Quasi elementary contractions of Fano manifolds

Cinzia Casagrande

Let X be a smooth complex **Fano variety** of dimension n. After boundedness for Fano varieties [3, 6], we know that X has only a finite number of possible topological types. What do we know about the topological invariants of X?

Recall that X is simply connected. We consider here its **second Betti number** b_2, which coincides with the **Picard number** ρ_X.

Let's look at the low dimensional cases. In dimension one there is just $\mathbb{P}^1$. If $n = 2$ then X is a Del Pezzo surface, and it is well known that $\rho_X \leq 9$.

Fano 3-folds have been classified by Iskovskikh, Mori, and Mukai. Thus we know that $\rho_X \leq 10$ in this case. In fact, more is true: *as soon as $\rho_X \geq 6$, X is a product of a Del Pezzo surface with $\mathbb{P}^1$.*

Starting from dimension 4, we do not have a bound on ρ_X. The known examples with largest Picard number are just *products of Del Pezzo surfaces with Picard number 9*, which gives $\rho_X = \frac{9}{2}n$.

Optimistically one could think that Fano varieties with large Picard number are simpler, maybe a product of lower dimensional varieties. This would yield a linear bound (in the dimension n) for ρ_X, in fact one could expect precisely:

$$\boxed{\rho_X \leq \frac{9}{2}n}$$

This is actually what happens in the toric case: if X is a smooth toric Fano variety of dimension n, then $\rho_X \leq 2n$, and equality holds if and only if n is even and X is $(S)^{\frac{n}{2}}$, S the blow-up of $\mathbb{P}^2$ in three non collinear points [4].

Let us also recall that there is a class of Fano varieties for which a stronger linear bound on ρ_X is expected. These are Fano varieties which do not contain curves of anticanonical degree 1, e.g. Fano varieties of index ≥ 2. In this case it is expected that $\rho_X \leq n$, with equality only for $(\mathbb{P}^1)^n$. This is a generalization of a conjecture by Mukai, and has been proved in dimension $n \leq 5$ and in the toric case [2, 1, 4].

A fruitful method to understand the geometry of a Fano variety X is to study its contractions under the viewpoint of Mori theory. By a **contraction** we mean a morphism with connected fibers $f \colon X \to Y$ onto a projective normal variety Y.

When ρ_X is large, a strategy is to look for a contraction

$$f \colon X \longrightarrow Y$$

of fiber type, namely with $\dim Y < \dim X$, and try to bound ρ_X in terms of ρ_Y and ρ_F, where F is a general fiber. There are (at least) two difficulties in this approach. First, Y may not be Fano, so that we do not know how to bound ρ_Y. Second, surely F is Fano, but in general ρ_F is much smaller than $\rho_X - \rho_Y$ (for instance any Del Pezzo surface S admits a contraction $S \to \mathbb{P}^1$ with general fiber $\mathbb{P}^1$).

This was our motivation to introduce the notion of *quasi elementary contraction of fiber type.* This is a contraction of fiber type as above, such that if $i\colon F \hookrightarrow X$ is a general fiber, then the image of $i_*\colon \mathcal{N}_1(F) \to \mathcal{N}_1(X)$ contains all numerical classes of curves contracted by f (here $\mathcal{N}_1(X)$ is the vector space of 1-cycles in X with real coefficients, modulo numerical equivalence). This implies:

$$\boxed{\rho_X \leq \rho_Y + \rho_F}$$

Recall that a contraction f is called *elementary* when $\rho_X - \rho_Y = 1$. In particular:

- *any elementary contraction of fiber type is quasi elementary*;
- a conic bundle $f\colon X \to Y$ is quasi elementary if and only if it is elementary;
- if a contraction is a smooth morphism (such as a projection $Y \times F \to Y$), then it is quasi elementary.

In [5] we define and study properties of quasi elementary contractions of smooth Fano varieties. It turns out that the target Y has always canonical and factorial singularities. When $\dim Y \leq 3$ these contractions have quite a good behaviour:

Theorem. *Let X be a smooth Fano variety, $f\colon X \to Y$ a quasi elementary contraction of fiber type, and F a general fiber.*

Suppose that $\dim Y = 2$. Then Y is a smooth Del Pezzo surface, f is equidimensional, and $\rho_X \leq \rho_Y + \rho_F \leq 9 + \rho_F$. If moreover $\rho_Y \geq 3$, then $X \simeq Y \times F$.

Suppose that $\dim Y = 3$. Then $\rho_X \leq \rho_Y + \rho_F \leq 10 + \rho_F$.

If moreover $\rho_Y \geq 4$, then Y is smooth and Fano.
If $\rho_Y \geq 6$, then $X \simeq S \times W$ and $Y \simeq S \times \mathbb{P}^1$, where S is a Del Pezzo surface and W is a smooth Fano variety.

The following are some applications to Fano 4-folds and 5-folds.

Corollary. *Let X be a smooth Fano 4-fold.*
If X has a non trivial quasi elementary contraction of fiber type, then $\rho_X \leq 18$, with equality if and only if $X \simeq S_1 \times S_2$, S_i Del Pezzo surfaces with $\rho_{S_i} = 9$.
If X has an elementary contraction onto a surface S and $\rho_X \geq 4$, then $X \simeq \mathbb{P}^2 \times S$.
If X has an elementary contraction onto a threefold and $\rho_X \geq 7$, then either $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times S$, or $X \simeq \mathbb{F}_1 \times S$, S a Del Pezzo surface.

Corollary. *Let X be a smooth Fano 5-fold.*
If X has two distinct elementary contractions of fiber type, then $\rho_X \leq 12$.
If X has an elementary contraction onto Y with $\dim Y \leq 3$, then $\rho_X \leq 11$.
Suppose that X has an elementary contraction $f\colon X \to Y$ with $\dim Y = 4$. If X has another elementary contraction φ of type (3,0), (4,0), (4,1), or such that $f(\mathrm{Exc}(\varphi)) = Y$, then $\rho_X \leq 12$.

Thus at least in dimension 4, we would get that the maximal Picard number is 18, if we were able to answer positively to the question:

Question. *Let X be a smooth Fano 4-fold. If ρ_X is large enough, does X admit a quasi elementary contraction of fiber type?*

Up to our knowledge, among the known examples of Fano 4-folds with no quasi elementary contractions, the one with largest Picard number has $\rho = 6$. Of course the same question is relevant in arbitrary dimension.

The proofs of these results are based on Mori theory; in particular we use many properties of contractions of Fano varieties shown by Wiśniewski [7]. We refer to [5] for more details.

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C. Casagrande - Università di Pisa, Dipartimento di Matematica - largo B. Pontecorvo, 5 - 56127 Pisa, Italy - casagrande@dm.unipi.it

Current address: Università di Pavia, Dipartimento di Matematica - via Ferrata, 1 - 27100 Pavia, Italy