Monads and Vector Bundles on Quadrics

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$X$ denotes a nonsingular subcanonical, irreducible ACM projective variety. A monad on $X$, is a complex of three vector bundles

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

such that $\alpha$ is injective and $\beta$ is surjective.

Throughout the note we often use the Horrocks correspondence between a bundle $E$ ($n \geq 3$) and the corresponding minimal monad (see [Ho] or [BH]).

We have a theorem about monads for rank $r$ bundles (see [Ml2]):

**Theorem 1.** On $X$ of dimension $n$ with $n > 3$, any minimal monad

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0,$$

such that $A$ or $C$ are not zero, for a rank $r$ ($r \geq 2$) bundle with $H^2_*(E) = H^{n-2}_*(E) = H^2_*(\wedge^2 E) = H^2_*(\wedge^2 E^\vee) = 0$, must satisfy the following conditions:

1. $H^1_*(\wedge^2 B) \neq 0$, $\beta_0(H^1_*(\wedge^2 B)) \geq \beta_0(H^0_*(S_2 C))$ and

   $$\beta_{0j}(H^1_*(\wedge^2 B)) \geq \beta_{0j}(H^0_*(S_2 C)) \quad \forall j \in \mathbb{Z}, \text{ if } C \text{ is not zero}.$$

2. $H^1_*(\wedge^2 B^\vee) \neq 0$, $\beta_0(H^1_*(\wedge^2 B^\vee)) \geq \beta_0(H^0_*(S_2 A^\vee))$ and

   $$\beta_{0j}(H^1_*(\wedge^2 B^\vee)) \geq \beta_{0j}(H^0_*(S_2 A^\vee)) \quad \forall j \in \mathbb{Z}, \text{ if } A \text{ is not zero}.$$

3. $H^2_*(\wedge^2 B) = H^2_*(\wedge^2 B^\vee) = 0$.

**Remark 2.** If $r = 2, 3$ we don’t need the hypothesis $H^2_*(\wedge^2 E) = H^2_*(\wedge^2 E^\vee) = 0$

By using monads we can improve Ottaviani’s criterion on quadrics (see [Ot1]) in the case of bundle with a small rank (see [Ml1]):

**Theorem 3.** Let $E$ a vector bundle on $Q_n$ ($n > 3$).

If $n$ is odd and rank $E < n - 1$, or if $n$ is even and rank $E < n$, then the following conditions are equivalent:
1. $\mathcal{E}$ splits into a direct sum of line bundles.

2. $H_1^1(Q, \mathcal{E} \otimes \Sigma) = H_2^2(Q, \mathcal{E}) = \cdots = H_2^{n-2}(Q, \mathcal{E}) = 0$.

3. $H_2^2(Q, \mathcal{E}) = \cdots = H_2^{n-2}(Q, \mathcal{E}) = H_n^1(Q, \mathcal{E} \otimes \Sigma) = 0$.

4. There exists an integer $j$, $2 \leq j \leq n-2$ such that $H_2^2(Q, \mathcal{E}) = \cdots = H_j(Q, \mathcal{E} \otimes \Sigma) = \cdots = H_n^{n-2}(Q, \mathcal{E}) = 0$.

Remark 4. This means that for instance a rank two bundle $\mathcal{E}$ on $Q_4$ splits if and only if $H_2^2(Q, \mathcal{E} \otimes \Sigma) = 0$.

This theorem is the equivalent on $Q_n$ of a result by Kumar, Peterson and Rao on $\mathbb{P}^n$ (see [KPR]).

Let us study more carefully the rank 2 bundles on $Q_n$ ($n > 3$) without inner cohomology (i.e. $H_2^2(\mathcal{E}) = \cdots = H_2^{n-2}(\mathcal{E}) = 0$).

We have the following result (see [Ml1]):

**Theorem 5.** For an indecomposable rank 2 bundle $\mathcal{E}$ on $Q_4$ with $H_1^1(\mathcal{E}) \neq 0$ and $H_2^2(\mathcal{E}) = 0$, the only possible minimal monad with $A$ or $C$ different from zero is (up to a twist)

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{S}'(1) \oplus S''(1) \rightarrow \mathcal{O}(1) \rightarrow 0,$$

and such a monad exists. We denote by $Z_4(1)$ the homology of our monad.

Remark 6. We can say then that there exist only three rank 2 bundles without inner cohomology in $Q_4$. They are $\mathcal{S}$, $\mathcal{S}'$ and $Z_4$ that is associated, by the Serre correspondence, to two disjoint planes, one in $\Lambda$ and one in $\Lambda'$.

**Corollary 7.** In higher dimension we have:

1. For an indecomposable rank 2 bundle $\mathcal{E}$ on $Q_5$ with $H_2^2(\mathcal{E}) = 0$ and $H_3^3(\mathcal{E}) = 0$, the only possible minimal monad with $A$ or $C$ not zero is (up to a twist)

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{S}_5(1) \rightarrow \mathcal{O}(1) \rightarrow 0,$$

and such a monad exists. We denote by $Z_5(1)$ the homology of our monad. $Z_5$ is a Cayley bundle (see [Ot2] for generalities on Cayley bundles).

2. For $n > 5$, no indecomposable bundle of rank 2 in $Q_n$ exists with $H_2^2(\mathcal{E}) = \cdots = H_n^{n-2}(\mathcal{E}) = 0$.

As a conclusion, the Kumar-Peterson-Rao theorem tells us that in $\mathbb{P}^n$ with $n > 3$ there are no rank 2 bundles without inner cohomology while in $Q_n$ with $n > 3$ there are 4 of them: precisely 3 in $Q_4$ and 1 in $Q_5$.

It is surprising that this classification of rank 2 bundle on $\mathbb{P}^n$ and $Q_n$ ($n > 3$) exactly agrees with the classification by Ancona, Peternell and Wisniewski of rank 2 Fano bundles (see [APW]).
Corollary 8. If $\mathcal{E}$ is a rank 2 bundle on $\mathbb{P}^n$ and $\mathcal{Q}_n$ ($n > 3$), then

$$
\mathcal{E} \text{ is a Fano bundle} \iff \mathcal{E} \text{ is without inner cohomology}.
$$

We can also classify rank three bundles without inner cohomology (see [Ml2]):

We call $\mathcal{G}_4$ and $\mathcal{P}_4$ the rank three bundles on $\mathcal{Q}_4$ which are the kernel and the cokernel of the monad (1).

We call $\mathcal{G}_5$ and $\mathcal{P}_5$ the rank three bundles on $\mathcal{Q}_5$ which are the kernel and the cokernel of the monad (2).

Theorem 9. On $\mathcal{Q}_n$ ($n > 3$) the only rank 3 bundles without inner cohomology are the following:

1. for $n = 4$, the ACM bundles $\mathcal{S}' \oplus \mathcal{O}(a)$ and $\mathcal{S}'' \oplus \mathcal{O}(a)$, $\mathcal{G}_4$, $\mathcal{P}_4$ and $\mathcal{Z}_4 \oplus \mathcal{O}(a)$.

2. For $n = 5$, $\mathcal{G}_5$, $\mathcal{P}_5$ and $\mathcal{Z}_5 \oplus \mathcal{O}(a)$.

References


