Monads and Vector Bundles on Quadrics

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X denotes a nonsingular subcanonical, irreducible ACM projective variety.
A monad on X, is a complex of three vector bundles

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$$

such that α is injective and β is surjective.
Throughout the note we often use the Horrocks correspondence between a bundle $\mathcal{E}$ ($n \geq 3$) and the corresponding minimal monad (see [Ho] or [BH]).

We have a theorem about monads for rank r bundles (see [Ml2]):

Theorem 1. *On X of dimension n with $n > 3$, any minimal monad*

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0,$$

such that $\mathcal{A}$ or $\mathcal{C}$ are not zero, for a rank r ($r \geq 2$) bundle with $H^2_*(\mathcal{E}) = H^{n-2}_*(\mathcal{E}) = H^2_*(\wedge^2\mathcal{E}) = H^2_*(\wedge^2\mathcal{E}^\vee) = 0$, must satisfy the following conditions:

1. $H^1_*(\wedge^2\mathcal{B}) \neq 0$, $\beta_0(H^1_*(\wedge^2\mathcal{B})) \geq \beta_0(H^0_*(S_2\mathcal{C}))$ *and*

   $\beta_{0j}(H^1_*(\wedge^2\mathcal{B})) \geq \beta_{0j}(H^0_*(S_2\mathcal{C}))$ $\forall j \in \mathbb{Z}$, *if $\mathcal{C}$ is not zero.*

2. $H^1_*(\wedge^2\mathcal{B}^\vee) \neq 0$, $\beta_0(H^1_*(\wedge^2\mathcal{B}^\vee)) \geq \beta_0(H^0_*(S_2\mathcal{A}^\vee))$ *and*

   $\beta_{0j}(H^1_*(\wedge^2\mathcal{B}^\vee)) \geq \beta_{0j}(H^0_*(S_2\mathcal{A}^\vee))$ $\forall j \in \mathbb{Z}$, *if $\mathcal{A}$ is not zero.*

3. $H^2_*(\wedge^2\mathcal{B}) = H^2_*(\wedge^2\mathcal{B}^\vee) = 0$.

Remark 2. *If $r = 2, 3$ we don't need the hypothesis $H^2_*(\wedge^2\mathcal{E}) = H^2_*(\wedge^2\mathcal{E}^\vee) = 0$*

By using monads we can improve Ottaviani's criterion on quadrics (see [Ot1]) in the case of bundle with a small rank (see [Ml1]):

Theorem 3. *Let $\mathcal{E}$ a vector bundle on $\mathcal{Q}_n$ ($n > 3$).*

If n is odd and rank $\mathcal{E} < n-1$, or if n is even and rank $\mathcal{E} < n$, then the following conditions are equivalent:

1. *$\mathcal{E}$ splits into a direct sum of line bundles.*
2. $H^1_*(\mathcal{Q}_n, \mathcal{E} \otimes \Sigma_*) = H^2_*(\mathcal{Q}_n, \mathcal{E}) = \cdots = H^{n-2}_*(\mathcal{Q}_n, \mathcal{E}) = 0.$
3. $H^2_*(\mathcal{Q}_n, \mathcal{E}) = \cdots = H^{n-2}_*(\mathcal{Q}_n, \mathcal{E}) = H^{n-1}_*(\mathcal{Q}_n, \mathcal{E} \otimes \Sigma_*) = 0.$
4. *There exists an integer j, $2 \leq j \leq n-2$ such that* $H^2_*(\mathcal{Q}_n, \mathcal{E}) = \cdots = H^j_*(\mathcal{Q}_n, \mathcal{E} \otimes \Sigma_*) = \cdots = H^{n-2}_*(\mathcal{Q}_n, \mathcal{E}) = 0.$

Remark 4. *This means that for instance a rank two bundle $\mathcal{E}$ on $\mathcal{Q}_4$ splits if and only if $H^2_*(\mathcal{Q}_n, \mathcal{E} \otimes \Sigma_*) = 0$.*

This theorem is the equivalent on $\mathcal{Q}_n$ of a result by Kumar, Peterson and Rao on $\mathbb{P}^n$ (see [KPR]).

Let us study more carefully the rank 2 bundles on $\mathcal{Q}_n$ $(n > 3)$ without inner cohomology (i.e. $H^2_*(\mathcal{E}) = ... = H^{n-2}_*(\mathcal{E}) = 0$).
We have the following result (see [Ml1]):

Theorem 5. *For an indecomposable rank 2 bundle $\mathcal{E}$ on $\mathcal{Q}_4$ with $H^1_*(\mathcal{E}) \neq 0$ and $H^2_*(\mathcal{E}) = 0$, the only possible minimal monad with $\mathcal{A}$ or $\mathcal{C}$ different from zero is (up to a twist)*

$$0 \to \mathcal{O} \to \mathcal{S}'(1) \oplus \mathcal{S}''(1) \to \mathcal{O}(1) \to 0, \quad (1)$$

and such a monad exists. We denote by $\mathcal{Z}_4(1)$ the homology of our monad.

Remark 6. *We can say then that there exist only three rank 2 bundles without inner cohomology in $\mathcal{Q}_4$. They are $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{Z}_4$ that is associated, by the Serre correspondence, to two disjoint planes, one in Λ and one in Λ'.*

Corollary 7. *In higher dimension we have:*

1. *For an indecomposable rank 2 bundle $\mathcal{E}$ on $\mathcal{Q}_5$ with $H^2_*(\mathcal{E}) = 0$ and $H^3_*(\mathcal{E}) = 0$, the only possible minimal monad with $\mathcal{A}$ or $\mathcal{C}$ not zero is (up to a twist)*

   $$0 \to \mathcal{O} \to \mathcal{S}_5(1) \to \mathcal{O}(1) \to 0. \quad (2)$$

   and such a monad exists. We denote by $\mathcal{Z}_5(1)$ the homology of our monad.
   $\mathcal{Z}_5$ is a Cayley bundle (see [Ot2] for generalities on Cayley bundles).

2. *For $n > 5$, no indecomposable bundle of rank 2 in $\mathcal{Q}_n$ exists with* $H^2_*(\mathcal{E}) = ... = H^{n-2}_*(\mathcal{E}) = 0.$

As a conclusion, the Kumar-Peterson-Rao theorem tells us that in $\mathbb{P}^n$ with $n > 3$ there are no rank 2 bundles without inner cohomology while in $\mathcal{Q}_n$ with $n > 3$ there are 4 of them: precisely 3 in $\mathcal{Q}_4$ and 1 in $\mathcal{Q}_5$.

It is surprising that this classification of rank 2 bundle on $\mathbb{P}^n$ and $\mathcal{Q}_n$ $(n > 3)$ exactly agrees with the classification by Ancona, Peternell and Wisniewski of rank 2 Fano bundles (see [APW]).

Corollary 8. *If $\mathcal{E}$ is a rank 2 bundle on $\mathbb{P}^n$ and $\mathcal{Q}_n$ $(n > 3)$, then*

$$\mathcal{E} \text{ is a Fano bundle} \Leftrightarrow \mathcal{E} \text{ is without inner cohomology.}$$

We can also classify rank three bundles without inner cohomolgy (see [Ml2]):
We call $\mathcal{G}_4$ and $\mathcal{P}_4$ the rank three bundles on $\mathcal{Q}_4$ which are the kernel and the cokernel of the monad (1).
We call $\mathcal{G}_5$ and $\mathcal{P}_5$ the rank three bundles on $\mathcal{Q}_5$ which are the kernel and the cokernel of the monad (2).

Theorem 9. *On $\mathcal{Q}_n$ $(n > 3)$ the only rank 3 bundles without inner cohomology are the following:*

1. *for $n = 4$, the ACM bundles $\mathcal{S}' \oplus \mathcal{O}(a)$ and $\mathcal{S}'' \oplus \mathcal{O}(a)$, $\mathcal{G}_4$, $\mathcal{P}_4$ and $\mathcal{Z}_4 \oplus \mathcal{O}(a)$.*

2. *For $n = 5$, $\mathcal{G}_5$, $\mathcal{P}_5$ and $\mathcal{Z}_5 \oplus \mathcal{O}(a)$.*

References

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