb{P}H^0(Pic^1(C), 3\Theta)^*_+$ can be injected in $\mathbb{P}Ext^1(\omega, \omega^{-1}) \cong \mathbb{P}^4$ as the hyper-plane whose points represent involution invariant extension classes, i.e. we have $\mathbb{P}H^0(Pic^1(C), 3\Theta)^*_+ \cong |\omega^3|^*_+$. Our aim is to describe the sub-locus of $\mathbb{P}Ext^1(\omega, \omega^{-1})$ which parametrizes strictly semi-stable involution invariant extension classes.
By applying a famous result of Bertram, we have the following lemma.

Lemma 0.0.6 *[Ber92] Let (e) be an extension class in $|\omega^3|^*_+$ and $Sec(C)$ the secant variety of $C \subset |\omega^3|^*_+$, then the vector bundle E_e is not semistable if and only if $e \in C$ and it is not stable if and only if $e \in Sec(C)$.*

This means that our locus will be the intersection of $Sec(C)$ with the involution-invariant hyperplane given by $\mathbb{P}H^0(Pic^1(C), 3\Theta)^*_+$. It turns out that the secant variety $Sec(C)$ is an hyper-surface in $\mathbb{P}Ext^1(\omega, \omega^{-1})$ and that $deg(Sec(C)) = 8$ but the intersection is a very particular one, as the next lemma shows.

Lemma 0.0.7 *The degree of $Sec(C) \cap \mathbb{P}H^0(Pic^1(C), 3\Theta)^*_+ \subset \mathbb{P}Ext^1(\omega, \omega^{-1})$ equals 4. The hyperplane $\mathbb{P}H^0(Pic^1(C), 3\Theta)^*_+$ is everywhere tangent to $Sec(C)$.*

Remark that now we have two quartic surfaces in $\mathbb{P}H^0(Pic^1(C), 3\Theta)^*_+$: our parameter space and the Weddle surface W associated to $Pic^1(C)$. With a little technical work we remarked that they both contain the same set of 25 distinct lines and thus are the *same* surface. This leads us to state the following Theorem.

Theorem 0.0.8 *Let C be a smooth genus 2 curve. The parameter space of strictly semistable involution invariant extension classes of ω by ω^{-1} is the Weddle surface $W \subset \mathbb{P}H^0(Pic^1(C), 3\Theta)^*_+$ associated to $Pic^1(C)$.*

Let now φ be the classifying map

$$\begin{aligned} \varphi : \mathbb{P}Ext^1(\omega, \omega^{-1}) &\dashrightarrow |2\Theta| \\ e &\mapsto \text{S-equivalence class of } E_e. \end{aligned}$$

In a forthcoming paper [Bol07a] we show that the map φ also defines a conic bundle over $\mathbb{P}^3 \cong |2\Theta|$. In fact for a general point $p \in \mathbb{P}^3$ the pre-image $\varphi^{-1}(p)$ consists of the intersection of three quadrics, that means C plus a conic. More precisely, let $S \subset \mathbb{P}Ext^1(\omega, \omega^{-1})$ be the cone over the twisted cubic $X \subset |\omega^3|^*_+ \subset \mathbb{P}Ext^1(\omega, \omega^{-1})$, we have proven the following theorem.

Theorem 0.0.9 *Let $Bl_S\mathbb{P}^4_\omega$ be the blow-up of $\mathbb{P}Ext^1(\omega, \omega^{-1})$ along the cone S and $\mathbb{P}^3_\mathcal{O}$ the blow-up of $\mathbb{P}^3 \cong |2\Theta|$ in the point corresponding to $[\mathcal{O} \oplus \mathcal{O}]$. Let moreover $Bl_\mathcal{O}K^0$ be the Blow-up of the Kummer surface $K^0 = Jac(C)/\pm Id \subset |2\Theta|$ in the origin. Then the rational map $\varphi : \mathbb{P}^4_\omega \dashrightarrow \mathbb{P}^3$ resolves to a morphism*

$$\tilde{\varphi} : Bl_S\mathbb{P}^4_\omega \longrightarrow \mathbb{P}^3_\mathcal{O}.$$

Furthermore the morphism $\tilde{\varphi}$ is a conic bundle whose degeneration locus is the surface $Bl_\mathcal{O}K^0 \subset \mathbb{P}^3_\mathcal{O}$.

Most of these results are contained in the paper [Bol07b], also available at http://arxiv.org/abs/math/0601251.

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