Slope inequalities for varieties fibred over curves

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We work over $\mathbb{C}$. A **fibred surface** (or fibration) is the data of

$$\begin{array}{ll} S & \text{a smooth projective surface with} \\ \downarrow f & \text{a surjective morphism with connected fibres} \\ B & \text{over a smooth projective curve.} \end{array}$$

The genus g of a general fibre F is called genus of the fibration. We call a fibration *relatively minimal* if the fibres contain no -1-curves. A relatively minimal fibration is said to be *semistable* if all the fibres are nodal curves (i.e. the fibres are semistable in the sense of Deligne and Mumford).

From now on $f\colon S \longrightarrow B$ is a *relatively minimal fibred surface of genus* $g \geq 2$. The *relative canonical sheaf* is the line bundle $\omega_f = \omega_X \otimes (f^*\omega_B)^{-1}$; and let K_f denote any associated divisor. We consider the following relative invariants, where $b = g(B)$:

$$K_f^2 = K_S^2 - 8(b-1)(g-1)$$
$$\chi_f = \deg f_*\omega_f = \chi\mathcal{O}_S - (b-1)(g-1)$$

These invariants are non-negative. Moreover, $\chi_f = 0$ if and only if f is locally trivial .

When f is not locally trivial, we can define the *slope of f* as

$$\boxed{s(f) := K_f^2/\chi_f.}$$

The problem we want to address is the **study of lower bounds for the slope**.

Of course $s(f) \geq 0$, but indeed a bigger lower bound for the slope holds, namely

$$s(f) \geq 4\frac{g-1}{g},$$

which is known as the *slope inequality*. It was proved with two different methods by Xiao [12] and by Cornalba-Harris [4] only in the semistable case. However, in [11] I have generalised the method of C-H, proving the slope inequality without restrictions. It is sharp, and the fibrations reaching it are in particular all hyperelliptic.

The importance of these kind of results is double.

- They are fundamental tools in the study of the "geography" of complex surfaces (for example, Pardini's recent proof of the Severi inequality for surfaces of maximal Albanese dimension in [9] makes an essential use of the slope inequality).
- The bounds on the slope have an application to the *positivity of divisors on the moduli space of stable curves of genus g* (for instance, in [6], the slope inequality is a key ingredient for attaching a conjecture on the nef cone of $\overline{\mathcal{M}}_{g,n}$).

We want to study the influence on lower bounds of the slope of other invariants of the fibred surface, such as

⋆ properties of the general fibres (gonality, Clifford index, involutions);

⋆ global properties such as the *relative irregularity*: $q_f := q(S) - b$.

It has been conjectured the existence of a lower bound as an increasing function of q_f, and of the gonality or the Clifford index of the general fibres, but very few results are known.

Relative irregularity and Clifford index

We define the Clifford index Cliff(f) of a fibred surface as the Clifford index of a general fibre:

$$\mathrm{Cliff}(F) = \min\{\deg L - 2(h^0(L) - 1) \mid L \in \mathrm{Pic}(F),\, h^0(L) \geq 2,\, h^1(L) \geq 2\}.$$

In [1] (joint work with M. A. Barja) we found a lower bound which is an increasing function of $m := \min\{\mathrm{Cliff}(f), q_f\}$:

$$s(f) \geq 4\frac{g-1}{g-[m/2]}.$$

This result is obtained applying both the C-H method and a the one of Xiao to some direct factor of $f_*\omega_f$ deriving from Fujita's decompositon

$$f_*\omega_f = \mathcal{A} \oplus \mathcal{O}_B^{\oplus q_f}.$$

A crucial result for the application of both methods is the proof of the *linear stability* for general projections of canonical curves.

Double cover fibrations

In a joint work with M. Cornalba [5] we treated the case of *double fibrations*, i.e. fibred surfaces such that the general fibres have an involution whose quotient has genus γ. We proved the sharp lower bound

$$s(f) \geq 4\frac{g-1}{g-\gamma},$$

for $g \geq 4\gamma + 1$ (for $g \leq 4\gamma$ the bound is known to be false). Moreover, we give a characterisation of the fibrations that reach it. This is an affirmative answer to a conjecture formulated in [3]. This result follows from an application of the slope inequality for fibred surfaces and of the Algebraic Index Theorem.

Trigonal fibrations

A first problem in the investigation of the influence of the gonality of the slope is the one of studying fibrations whose general fibre is trigonal, i.e. *trigonal fibrations*. The main known results are due to Konno [8] and Stankova-Frenkel [10]. In [2] we prove the following result, which was conjectured by Stankova-Frenkel for the semistable case.

Let $f \colon S \longrightarrow B$ be a relatively minimal fibred surface such that the general fibre C is either:

- *a trigonal curve of even genus $g \geq 6$ and zero Maroni invariant;*
- *a curve of genus 6 with a g_5^2.*

Then

$$s_f \geq \frac{5g-6}{g},$$

and this bound is sharp.

The Maroni invariant: If C is a trigonal curve of genus g, its canonical image in $\mathbb{P}^{g-1}$ lives in a rational normal scroll Σ which is the intersection of quadrics containing C; Σ is isomorphic to a Hirzebruch surface $F_c = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(c))$, embedded in $\mathbb{P}^{g-1}$ as a surface of minimal degree. The number c has the same parity as the genus $g = g(C)$ and it is called the *Maroni invariant of* C. A general trigonal curve C of genus g has Maroni invariant 0 (resp. 1) if g is even (resp. odd).

The idea of the proof is the following. We can embed the relative canonical image of S in a 3-fold W (the relative hyperquadric hull) fibred over B, whose fibre over general $t \in B$ is the rational normal scroll associated to the fibre of S over t.

$$\begin{array}{ccccc} S & \dashrightarrow & W & \dashrightarrow & \mathbb{P}_B(f_*\omega_f) \\ & \searrow^{f} & \downarrow{\scriptstyle\phi} & \swarrow & \\ & & B & & \end{array}$$

We obtain the result applying the C-H method to this setting.

In this case the application of the method of Xiao gives a different inequality, which is weaker but holds for any trigonal fibration [8].

Slope inequality for families of canonical varieties

Let $\phi\colon X \longrightarrow B$ be a fibration of a normal $\mathbb{Q}$-factorial variety with at most canonical singularities over a curve. Under these assumptions K_X and $K_\phi = K_X - \phi^* K_B$ are Weil, $\mathbb{Q}$-Cartier divisor and we can consider its associated divisorial sheaves ω_X and ω_ϕ. In this case the generalised C-H method implies the following result [2].

Suppose that the canonical sheaf ω_X is ϕ-nef, and that on a general fibre F the canonical divisor $\omega_F = \omega_{\phi|F}$ induces a Hilbert semistable map which is finite on the image of F. Then the following inequality holds

$$K_\phi^n \geq n \, \frac{d}{p_g(F)} \deg \phi_* \omega_\phi,$$

where $p_g(F) = h^0(F, K_F)$ and d is the degree of the canonical image of the general fibre F in $\mathbb{P}^{p_g(F)-1}$.

It has to be stressed that the assumption of Hilbert stability is very difficult to verify, even for surfaces. However, some results can be obtained for special cases, such as fibrations whose general fibres are surfaces with p_g small.

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