Geometry over fields

ABSTRACT. This notes provide a short summary of the main results contained in Chapter 3 of the book Geometry: Euclid and Beyond, by Robin Hartshorne.

1. The Cartesian plane over a field

DEFINITION 1.1. A field is a set F with two operations, $+$ and $\cdot$, such that

(1) $(F, +)$ is an abelian group; we denote by 0 its identity element.
(2) Setting $F^* = F \setminus \{0\}$, then $(F^*, \cdot)$ is an abelian group; we denote by 1 its identity element.
(3) The distributive law holds:

$$a(b+c) = ab + ac \qquad \forall a, b, c \in F.$$

DEFINITION 1.2. The *cartesian plane* over the field F, denoted by Π_F is the set F^2 of ordered pairs of elements of F, which are called the *points* of Π_F. A *line* is a subset of Π_F defined by a linear equation $ax + by + c = 0$, with $a, b, c \in F$ such that $(a, b) \neq (0, 0)$. If $b \neq 0$ the line can be written as $y = mx + q$, and m is called the *slope* of the line. If $b = 0$, then we say that the slope of the line is ∞.

PROPOSITION 1.3. *For any field F, the cartesian plane Π_F satisfies Hilbert's axioms* (I1), (I2), (I3) *and* (P).

PROOF. The usual formula for finding the line by two points in the real cartesian plane works in any field, so, given two points, we can always find a (unique) line passing by them, and (I1) holds.

Since any field F has at least the two distinct elements $0, 1$, by putting $x = 0, 1$ if the line has the form $y = mx + b$, or by putting $y = 0, 1$, if the line is $x = c$, we obtain two points on any line, and (I2) holds.
(13) says that there exist three noncollinear points. Indeed, we can always take (0,0), (0,1), (1,0), and we can see easily that these do not lie on any line.
In the plane Π_F, we see immediately that two lines are parallel if and only if they have the same slope. So given a line ℓ, let its slope be m. Then the familiar "point-slope" formula of analytic geometry shows that there is a unique line of slope m passing through a given point A. This will be the unique parallel to ℓ passing by A, so (P) holds in the stronger form, and Π_F is an affine plane. □

2. Ordered fields and betweenness

DEFINITION 2.1. An ordered field (F, P) is a pair where F is a field, and $P \subset F$ - called the subset of *positive* elements, satisfies:

(1) If $a, b \in P$, then $a + b \in P$ and $ab \in P$.
(2) $\forall a \in F$ one and only one of the following holds: $\quad a \in P, \quad a = 0, \quad -a \in P$.

Some straightforward properties of an ordered field are listed in the following proposition:

PROPOSITION 2.2. *If (F, P) is an ordered field, then*

a) $1 \in P$.
b) *F has characteristic zero.*
c) *For every $a \neq 0$, $a^2 \in P$*

PROOF. In any field, $1 \neq 0$, so either $1 \in P$ or $-1 \in P$. In the second case, by property (1) we have $(-1)(-1) = 1 \in P$, contradicting property (2).

Take $x \in P$; then the sum $x + x + x + \cdots + x = nx$ is also in P for any n. In particular, such a sum is never zero, so F has characteristic 0.

Finally, if $a \neq 0$ then either $a \in P$ or $-a \in P$. If $a \in P$, then $a^2 \in P$ by property (1). If $-a \in P$, then $(-a)(-a) = a^2 \in P$, again by property (1). □

REMARK 2.3. It is easy to show that, if (F, P) is an ordered field, then the relation defined as $a > b$ if and only if $a - b \in P$ is a strict total order on F. Clearly an element $a \in P$ if and only if $a > 0$.

PROPOSITION 2.4 (15.3). *If F is a field, and if there is a notion of betweenness in the Cartesian plane Π_F satisfying Hilbert's axioms* (B1)-(B4)*, then F is an ordered field. Conversely, if (F, P) is an ordered field, we can define betweenness in Π_F so as to satisfy* (B1)-(B4).

The idea of the proof is the following: if Π_F has a notion of betweenness as above we define $P \subset F$ to be the set

$$\{a \in F \mid (a, 0) \text{ is on the same side of } (0,0) \text{ as } (1,0) \text{ in the line } y = 0\}.$$

Conversely, if F is an ordered field, given $A = (a_1, a_2)$, $B = (b_1, b_2)$ and $C = (c_1, c_2)$ then $A * B * C$ if one of the following holds:

- $a_1 < b_1 < c_1$;
- $a_1 > b_1 > c_1$;
- $a_1 = b_1 = c_1$ and $a_2 < b_2 < c_2$ or $a_2 > b_2 > c_2$.

Equivalently $A * B * C$ if and only if there exists $\lambda \in F$ with $0 < \lambda < 1$ such that

$$\lambda a_i + (1 - \lambda) c_i = b_i \qquad i = 1, 2.$$

3. Congruences of segments and angles

Given an ordered field (F, P) and points $A = (a_1, a_2)$, $B = (b_1, b_2)$ in Π_F we can define

$$\operatorname{dist}^2(A, B) = (a_1 - b_1)^2 + (a_2 - b_2)^2,$$

and thus a relation of congruence of segments:

$$\overline{AB} \cong \overline{CD} \quad \text{if and only if} \quad \operatorname{dist}^2(A, B) = \operatorname{dist}^2(C, D).$$

PROPOSITION 3.1. *The relation of congruence of segments just defined satisfies Hilbert's axioms* (C2)-(C3).

PROOF. The validity of (C2) follows directly from the definition. To prove (C3), assume that we have $A * B * C$ and $D * E * F$ with $\overline{AB} \simeq \overline{DE}$ and $\overline{BC} \simeq \overline{EF}$. There exist λ, μ with $0 < \lambda < 1$ and $0 < \mu < 1$ such that

$$\lambda a_i + (1 - \lambda) c_i = b_i \qquad i = 1, 2.$$

$$\mu d_i + (1-\mu) f_i = e_i \qquad i = 1, 2.$$

From the congruence of segments we get

$$(1-\lambda)^2[(a_1-c_1)^2 + (a_2-c_2)^2] = (1-\mu)^2[(d_1-f_1)^2 + (d_2-f_2)^2]$$

$$\lambda^2[(a_1-c_1)^2 + (a_2-c_2)^2] = \mu^2[(d_1-f_1)^2 + (d_2-f_2)^2]$$

From which we get

$$\left(\frac{1-\lambda}{\lambda}\right)^2 = \left(\frac{1-\mu}{\mu}\right)^2.$$

Let us notice that, if $x, y \in P$ and $x^2 = y^2$ then $x = y$. In fact we can write $0 = x^2 - y^2 = (x-y)(x+y)$ and the second factor cannot be zero since it belongs to P. It follows then that $\lambda = \mu$, hence

$$\operatorname{dist}^2(A, C) = [(a_1-c_1)^2 + (a_2-c_2)^2] = [(d_1-f_1)^2 + (d_2-f_2)^2] = \operatorname{dist}^2(D, F). \quad \square$$

REMARK 3.2. We cannot expect (C1) to hold in the Cartesian plane on every ordered field: take for instance $F = \mathbb{Q}$, $A = (0,0)$, $B = (1,0)$ and the ray r originating at $(0,0)$ and passing by $(1,1)$. The point E on $\overrightarrow{CD}$ such that $\overline{OE} \simeq \overline{AB}$ has coordinates $(\sqrt{2}/2, \sqrt{2}/2)$, so it is not in $\Pi_{\mathbb{Q}}$.

Given an angle α, defined by two rays r, r' with a common point, lying on lines ℓ, ℓ', of slopes m, m', we say that α is a *right* angle if either $mm' = -1$ or, up to exchange the lines, $(m, m') = (0, \infty)$. We say that α is an *acute* angle if it is contained in the interior of a right angle, an *obtuse* angle if it contains a right angle in its interior.
We define the tangent of α in the following way:

$$\tan\alpha := \begin{cases} \infty & \text{if } \alpha \text{ is a right angle} \\[2ex] \left|\dfrac{m'-m}{1+mm'}\right| & \text{if } \alpha \text{ is an acute angle} \\[2ex] -\left|\dfrac{m'-m}{1+mm'}\right| & \text{if } \alpha \text{ is an obtuse angle} \end{cases}$$

with the conventions

$$\frac{\infty - m}{1 + \infty \cdot m} = \frac{1}{m} \qquad\qquad \frac{m' - \infty}{1 + \infty \cdot m'} = -\frac{1}{m'}.$$

We say that two angles α and β are congruent if and only if $\tan\alpha = \tan\beta$.

PROPOSITION 3.3. *The relation of congruence of angles just defined satisfies Hilbert's axioms* (C4)-(C5).

PROOF. Suppose we are given an angle α and a ray emanating from a point A with slope m We must find a line passing through A with slope m' such that

$$\tan(\alpha) = \mp \left|\frac{m-m'}{1+mm'}\right|$$

where the sign is adjusted according to whether α is acute or obtuse. This gives equations that are linear in m', and so can be solved in F and (C4) holds. The validity of (C5) is clear from the definition. □

3.1. Pytagorean fields.

DEFINITION 3.4. A field F is called *Pythagorean* if for every $a \in F$ there exists $b \in F$ such that $b^2 = 1 + a^2$.

EXAMPLE 3.5. *The field $\mathbb{Q}$ of rational numbers is not Pythagorean (take, for instance, $a = 1$).*

PROPOSITION 3.6 (16.1 - 17.2 - 17.1). *If F is a Pythagorean, ordered field, then the relations of congruence of segments and angles defined above in Π_F satisfy Hilbert's axioms* (C1)-(C6).

In order to prove (C6), we use the fact that it is equivalent to (ERM) - see section 5.

COROLLARY 3.7. *If F is a Pythagorean, ordered field, then Π_F is an Hilbert plane satisfying* (P).

4. Euclidean fields and planes

DEFINITION 4.1. An ordered field (F, P) is called *Euclidean* if for every $a \in P$ there exists $b \in F$ such that $b^2 = a$.

REMARK 4.2. Clearly, an Euclidean field is also a Pythagorean field. The converse does not hold, but there are no easy counterexamples.

PROPOSITION 4.3 (16.2). *If F is an ordered field and Π_F is the Cartesian plane over F then the following are equivalent:*

(1) *F is Euclidean;*
(2) *Π_F satisfies the circle-circle intersection property* (E);

(3) Π_F *satisfies the line-circle intersection property* (LCI) *(see Proposition 11.6).*

DEFINITION 4.4. An Euclidean plane is an Hilbert plane satisfying (P) and (E).

COROLLARY 4.5. *If F is an ordered Euclidean field and Π_F is the Cartesian plane over F then Π_F is an Euclidean plane.*

THEOREM 4.6 (10.4 - 11.8 - 12.1 - 12.3 - 12.4 - 12.5). *Euclid's propositions contained in Books I-IV are valid in an Euclidean plane.*

5. Archimedean fields

DEFINITION 5.1. An ordered field F is called *Archimedean* if, for every $a \in F$ there exists $n \in \mathbb{N}$ such that $n \cdot 1 > a$.

DEFINITION 5.2. An Hilbert plane Π satisfies (A) if, given two segments $\overline{AB}$ and $\overline{CD}$, there exists $n \in \mathbb{N}$ such that $n\overline{AB} > \overline{CD}$ where $n\overline{AB}$ is a segment obtained by adding n copies of the segment $\overline{AB}$.

It can be shown that

PROPOSITION 5.3 (15.4 -15.5). *An ordered field is Archimedean if and only if Π_F, the cartesian plane over F, satisfies* (A)*. Moreover, in this case, F is isomorphic (as an ordered field) to a subfield of $\mathbb{R}$.*

EXAMPLE 5.4. *Let us give an example of a non-Archimedean field. Set $F := \mathbb{R}(t)$, the quotient field of $\mathbb{R}[t]$, the ring of polynomials in one variable with real coefficients. An element φ of F can be written as a quotient of two polynomials $f(t), g(t)$, with $g(t) \not\equiv 0$; considering it as a function $\varphi(t)$, defined over $\mathbb{R}$ minus the zeroes of g, we can define $P \subset F$ as follows:*

$$\varphi \in P \quad \Longleftrightarrow \quad \exists x_0 \in \mathbb{R} \quad \text{t.c.} \quad \varphi(t) > 0 \quad \forall t > x_0.$$

It is straightforward to verify that P satisfies properties (1) and (2) in Definition 2.1, so it defines an order in F.
Tale now $\varphi(t) = t$; it is clear that $t - n \in P$ for every $n \in \mathbb{N}$, so $t > n \quad \forall n \in \mathbb{N}$, hence F is not Archimedean.

6. Rigid motions and SAS

DEFINITION 6.1. Let Π be a geometry consisting of the undefined notions of points, lines, betweenness and congruence of line segments and angles, which may or may not satisfy various of Hilbert's axioms. A *rigid motion* of Π is a bijection $\varphi : \Pi \to \Pi$ such that:

(1) φ sends lines into lines.
(2) φ preserves betweenness of collinear points.
(3) For any two points A, B we have $\overline{AB} \simeq \overline{\varphi(A)\varphi(B)}$.
(4) For any angle α, we have $\alpha \simeq \varphi(\alpha)$.

It is clear that the composition of two rigid motions is again a rigid motion, and it is not difficult to show that the inverse of a rigid motion is again a rigid motion, hence the set of rigid motions of Π is a group with respect to composition.

Let Π be a geometry consisting of the undefined notions of point, line, betweenness, and congruence of line segments and angles. Then Π satisfies (ERM) if

(1) For any two points $A, A' \in \Pi$, there is a rigid motion φ s.t. $\varphi(A) = A'$.
(2) For any three points O, A, A', there is a rigid motion φ s.t. $\varphi(O) = O$ and φ sends the ray $\overrightarrow{OA}$ to the ray $\overrightarrow{OA}'$.
(3) For any line ℓ, there is a rigid motion φ s.t. φ(P) = P for all $P \in \ell$ and φ interchanges the two sides of ℓ.

It can be shown that

PROPOSITION 6.2 (17.1, 17.4). *Let* Π *be a geometry satisfying* (I1)-(I3), (B1)-(B4), (C1)-(C5). *Then* Π *satisfies* (C6) *if and only if it satisfies* (ERM).

PROOF. We saw in the classroom this proof, which can be found on the textbook, combining the proofs of Propositions 17.1 and 17.4. $\square$