NILMANIFOLDS: COMPLEX STRUCTURES, GEOMETRY AND DEFORMATIONS

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The aim of this work was to understand the deformations in the large of a certain class of compact, complex manifolds.

We say that two compact, complex manifolds X and X' are directly deformation equivalent $X \sim_{def} X'$ if there exists an irreducible, flat family $\pi : \mathcal{M} \to \mathcal{B}$ of compact, complex manifolds over an analytic space $\mathcal{B}$ such that $X \cong \pi^{-1}(b)$ and $X' \cong \pi^{-1}(b')$ for some points $b, b' \in \mathcal{B}$. The manifold X is said to be a deformation in the large of X' if both are in the same equivalence class with respect to the equivalence relation generated by $\sim_{def}$.

The problem of determining the deformations of a given complex manifold is very difficult in general; but while there is a general method due to Kuranishi, Kodaira and Spencer to tackle small deformations there is no general approach to deformations in the large.

From Tori to Nilmanifolds. Even the seemingly natural fact that any deformation in the large of a complex torus is again a complex torus has been fully proved only in 2002 by Catanese. In [Cat04] he studies more in general deformations in the large of principal holomorphic torus bundles, especially bundles of elliptic curves. This was the starting point for our research.

It turns out that the right context to generalise Catanese's results is the theory of left invariant complex structures on nilmanifolds, i.e., compact quotients of nilpotent real Lie groups [CF06].

Many (counter-)examples in complex differential geometry have been constructed from nilmanifolds:

- Thurston's example of a manifold which admits a complex structure and a symplectic structure but no Kähler structure.
- Guan's example of a simply connected, non-kählerian, holomorphic symplectic manifold.
- Manifolds with arbitrarily non degenerating Frölicher spectral sequence [Rol07]. This answers a question mentioned in the book of Griffith and Harris.

In fact, a nilmanifold M admits a Kähler structure if and only if it is a complex torus [BG88].

There are too many nilmanifolds. Even if every (iterated) principal holomorphic torus bundle can be regarded as a nilmanifold, the converse is far from true. Moreover it turns out that even a small deformation of a principal holomorphic torus bundle may not admit such a structure.

Recall that a Kodaira surface is a non trivial principal holomorphic bundle of elliptic curves over an elliptic curve.

Example 1 — We endow $\mathbb{C}^3$ with a real Lie group structure given by the multiplication law

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} * \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} := \begin{pmatrix} z_1 + w_1 \\ z_2 + w_2 + \bar{z}_1 w_1 \\ z_3 + w_3 - \frac{1}{2}(z_2 + \bar{z}_2) w_1 \end{pmatrix}.$$

Note that multiplication on the left with some fixed element in $\mathbb{C}^3$ is holomorphic even though $(\mathbb{C}^3, *)$ is not a complex Lie group.

For $\lambda \in \mathbb{R}$ we consider the discrete subgroup Γ_λ generated by the elements

$$e_1, ie_1, e_2, ie_2, e_3 + \lambda i e_2, ie_3$$

and, by taking the quotient, we obtain a family of compact, complex manifolds

$$M_\lambda := \Gamma_\lambda \backslash \mathbb{C}^3, \qquad \lambda \in \mathbb{R}.$$

The last three generators act only by translation, while the first three are not contained in the centre of $(\mathbb{C}^3, *)$.

If $\lambda = 0$ (or more in general if $\lambda \in \mathbb{Q}$) then Γ_λ acts by translations on the complex subspace $V := \{z_1 = z_2 = 0\}$ and, since this action is compatible with the projection map

$$\begin{array}{ccc} \mathbb{C}^3 & \longrightarrow & \mathbb{C}^3/V \\ \downarrow & & \downarrow \\ M_\lambda & \xrightarrow{\pi_\lambda} & S \end{array},$$

we get a principal holomorphic bundle of elliptic curves over a Kodaira surface S.

On the other hand, if $\lambda \notin \mathbb{Q}$ then there is no complex subspace of $\mathbb{C}^3$ on which Γ_λ acts by translations and hence there is no principal holomorphic torus bundle structure.

Addressed questions.

1) What are the **small deformations** of nilmanifolds with left invariant complex structure.
2) When has such a nilmanifold a **geometric description** as an (iterated) principal holomorphic torus bundle?
3) Can we determine all **deformations in the large** of (iterated) principal holomorphic torus bundles?

Small deformations. A fairly complete answer to the first question is given by the following result:

Theorem A — *Let $M = \Gamma\backslash G$ be a nilmanifold with left-invariant complex structure J. If the Dolbeault cohomology $H^{p,q}(M, J)$ can be calculated using left-invariant differential forms then all small deformations of (M, J) are again nilmanifolds with left-invariant complex structure.*

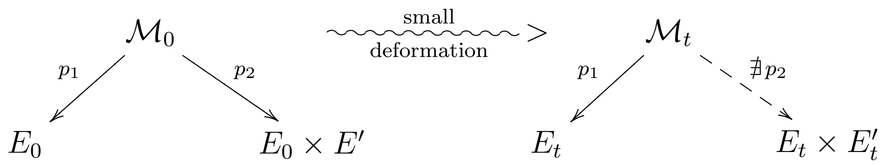
The condition on Dolbeault cohomology is satisfied if (M, J) is an iterated principal holomorphic torus bundle or if J is generic (see [CF01, CFGU00]) and conjecturally holds true for all left-invariant complex structures.

The strategy of the proof is to show that the Kuranishi family can be described using only left-invariant differential forms generalising results of [CFP06].

Stable geometries. In order to study deformations in the large we need more control over the geometry – the existence of a so-called stable torus bundle series.

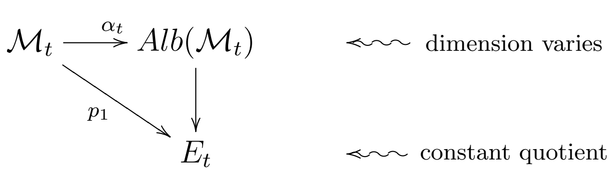
Example 2 — Let $\pi : S \to E_0$ be a Kodaira surface, i.e., a non-trivial principal bundle of elliptic curves and let E' be an elliptic curve. We consider a family of nilmanifolds $\mathcal{M} \to \Delta$ such that $\mathcal{M}_0 = S \times E'$.

After a general small deformation the projection to the product of curves will vanish, while the projection induced by π always remains:



Hence the right approach is to study M_t as a principal 2-torus bundles over an elliptic curve.

Analysing the Albanese map it turns out that



Note that even the $\mathcal{C}^\infty$-map underlying p_1 does not change.

This is the simplest example of a stable torus bundle series and we determined several condition under which these exist.

Deformations in the large. Studying deformations in the large of such constant quotients (if they exist) instead of the whole Albanese variety we can generalise the results in [Cat04, CF06]. For technical reasons it is better to formulate the results in the language of Lie theory.

Theorem B — *Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and let $\Gamma \subset G$ be a lattice such that the following holds:*

1) $\mathfrak{g}$ *admits a stable torus bundle series* $(\mathcal{S}^i\mathfrak{g})_{i=0,\dots,t}$.
2) *The nilmanifolds of the type* $(\mathcal{S}^{t-1}\mathfrak{g}, J, \Gamma \cap \exp(\mathcal{S}^{t-1}\mathfrak{g}))$ *constitute a good fibre class. (Examples are Tori or Kodaira surfaces.)*

Then any deformation in the large M' of a nilmanifold with left-invariant complex structure of type $M = (\Gamma\backslash G, J)$ carries a left-invariant complex structure.

In complex dimension 3 there are only 16 cases to check [Sal01]:

Theorem C — *Let $M = (\Gamma\backslash G, J)$ be an iterated, principal holomorphic torus bundle which has complex dimension at most 3.*

If G is not isomorphic to the real Lie group described in Example 1 then every deformation in the large of M is also an iterated principal holomorphic torus bundle.

In higher dimension there are several conditions on the structure of the Lie group under which the same conclusion as in Theorem C holds.

Final remarks. These results are part of my PhD Thesis [Rol07]. I would like to express my gratitude to my adviser Fabrizio Catanese for suggesting this research, constant encouragement and several helpful discussions.

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