Open Problems

Algebraic Geometry in Higher Dimensions

Levico Terme, June 2007

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Problems presented by E. Ballico

Problem 1

Question *Describe all the semistable torsion free sheaves on the integral projective curves C with $p_a(C) = 1$ having an ordinary cusp.*

For the case of nodal curves with $p_a = 1$ see Mozgovoy's paper math.AG/0410190

Problems presented by C. Bocci

Problem 2

Fix integers $k > 0$, $m \geq 3$, $a_i, n_i > 0$ for $1 \leq i \leq m$.
Consider the Segre-Veronese embedding

$$\prod_{i=1}^{m} \mathbb{P}^{k^{n_i}-1} \hookrightarrow \mathbb{P}^N$$

given by the line bundle $\mathcal{O}(a_1, \ldots, a_m)$. In particular $N+1 = \prod \binom{k^{n_i}+a_i-1}{a_i}$.
Set $Z := \mathrm{Sec}^{k-1}(X)$.

Question *Can we find a Groebner basis for the homogeneous ideal of Z?*

Experts know how to reduce the general case to the case where $\forall i\ n_i = 1$.

The good news is that even the case $m = 3$ will be interesting. There is work in progress by Allmar and Rhodes for $k = 4$; this case is linked to the DNA.

The bad news is that the next interesting case for phylogenetic trees is $k = 21$: aminoacids correspond to $\mathbb{P}^{20} \times \mathbb{P}^{20} \times \mathbb{P}^{20}$. The next one is $k = 61$ (codons) and they seems so much more difficult than the case $k = 4$ that mathematics for arbitrarily k should be used.

Problems presented by F. Campana

Problem 3

Let $S_d \subset \mathbb{P}^3$ be any smooth hypersurface of degree $d > 4$.

Question *For which pairs (d, m) with $4 < d < m$ does there exist surfaces S_d, S_m and a rational dominant map $S_m \dashrightarrow S_d$?*

Obviously, this is the case if d divides m (Fermat surfaces). Are there other pairs (d, m)?

Problem 4

Let (X/Δ) be a smooth projective orbifold. Thus $\Delta = \sum_j(1 - \frac{1}{m_j}).D_j$ is an effective $\mathbb{Q}$-divisor, with $1 < m_j \leq +\infty$ integers, the union $|\Delta|$ of the $D_j's$ being a normal crossing divisor.

Let $C \subset X$ be an irreducible rational curve with normalisation $n : \mathbb{P}^1 \to X$ such that C not included in $|\Delta|$. For all $j, p \in \mathbb{P}^1$ define $t_{j,p}$ by: $n^*(D_j) = \sum_{p\in\mathbb{P}^1} t_{j,p}.\{p\}$, and $\mu_p := max_j\{\lceil m_j/t_{j,p}\rceil\}; j$ such that: $t_{j,p} > 0$.

Let next $\Delta' := \sum_{p\in\mathbb{P}^1}(1 - \frac{1}{\mu_p}).\{p\}$: this is the smallest orbifold divisor on $\mathbb{P}^1$ making $n : (\mathbb{P}^1/\Delta') \to (X/\Delta)$ an orbifold morphism.

For example, if C has all of its contacts with each D_j of order at least m_j, then $\Delta' = \emptyset$.

Definition: C is an orbifold rational curve (ORC) if:
$\deg(K_{\mathbb{P}^1} + \Delta') = -2 + \sum_p(1 - \frac{1}{\mu_p}) < 0$.

Conjecture *If $\kappa(X/\Delta) = -\infty$, then (X/Δ) is uniruled, ie: (generically) covered by ORC's*

The conjecture is proved for surfaces if all $m_j's = +\infty$ by Keel-McKernan.

Specific example: $X = \mathbb{P}^2$, $\Delta = 2/3(L_3 + M_3) + 4/5L_5 + 6/7L_7$, with L_3, M_3, L_5, L_7 four lines in general position, is Fano, since $2/3 + 1/5 + 1/7 = 1 + 1/105 > 1$.

Question *Check that for small degrees N the ORC's do not cover $\mathbb{P}^2$ (are finite in number)*

Let $N = 105k, k > 0$. Are there ORC's of degree N having all their contacts with L_d, M_d (if $d = 3$) of order d, for $d = 3, 5, 7$? A simple parameter count gives $N(1/3 + 1/3 + 1/5 + 1/7 - 1) = k$-dimensional such families.

Question *Check that the resulting parametrising polynomials are coprime.*

Question *Describe other more conceptual constructions of such ORC's in this case.* Using deformation theory as Keel-McKernan do?

Problems presented by F. Catanese

Problem 5

Assume $X = \mathcal{H}^n/\Gamma$ is smooth compact, where $\mathcal{H}$ is the upper half plane. Then $\frac{dz_1 \cdots dz_n}{dz_1 \wedge \cdots \wedge dz_n}$ descends to a nonzero section

$$\theta \in H^0(S^n \Omega^1_X(-K_X) \otimes \eta)$$

where $\eta^2 \cong \mathcal{O}_X$.

For $n = 2$ (Catanese-Franciosi) X is a quotient of $\mathcal{H}^2/\Gamma$ if and only if $P_2(X) > 0$, there exists η with $\eta^2 \cong \mathcal{O}_X$, and there exists $\theta \in H^0(S^n \Omega^1_X(-K_X) \otimes \eta)$, $\theta \not\equiv 0$.

In fact one deduces from the above conditions that there is an étale cover X' of degree 2 such that $T_{X'}$ splits, and then one uses Beauville-Yau.

Question *Is there an elegant characterization in higher dimension?*

Problem 6

Let X_t, $t \in T$ be a flat family of projective varieties with canonical singularities.

Question *Under which conditions the rational connected reduction form a flat family?*

E.g. assume X_t has a rational reduction which has general type and has S_t as canonical model. Then do the S_t' form a flat family?

Problem 7

Assume a variety X^n is defined over $\overline{\mathbb{Q}}$.

Question *Is there a Belyi type theorem for dimension greater or equal than 2?*

Problems presented by C. Fontanari

Problem 8

Conjecture (Looijenga) *Is $\mathcal{M}_g$ union of $g-1$ affine subsets?*

Evidences:

(i) For $g = 2$, $\mathcal{M}_2$ is affine;

(ii) For $g = 3$, $\mathcal{M}_3 = U_1 \cup U_2$ where

- $U_1 = \mathcal{M}_3 \setminus$ the hyperelliptic locus
- $U_2 = \mathcal{M}_3 \setminus$ plane quartics with hyperflex

are affine;

(iii) Looijenga conjecture implies a theorem by Harer (Inv. Math. 1986) on the cohomological dimension of $\mathcal{M}_g$.

As remarked by Looijenga few days ago even the case $g = 4$ is still open.

Conjecture (Accola-Rauch $p-2$ conjecture) *Are there $p-2$ hypersurfaces in the Teichmüller space $\mathcal{T}_p$ (or in whatever covering of $\mathcal{M}_g$) intersecting precisely in the hyperelliptic locus?*

Evidence: If $p \leq 5$ then the vanishing of $p-2$ thetanulls characterizes hyperellipticity.

Remark (Schneider, Math. Nach. 2007): the above evidence fails for $p = 6$.

Question (By Fontanari and Previato) *For every $g \geq 3$ is there a geometrically defined effective divisor E on $\overline{\mathcal{M}}_g$ such that*

$$\operatorname{Supp} E \cap H_g \cap \mathcal{M}_g = \emptyset?$$

Here H_g is the hyperelliptic locus

Example: for $g = 3$ take $E_3 := \{\text{plane quartics with hyperflex}\}$. Then $E_3 \cap H_3 = \emptyset$. But $E(1) := \{C \in \mathcal{M}_g : h^0(C, (g+1)P) \geq 3\}$ is irreducible for $g \geq 4$ (Cukierman).

Motivation: $p-2$ conjecture + affirmative answer to this question imply (modulo ampleness, which seems to be a minor problem) Looijenga conjecture.

Problem 9

Question *How to characterize $\mathbb{P}^n$ among singular varieties?*

State of the art:

(1) A deep result by Chen-Tseng, generalizing the analogous result for the smooth case by Cho-Miyaoka-Shepherd Barron/Kebekus:

If X is a projective variety with L.C.I.Q. singularities such that $\forall C \subset X$, $-K_X \cdot C \geq n+1$ then $X \cong \mathbb{P}^n$.

(2) An easy remark by Ballico-Fontanari, following Ishihara and Campana-Peternell (Beutrage Alg. Geom., to appear):

Let X be a projective variety with at worst log terminal singularities, $E \subset T_X$ an ample vector subbundle of rank r.

(i) If $r = n$ then $X \cong \mathbb{P}^n$

(ii) If $r = n-1$ and X has only isolated siingularities then either $X \cong \mathbb{P}^n$ or $n = 2$ and X is a quadric cone.

Problems presented by F. Malaspina

Problem 10

A vector bundle E on a Fano manifold X is said to be Fano if $\mathbb{P}(E)$ is a Fano manifold.

A vector bundle E on $\mathbb{P}^n$ or on the smooth hyperquadric $Q_n \subset \mathbb{P}^{n+1}$ is said to have no inner cohomology if $h^i(E(t)) = 0$, $\forall t \in \mathbb{Z}$ and $\forall 2 \leq i \leq n-2$.

Long time ago Ancona, Peternell, Wiśniewski give a classification of rank 2 Fano vector bundles on $\mathbb{P}^n$ or Q_n for $n \geq 4$. They arise only when $n = 4, 5$ and in these cases they coincide with the vector bundles with no inner cohomology.

Question *Which is the connection between Fano vector bundles and bundles with no inner cohomology in the rank* 3 *case?*

Question *What happens for other (even singular) $n-$dimensional Fano varieties?*

In this case "no inner cohomology" may become "vanishing of $h^i(E \otimes L^{\otimes k})$ for all $k \in \mathbb{Z}$, for all $2 \leq i \leq n-2$, and all ample line bundles L.

Question *Is it possible to classify all rank two vector bundles on Q_4 not coming from $\mathbb{P}^4$ (over $\mathbb{C}$).*

Problems presented by E. Schlesinger

Problem 11

Question *Is every arithmetically Cohen–Macaulay subscheme of* $\mathbb{P}^N$ *glicci? In particular, is every zero-scheme in* $\mathbb{P}^3$ *glicci?*

A subscheme of $\mathbb{P}^N$ is arithmetically Cohen–Macaulay (ACM) if its homogeneous coordinate ring is Cohen-Macaulay. Thus zero dimensional subschemes are ACM, and a smooth positive dimensional subvariety of $\mathbb{P}^N$ is ACM if and only if it is projectively normal.

A subscheme of $\mathbb{P}^N$ is arithmetically Gorenstein (AG) if its homogeneous coordinate ring is a Gorenstein ring. A complete intersection is AG, but the converse is true only in codimension ≤ 2.

Two subschemes V and W of $\mathbb{P}^N$ of pure dimension n are directly linked (resp. G-linked) if, roughly speaking, their union is a complete intersection (resp. AG). Liaison (G-liaison) is the equivalence relation generated by direct links (G-links).

A theorem of Gaeta asserts that a codimension 2 ACM subscheme of $\mathbb{P}^N$ is licci, i.e., in the *li*aison *c*lass of a *c*omplete *i*ntersection.

In codimension > 2, liaison classes are too small to be useful, for example five points in general position in $\mathbb{P}^3$ are not licci, and G-liaison has been introduced to remedy this problem [**4**]. It is still an open problem whether in codimension > 2 every arithmetically Cohen–Macaulay subscheme of $\mathbb{P}^N$ is glicci, i.e., in the G-liaison class of a complete intersection. In particular, it is not known whether every zero-scheme in $\mathbb{P}^3$ is glicci.

Some related results are:

(1) Every determinantal subscheme is glicci: a special case is proven in [**4**], the general case is treated in [**1**].
(2) Every Borel-fixed ACM monomial ideal is glicci. In particular, every ACM subscheme is glicci up to flat deformation [**5**].
(3) A set of n general points on a smooth quadric or cubic surface in $\mathbb{P}^3$ is glicci; in particular, a set of $n \leq 19$ general points in $\mathbb{P}^3$ is glicci [**2**].
(4) On the other hand, it is not known whether a set of 20 points in general position in $\mathbb{P}^3$ is glicci, and a set of $n \geq 56$ points in general position in $\mathbb{P}^3$ admits no direct G-liaison to a set of points of smaller degree [**3**].

Part of the difficulty of the problem stems from the limited understanding we have of AG schemes. For example

Question

Is there an AG zero dimensional subscheme of $\mathbb{P}^3$ *of degree* 30 *containing* 20 *points in general position ?*

References

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[2] Hartshorne, R., Some examples of Gorenstein liaison in codimension three, *Collect. Math.* **53** (2002) 21–48.

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[5] Migliore, J. C., and Nagel, U., Monomial ideals and the Gorenstein liaison class of a complete intersection, *Compositio Math.* **133** (2002), 25–36.