Fano manifolds whose elementary contractions are smooth $\mathbb{P}^1$-fibrations

Gianluca Occhetta

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Motivation

Fano bundles

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Later the assumption $b_4 = 1$ was removed by Watanabe.

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- $\mathcal{Q}$ universal quotient bundle.

Varieties with two $\mathbb{P}^1$-bundle structures

A different perspective

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Theorem

A Fano manifold with Picard number 2 and two $\mathbb{P}^1$-bundle structures is isomorphic to a complete flag manifold of type $A_1 \times A_1$, A_2, B_2 or G_2.

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Theorem

X Fano manifold whose elementary contractions are smooth $\mathbb{P}^1$-fibrations such that X is not a product. If $\dim X \neq 24$ then X is a complete flag manifold.

Lie algebras and complete flags

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Cartan decomposition

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The spaces $\mathfrak{g}_\alpha$ are defined by

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h, g] = \alpha(h) g, \text{ for every } h \in \mathfrak{h}\} ,$$

and the elements $\alpha \in \mathfrak{h}^\vee \setminus \{0\}$ such that $\mathfrak{g}_\alpha \neq 0$ are called roots of $\mathfrak{g}$. The (finite) set Φ of such elements is called root system of $\mathfrak{g}$.

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A basis of $\mathfrak{h}^\vee$ formed by elements of Φ such that the coordinates of every element of Φ are integers, all ≥ 0 or all ≤ 0 is a system of simple roots of $\mathfrak{g}$.

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$W \subset \mathrm{Gl}(E)$ generated by $\{\sigma_\alpha, \alpha \in \Phi\}$ is the Weyl group of $\mathfrak{g}$.

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In particular the possible 2×2 principal minors are (up to transposition)

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

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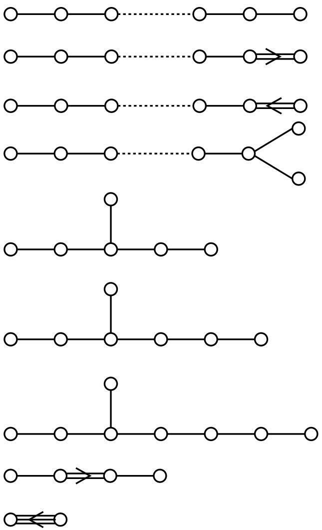
Dynkin diagrams

of (semi)simple Lie algebras



A_n	SL_{n+1}	CLASSICAL
B_n	SO_{2n+1}	
C_n	Sp_{2n}	
D_n	SO_{2n}	
E_6		EXCEPTIONAL
E_7		
E_8		
F_4		
G_2		

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- Elementary contractions of G/B are $\mathbb{P}^1$-bundles.
- Every rational homogenous manifold is dominated by a complete flag manifold.
- If $f : Z \to G/B$ is a surjective morphism from a rational homogenous manifold, then $Z = G/B \times Z'$, and f is the projection.

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The Cartan matrix of $\mathcal{D}$ is the intersection matrix $-K_i \cdot C_j$.

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Reflection group

Relative duality

Lemma

$\pi : M \to Y$ smooth $\mathbb{P}^1$-fibration (M, Y smooth).
Γ fiber, K relative canonical, D divisor on M, $l := D \cdot \Gamma$.

$$\begin{aligned} H^i(M,D) &\cong H^{i+\mathrm{sgn}(l+1)}(M, D+(l+1)K), \forall i \in \mathbb{Z},\ se\ l \neq -1 \\ H^i(M,D) &\cong \{0\}\ \text{for every } i \in \mathbb{Z},\ \text{if } l = -1. \end{aligned}$$

Corollary

$\chi(M,D) = -\chi(M, D+(l+1)K)$

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For every contraction $\pi_i : X \to X$ let us consider the affine involution $r_i' : N^1(X) \to N^1(X)$

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Setting $T(D) := D + K_X/2$ the maps $r_i := T^{-1} \circ r_i' \circ T$ are liner involutions of $N^1(X)$ given by

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We have $r_i(K_i) = -K_i$ and r_i fixes pointwise the hyperplane

$$M_i := \{D | D \cdot \Gamma_i = 0\} \subset N^1(X).$$

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Let $W \subset \mathrm{Gl}(N^1(X))$ be the group generated by the r_i's.

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Let $\chi_X : N^1(X) \to \mathbb{R}$ be the polynomial (of degree $\leq \dim X$) such that

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Corollary

For every $\mathbb{R}$-divisor D and every $w \in W$ $\quad \chi_X^T(D) = \pm\chi_X^T(w(D)).$

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and let $\chi_X^T := \chi_X \circ T$.

Lemma

For every $\mathbb{R}$-divisor D and every r_i $\quad \chi_X^T(D) = -\chi_X^T(r_i(D))$.

Corollary

For every $\mathbb{R}$-divisor D and every $w \in W$ $\quad \chi_X^T(D) = \pm\chi_X^T(w(D))$.

Theorem

W is a finite group.

Cartan matrix

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The Cartan matrix $\mathcal{A}$ of this root system is the $n \times n$ matrix with entries $a_{ij} := -K_i \cdot \Gamma_j \in \mathbb{Z}$.

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Using geometric properties of X one can show that a_{ij} with $i \neq j$ are nonpositive.

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X \arrow[d, "\pi_i"] \\ \Gamma_j \arrow[u, bend left, "s_j"] \arrow[ur] \arrow[r, "\pi_i|_{\Gamma_j}"'] & X_i \end{tikzcd}$$

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As a consequence we can show

Theorem

Every connected component of $\mathcal{D}$ is one of the following: A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2.

Homogeneous models

We associate with X a semi simple Lie group G, determined by $\mathcal{D}$.

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Proposition

$\Lambda \subset \mathrm{Pic}(X)$ generated by the K_i's.

- $\dim X = \dim G/B$;
- $h^i(X, D) = h^i(G/B, \psi(D))$ for every $D \in \Lambda,\ i \in \mathbb{Z}$.

Bott-Samelson varieties

$x \in X$, $\ell = (l_1, \dots, l_r)$, $l_i \in D$, and $\ell[s] := (l_1, \dots, l_{r-s})$.

We introduce manifolds $Z_{\ell[s]}$, with morphisms $f_{\ell[s]} : Z_{\ell[s]} \to X$, called Bott-Samelson varieties associated with the subsequences $\ell[s]$, in the following way:

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the map $f_{\ell[s+1]}$ factors via $Z_{\ell[s]}$, and gives a section $\sigma_{\ell[s+1]}$ di $p_{\ell[s+1]}$.

In particular $p_{\ell[s+1]}$ is a $\mathbb{P}^1$-bundle, given by the projectivization of an extension $\mathcal{F}_{\ell[s]}$ of $\mathcal{O}_{Z_{\ell[s+1]}}$ with $\mathcal{O}_{Z_{\ell[s+1]}}(f^*_{\ell[s+1]}(K_{l_{r-s}}))$:

$$0 \to \mathcal{O}_{Z_{\ell[s+1]}}(f^*_{\ell[s+1]}(K_{l_{r-s}})) \longrightarrow \mathcal{F}_{\ell[s]} \longrightarrow \mathcal{O}_{Z_{\ell[s+1]}} \to 0,$$

determined by $\zeta_{\ell[s]} \in H^1(Z_{\ell[s+1]}, f^*_{\ell[s+1]}(K_{l_{r-s}}))$.

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Given $w \in W$, lits length $\lambda(w)$ is the minimum t such that $w = r_{i_1} \circ \cdots \circ r_{i_t}$; such an expression for w is called *reduced*.
In W there exists a unique w_0 of length $\dim X$, and all the other elements are shorter.

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In the homogeneous case $\dim f_\ell(Z_\ell) = \lambda(w(\ell))$; moreover, if $w(\ell)$ is reduced then $f_\ell : Z_\ell \to f(Z_\ell)$ is birational.

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We show that the same properties hold in general.

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Uniqueness

X, with connected diagram $\mathcal{D}$, G/B homogeneus model, ℓ sequence.

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Let ℓ_0 be a list corresponding to the longest element in W: $w(\ell_0) = w_0$.
If $Z_{\ell_0} \simeq \overline{Z}_{\ell_0}$, then $X \simeq G/B$.

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Proposizione

If $\mathcal{D} \neq \mathrm{F}_4, \mathrm{G}_2$ there exists $\ell = (l_1, \ldots, l_m)$ with $w(\ell) = w_0$ such that $Z_{\ell[s]} \simeq \overline{Z}_{\ell[s]}$ for every $s = 0, \ldots, m-1$.

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Z_ℓ is defined by the extension

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So we have to show that if the index j appears in $\ell[1]$ then $h^1(Z_{\ell[1]}, f^*_\ell(K_j)) \leq 1$.

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- F_4: no expression (there are 2144892) of w_0 works. We are working on an ad hoc argument.

Campana-Peternell Conjecture

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- Examples:

Homogeneous manifolds $\to$ Abelian

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Theorem (Demailly, Peternell and Schneider (1994))

$$T_X \textit{ nef} \Rightarrow \begin{cases} X \xleftarrow{\text{étale}} X' \xrightarrow{F} A \\ A \textit{ Abelian, } F \textit{ Fano, } T_F \textit{ nef} \end{cases}$$

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A possible alternative strategy is:

1. Prove the conjecture for CP-manifolds with "maximal" Picard number.
2. Show that any CP-manifold is dominated by a CP-manifold with "maximal" Picard number.