A characterization of complete flag manifolds

Gianluca Occhetta

with R. Muñoz, L.E. Solá Conde, K. Watanabe and J. Wiśniewski

Busan, January 2016

Outline

Campana-Peternell Conjecture

Positivity of the tangent bundle

X smooth complex projective variety.

Positivity of the tangent bundle

X smooth complex projective variety.

Theorem [Mori (1979)]

T_X ample $\Leftrightarrow X = \mathbb{P}^m$.

Positivity of the tangent bundle

X smooth complex projective variety.

Theorem [Mori (1979)]

T_X ample $\Leftrightarrow X = \mathbb{P}^m$.

- T_X nef $\Rightarrow$??

Positivity of the tangent bundle

X smooth complex projective variety.

Theorem [Mori (1979)]

T_X ample $\Leftrightarrow X = \mathbb{P}^m$.

- T_X nef $\Rightarrow$??
- Examples:

$$\begin{tikzcd} & \text{Abelian} \\ \text{Homogeneous manifolds} \arrow[ur] \arrow[dr] & \\ & \text{Rational} \end{tikzcd}$$

Positivity of the tangent bundle

X smooth complex projective variety.

Theorem [Mori (1979)]

T_X ample $\Leftrightarrow X = \mathbb{P}^m$.

- T_X nef $\Rightarrow$??
- Examples:

$$\begin{tikzcd} & \text{Abelian} \\ \text{Homogeneous manifolds} \arrow[ur] \arrow[dr] & \\ & \text{Rational} \end{tikzcd}$$

Theorem [Demailly, Peternell and Schneider (1994)]

$$T_X \text{ nef} \Rightarrow \begin{cases} \begin{tikzcd} X & X' \arrow[l, "\text{étale}"'] \arrow[r, "F"] & A \end{tikzcd} \\ A \text{ Abelian, } F \text{ Fano, } T_F \text{ nef} \end{cases}$$

Campana-Peternell Conjecture

Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

Campana-Peternell Conjecture

Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

Results:

☑ $\dim X = 3$ [Campana & Peternell (1991)]

Campana-Peternell Conjecture

Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

Results:

- ☑ $\dim X = 3$ [Campana & Peternell (1991)]
- ☑ $\dim X = 4$ [CP (1993), Mok (2002), Hwang (2006)]

Campana-Peternell Conjecture

Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

Results:

- ☑ $\dim X = 3$ [Campana & Peternell (1991)]
- ☑ $\dim X = 4$ [CP (1993), Mok (2002), Hwang (2006)]
- ☑ $\dim X = 5$ and $\rho_X > 1$ [Watanabe (2012)]

Campana-Peternell Conjecture

Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

Results:

- ☑ $\dim X = 3$ [Campana & Peternell (1991)]
- ☑ $\dim X = 4$ [CP (1993), Mok (2002), Hwang (2006)]
- ☑ $\dim X = 5$ and $\rho_X > 1$ [Watanabe (2012)]
- ☑ $\dim X = 5$ [Kanemitsu (2015)]

Campana-Peternell Conjecture

Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

Results:

- ☑ $\dim X = 3$ [Campana & Peternell (1991)]
- ☑ $\dim X = 4$ [CP (1993), Mok (2002), Hwang (2006)]
- ☑ $\dim X = 5$ and $\rho_X > 1$ [Watanabe (2012)]
- ☑ $\dim X = 5$ [Kanemitsu (2015)]
- ☑ T_X big and 1-ample [Solá-Conde & Wiśniewski (2004)]

Campana-Peternell Conjecture

Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

Results:

- ☑ $\dim X = 3$ [Campana & Peternell (1991)]
- ☑ $\dim X = 4$ [CP (1993), Mok (2002), Hwang (2006)]
- ☑ $\dim X = 5$ and $\rho_X > 1$ [Watanabe (2012)]
- ☑ $\dim X = 5$ [Kanemitsu (2015)]
- ☑ T_X big and 1-ample [Solá-Conde & Wiśniewski (2004)]
- ☑ X horospherical [Li (2015)]

Fano bundles

Fano bundles

A vector bundle $\mathcal{E}$ on a smooth complex projective variety X is a Fano bundle iff $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.

Fano bundles

A vector bundle $\mathcal{E}$ on a smooth complex projective variety X is a Fano bundle iff $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.

If $\mathcal{E}$ is a Fano bundle on X then X is a Fano manifold.

Fano bundles

A vector bundle $\mathcal{E}$ on a smooth complex projective variety X is a Fano bundle iff $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.

If $\mathcal{E}$ is a Fano bundle on X then X is a Fano manifold.

Fano bundles of rank 2 on $\mathbb{P}^m$ and $\mathbb{Q}^m$ have been classified in the '90s (Ancona, Peternell, Sols, Szurek, Wiśniewski).

Fano bundles

A vector bundle $\mathcal{E}$ on a smooth complex projective variety X is a Fano bundle iff $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.

If $\mathcal{E}$ is a Fano bundle on X then X is a Fano manifold.

Fano bundles of rank 2 on $\mathbb{P}^m$ and $\mathbb{Q}^m$ have been classified in the '90s (Ancona, Peternell, Sols, Szurek, Wiśniewski).

Generalization: Classification of Fano bundles of rank 2 on (Fano) manifolds with $b_2 = b_4 = 1$ (MOS, 2012).

Fano bundles

A vector bundle $\mathcal{E}$ on a smooth complex projective variety X is a Fano bundle iff $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.

If $\mathcal{E}$ is a Fano bundle on X then X is a Fano manifold.

Fano bundles of rank 2 on $\mathbb{P}^m$ and $\mathbb{Q}^m$ have been classified in the '90s (Ancona, Peternell, Sols, Szurek, Wiśniewski).

Generalization: Classification of Fano bundles of rank 2 on (Fano) manifolds with $b_2 = b_4 = 1$ (MOS, 2012).

As a special case we have the classification of Fano manifolds of Picard number two (and $b_4 = 2$) with two $\mathbb{P}^1$-bundle structures.

Fano bundles

A vector bundle $\mathcal{E}$ on a smooth complex projective variety X is a Fano bundle iff $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.

If $\mathcal{E}$ is a Fano bundle on X then X is a Fano manifold.

Fano bundles of rank 2 on $\mathbb{P}^m$ and $\mathbb{Q}^m$ have been classified in the '90s (Ancona, Peternell, Sols, Szurek, Wiśniewski).

Generalization: Classification of Fano bundles of rank 2 on (Fano) manifolds with $b_2 = b_4 = 1$ (MOS, 2012).

As a special case we have the classification of Fano manifolds of Picard number two (and $b_4 = 2$) with two $\mathbb{P}^1$-bundle structures.

Later the assumption on b_4 was removed by Watanabe (2013).

Fano bundles

A vector bundle $\mathcal{E}$ on a smooth complex projective variety X is a Fano bundle iff $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.

If $\mathcal{E}$ is a Fano bundle on X then X is a Fano manifold.

Fano bundles of rank 2 on $\mathbb{P}^m$ and $\mathbb{Q}^m$ have been classified in the '90s (Ancona, Peternell, Sols, Szurek, Wiśniewski).

Generalization: Classification of Fano bundles of rank 2 on (Fano) manifolds with $b_2 = b_4 = 1$ (MOS, 2012).

As a special case we have the classification of Fano manifolds of Picard number two (and $b_4 = 2$) with two $\mathbb{P}^1$-bundle structures.

Later the assumption on b_4 was removed by Watanabe (2013).

Finally the assumption "$\mathbb{P}^1$-bundle" was replaced by "smooth $\mathbb{P}^1$-fibration" (MOSWa 2013).

Varieties with two $\mathbb{P}^1$-fibrations

Theorem

A Fano manifold with Picard number 2 whose elementary contractions are $\mathbb{P}^1$-fibrations is isomorphic to one of the following

Varieties with two $\mathbb{P}^1$-fibrations

Theorem

A Fano manifold with Picard number 2 whose elementary contractions are $\mathbb{P}^1$-fibrations is isomorphic to one of the following

- $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O})$

Varieties with two $\mathbb{P}^1$-fibrations

Theorem

A Fano manifold with Picard number 2 whose elementary contractions are $\mathbb{P}^1$-fibrations is isomorphic to one of the following

- $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O})$
- $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$

Varieties with two $\mathbb{P}^1$-fibrations

Theorem

A Fano manifold with Picard number 2 whose elementary contractions are $\mathbb{P}^1$-fibrations is isomorphic to one of the following

- $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O})$
- $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$
- $\mathbb{P}_{\mathbb{P}^3}(\mathcal{N}) = \mathbb{P}_{\mathbb{Q}^3}(\mathcal{S})$ - $\mathcal{N}$ Null-correlation , $\mathcal{S}$ Spinor

Varieties with two $\mathbb{P}^1$-fibrations

Theorem

A Fano manifold with Picard number 2 whose elementary contractions are $\mathbb{P}^1$-fibrations is isomorphic to one of the following

- $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O})$
- $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$
- $\mathbb{P}_{\mathbb{P}^3}(\mathcal{N}) = \mathbb{P}_{\mathbb{Q}^3}(\mathcal{S})$ - $\mathcal{N}$ Null-correlation , $\mathcal{S}$ Spinor
- $\mathbb{P}_{\mathbb{Q}^5}(\mathcal{C}) = \mathbb{P}_{K(G_2)}(\mathcal{Q})$ - $\mathcal{C}$ Cayley, $\mathcal{Q}$ universal quotient.

Varieties with two $\mathbb{P}^1$-fibrations

Theorem

A Fano manifold with Picard number 2 whose elementary contractions are $\mathbb{P}^1$-fibrations is isomorphic to one of the following

- $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O})$
- $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$
- $\mathbb{P}_{\mathbb{P}^3}(\mathcal{N}) = \mathbb{P}_{\mathbb{Q}^3}(\mathcal{S})$ - $\mathcal{N}$ Null-correlation , $\mathcal{S}$ Spinor
- $\mathbb{P}_{\mathbb{Q}^5}(\mathcal{C}) = \mathbb{P}_{K(G_2)}(\mathcal{Q})$ - $\mathcal{C}$ Cayley, $\mathcal{Q}$ universal quotient.

All the varieties appearing in the list are rational homogeneous!

A generalization

- Classify Fano manifolds whose elementary contractions are $\mathbb{P}^1$-bundles - or just smooth $\mathbb{P}^1$-fibrations.

A generalization

- Classify Fano manifolds whose elementary contractions are $\mathbb{P}^1$-bundles - or just smooth $\mathbb{P}^1$-fibrations.
- The vector bundle approach seems difficult to apply to this more general situation.

A generalization

- Classify Fano manifolds whose elementary contractions are $\mathbb{P}^1$-bundles - or just smooth $\mathbb{P}^1$-fibrations.
- The vector bundle approach seems difficult to apply to this more general situation.
- Is it possible to prove directly that these varieties are rational homogeneous?

Rational Homogeneous Manifolds

Lie Algebras

- G semisimple Lie group,

Lie Algebras

- G semisimple Lie group,
- $\mathfrak{g}$ associated Lie algebra,

Lie Algebras

- G semisimple Lie group,
- $\mathfrak{g}$ associated Lie algebra,
- n rank of $\mathfrak{g}$.

Lie Algebras

- G semisimple Lie group,
- $\mathfrak{g}$ associated Lie algebra,
- n rank of $\mathfrak{g}$.

With $\mathfrak{g}$ is associated a square $n \times n$ matrix $A = [a_{ij}]$, called Cartan matrix, which encodes the structure of $\mathfrak{g}$;

Lie Algebras

- G semisimple Lie group,
- $\mathfrak{g}$ associated Lie algebra,
- n rank of $\mathfrak{g}$.

With $\mathfrak{g}$ is associated a square $n \times n$ matrix $A = [a_{ij}]$, called Cartan matrix, which encodes the structure of $\mathfrak{g}$;

A and all its principal minors are positive definite and moreover

- $a_{ii} = 2$ for every i,
- $a_{ij} = 0$ iff $a_{ji} = 0$,
- if $a_{ij} \neq 0$, $i \neq j$, then $a_{ij}, a_{ji} \in \mathbb{Z}^-$ and $a_{ij}a_{ji} = 1, 2$ or 3.

Lie Algebras

- G semisimple Lie group,
- $\mathfrak{g}$ associated Lie algebra,
- n rank of $\mathfrak{g}$.

With $\mathfrak{g}$ is associated a square $n \times n$ matrix $A = [a_{ij}]$, called Cartan matrix, which encodes the structure of $\mathfrak{g}$;

A and all its principal minors are positive definite and moreover

- $a_{ii} = 2$ for every i,
- $a_{ij} = 0$ iff $a_{ji} = 0$,
- if $a_{ij} \neq 0$, $i \neq j$, then $a_{ij}, a_{ji} \in \mathbb{Z}^-$ and $a_{ij}a_{ji} = 1, 2$ or 3.

Example (n=2)

The Cartan matrices of rank 2 semi simple Lie algebras are

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Dynkin diagrams

With the matrix A is associated a finite Dynkin diagram $\mathcal{D}$, in the following way

Dynkin diagrams

With the matrix A is associated a finite Dynkin diagram $\mathcal{D}$, in the following way

- $\mathcal{D}$ is a graph with n nodes,

Dynkin diagrams

With the matrix A is associated a finite Dynkin diagram $\mathcal{D}$, in the following way

- $\mathcal{D}$ is a graph with n nodes,
- the nodes i and j are joined by $a_{ij}a_{ji}$ edges,

Dynkin diagrams

With the matrix A is associated a finite Dynkin diagram $\mathcal{D}$, in the following way

- $\mathcal{D}$ is a graph with n nodes,
- the nodes i and j are joined by $a_{ij}a_{ji}$ edges,
- if $|a_{ij}| > |a_{ji}|$ the edges are directed towards the node i.

Dynkin diagrams

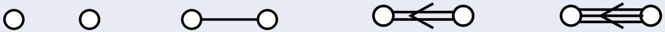
With the matrix A is associated a finite Dynkin diagram $\mathcal{D}$, in the following way

- $\mathcal{D}$ is a graph with n nodes,
- the nodes i and j are joined by $a_{ij}a_{ji}$ edges,
- if $|a_{ij}| > |a_{ji}|$ the edges are directed towards the node i.

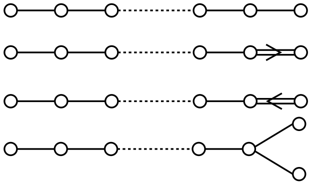
Example (n=2)

The Dynkin diagrams of rank 2 Lie algebras are

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$



Dynkin diagrams



A_n SL_{n+1}

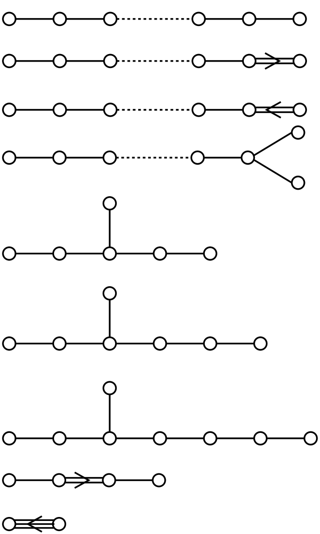
B_n SO_{2n+1}

C_n Sp_{2n}

D_n SO_{2n}

CLASSICAL

Dynkin diagrams



A_n	SL_{n+1}	
B_n	SO_{2n+1}	CLASSICAL
C_n	Sp_{2n}	
D_n	SO_{2n}	
E_6		
E_7		EXCEPTIONAL
E_8		
F_4		
G_2		

Rational homogeneous manifolds

Subgroups $P \subset G$ s.t. G/P is projective are called parabolic.

Rational homogeneous manifolds

Subgroups $\mathsf{P} \subset \mathsf{G}$ s.t. G/P is projective are called parabolic.

A parabolic subgroup is given by the choice of a set of nodes, and the variety G/P is denoted by marking these nodes.

Rational homogeneous manifolds

Subgroups $P \subset G$ s.t. G/P is projective are called parabolic.

A parabolic subgroup is given by the choice of a set of nodes, and the variety G/P is denoted by marking these nodes.

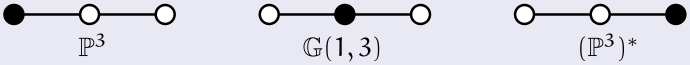
$G = SL(4)$

Rational homogeneous manifolds

Subgroups $P \subset G$ s.t. G/P is projective are called parabolic.

A parabolic subgroup is given by the choice of a set of nodes, and the variety G/P is denoted by marking these nodes.

$G = SL(4)$

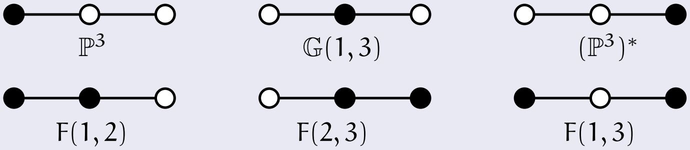


Rational homogeneous manifolds

Subgroups $P \subset G$ s.t. G/P is projective are called parabolic.

A parabolic subgroup is given by the choice of a set of nodes, and the variety G/P is denoted by marking these nodes.

$G = SL(4)$

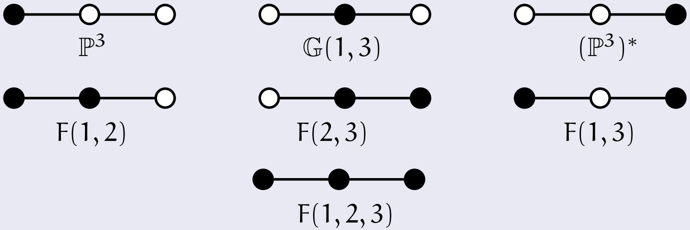


Rational homogeneous manifolds

Subgroups $P \subset G$ s.t. G/P is projective are called parabolic.

A parabolic subgroup is given by the choice of a set of nodes, and the variety G/P is denoted by marking these nodes.

$G = SL(4)$

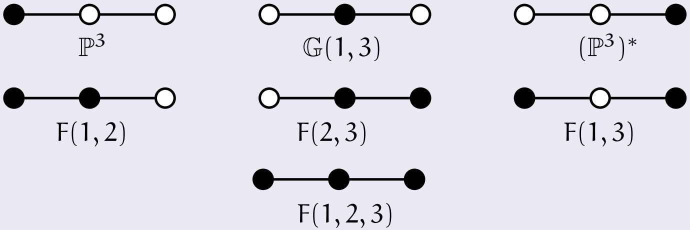


Rational homogeneous manifolds

Subgroups $P \subset G$ s.t. G/P is projective are called parabolic.

A parabolic subgroup is given by the choice of a set of nodes, and the variety G/P is denoted by marking these nodes.

$G = SL(4)$



So a rational homogeneous (RH) manifold is given by a marked Dynkin diagram $(\mathcal{D}, \mathcal{I})$.

Cone and contractions

X RH given by $(\mathcal{D}, \mathcal{I})$.

Cone and contractions

X RH given by $(\mathcal{D}, \mathcal{I})$.

1. X is a Fano manifold;

Cone and contractions

X RH given by $(\mathcal{D}, \mathcal{I})$.

1. X is a Fano manifold;
2. The Picard number ρ_X of X is $\#I$;

Cone and contractions

X RH given by $(\mathcal{D}, \mathcal{I})$.

1. X is a Fano manifold;
2. The Picard number ρ_X of X is $\#I$;
3. The cone $NE(X)$ is simplicial, and its faces correspond to proper subsets $J \subsetneq I$;

Cone and contractions

X RH given by $(\mathcal{D}, \mathcal{I})$.

1. X is a Fano manifold;
2. The Picard number ρ_X of X is $\#I$;
3. The cone $NE(X)$ is simplicial, and its faces correspond to proper subsets $J \subsetneq I$;
4. Every contraction $\pi : X \to Y$ is of fiber type and smooth.

Cone and contractions

X RH given by $(\mathcal{D}, \mathcal{I})$.

1. X is a Fano manifold;
2. The Picard number ρ_X of X is $\#I$;
3. The cone $NE(X)$ is simplicial, and its faces correspond to proper subsets $J \subsetneq I$;
4. Every contraction $\pi : X \to Y$ is of fiber type and smooth.
5. Y is RH with marked Dynkin diagram $(\mathcal{D}, \mathcal{J})$,

Cone and contractions

X RH given by $(\mathcal{D}, \mathcal{I})$.

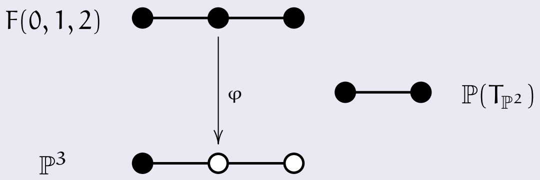
1. X is a Fano manifold;
2. The Picard number ρ_X of X is $\#I$;
3. The cone $NE(X)$ is simplicial, and its faces correspond to proper subsets $J \subsetneq I$;
4. Every contraction $\pi : X \to Y$ is of fiber type and smooth.
5. Y is RH with marked Dynkin diagram $(\mathcal{D}, \mathcal{J})$,
6. Every fiber is RH with marked Dynkin diagram $(\mathcal{D} \setminus \mathcal{J}, \mathcal{I} \setminus \mathcal{J})$.

Cone and contractions

X RH given by $(\mathcal{D},\mathcal{I})$.

1. X is a Fano manifold;
2. The Picard number ρ_X of X is $\#I$;
3. The cone $NE(X)$ is simplicial, and its faces correspond to proper subsets $J \subsetneq I$;
4. Every contraction $\pi : X \to Y$ is of fiber type and smooth.
5. Y is RH with marked Dynkin diagram $(\mathcal{D},\mathcal{J})$,
6. Every fiber is RH with marked Dynkin diagram $(\mathcal{D}\setminus\mathcal{J},\mathcal{I}\setminus\mathcal{J})$.

Example



Complete flag manifolds

A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. The corresponding parabolic subgroup B is called a Borel subgroup.

Complete flag manifolds

A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. The corresponding parabolic subgroup B is called a Borel subgroup.

- Every RH manifold is dominated by a complete flag manifold.

Complete flag manifolds

A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. The corresponding parabolic subgroup B is called a Borel subgroup.

- Every RH manifold is dominated by a complete flag manifold.
- $p_i : G/B \to G/P^i$ contractions corresponding to the unmarking of one node are $\mathbb{P}^1$-fibrations.

Complete flag manifolds

A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. The corresponding parabolic subgroup B is called a Borel subgroup.

- Every RH manifold is dominated by a complete flag manifold.
- $p_i : G/B \to G/P^i$ contractions corresponding to the unmarking of one node are $\mathbb{P}^1$-fibrations.
- If Γ_i is a fiber of p_i, and K_i the relative canonical, the intersection matrix $[-K_i \cdot \Gamma_j]$ is the Cartan matrix.

Complete flag manifolds

A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. The corresponding parabolic subgroup B is called a Borel subgroup.

- Every RH manifold is dominated by a complete flag manifold.
- $p_i : G/B \to G/P^i$ contractions corresponding to the unmarking of one node are $\mathbb{P}^1$-fibrations.
- If Γ_i is a fiber of p_i, and K_i the relative canonical, the intersection matrix $[-K_i \cdot \Gamma_j]$ is the Cartan matrix.

Example (A_n)

If $\mathcal{D} = A_n$, then G/B is the manifold parametrizing complete flags of linear subspaces in $\mathbb{P}^n$.

Theorem

A Fano manifold with Picard number 2 whose elementary contractions are $\mathbb{P}^1$-fibrations is isomorphic to one of the following

Theorem

A Fano manifold with Picard number 2 whose elementary contractions are $\mathbb{P}^1$-fibrations is isomorphic to one of the following

$$\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}) \qquad \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}) \qquad \mathbb{P}_{\mathbb{P}^3}(\mathcal{N}) \qquad \mathbb{P}_{\mathbb{Q}^5}(\mathcal{C})$$

Theorem

A Fano manifold with Picard number 2 whose elementary contractions are $\mathbb{P}^1$-fibrations is isomorphic to one of the following

$\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O})$

$\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$

$\mathbb{P}_{\mathbb{P}^3}(\mathcal{N})$

$\mathbb{P}_{\mathbb{Q}^5}(\mathcal{C})$

Theorem

A Fano manifold with Picard number 2 whose elementary contractions are $\mathbb{P}^1$-fibrations is a complete flag manifold.

Fano manifolds whose elementary contractions are smooth $\mathbb{P}^1$-fibrations

Relative duality

$\pi : M \to Y$ smooth $\mathbb{P}^1$-fibration. Γ fiber, K relative canonical

Lemma

Let D *be a divisor on* M *and set* $l := D \cdot \Gamma + 1$. *Then,* $\forall i \in \mathbb{Z}$

$$
\begin{aligned}
H^i(M, D) &\cong H^{i-1}(M, D + lK) && \text{if } l < 0 \\
H^i(M, D) &\cong \{0\} && \text{if } l = 0 \\
H^i(M, D) &\cong H^{i+1}(M, D + lK) && \text{if } l > 0
\end{aligned}
$$

In particular $\chi(M, D) = -\chi(M, D + lK)$ *for any* D.

Idea of Proof

I - Finding the intersection matrix

- X Fano manifold with Picard number 2.

Idea of Proof

I - Finding the intersection matrix

- X Fano manifold with Picard number 2.
- $\pi_i : X \to X_i$ elementary contration.

Idea of Proof

I - Finding the intersection matrix

- X Fano manifold with Picard number 2.
- $\pi_i : X \to X_i$ elementary contration.
- K_i relative canonical, Γ_i fiber of π_i.

Idea of Proof

I - Finding the intersection matrix

- X Fano manifold with Picard number 2.
- $\pi_i : X \to X_i$ elementary contration.
- K_i relative canonical, Γ_i fiber of π_i.

Intersection matrix $[-K_i \cdot \Gamma_j]$:

$$A := \begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix} \qquad a, b \leq 0$$

Idea of Proof

I - Finding the intersection matrix

- X Fano manifold with Picard number 2.
- $\pi_i : X \to X_i$ elementary contration.
- K_i relative canonical, Γ_i fiber of π_i.

Intersection matrix $[-K_i \cdot \Gamma_j]$:

$$A := \begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix} \qquad a, b \leq 0$$

Claim

$\det A > 0$.

Idea of Proof

I - Finding the intersection matrix

- X Fano manifold with Picard number 2.
- $\pi_i : X \to X_i$ elementary contration.
- K_i relative canonical, Γ_i fiber of π_i.

Intersection matrix $[-K_i \cdot \Gamma_j]$:

$$A := \begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix} \qquad a, b \leq 0$$

Claim

$\det A > 0$.

Assume $\det A < 0$.

Idea of Proof

I - Finding the intersection matrix

- X Fano manifold with Picard number 2.
- $\pi_i : X \to X_i$ elementary contration.
- K_i relative canonical, Γ_i fiber of π_i.

Intersection matrix $[-K_i \cdot \Gamma_j]$:

$$A := \begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix} \qquad a, b \leq 0$$

Claim

$\det A > 0$.

Assume $\det A < 0$.

If H is an ample line bundle and we write

$$H = \alpha K_1 + \beta K_2.$$

from $\det A < 0$ we get $\alpha, \beta > 0$.

Let D_0 be a very ample line bundle, set $d_0 = D_0 \cdot \Gamma_1 + 1$ and apply relative duality with respect to π_1

$$h^0(D_0) = h^1(D_0 + d_0 K_1)$$

Let D_0 be a very ample line bundle, set $d_0 = D_0 \cdot \Gamma_1 + 1$ and apply relative duality with respect to π_1

$$h^0(D_0) = h^1(D_0 + d_0K_1)$$

Set $E_1 := D_0 + d_0K_1$.

Let D_0 be a very ample line bundle, set $d_0 = D_0 \cdot \Gamma_1 + 1$ and apply relative duality with respect to π_1

$$h^0(D_0) = h^1(D_0 + d_0 K_1)$$

Set $E_1 := D_0 + d_0 K_1$.

By construction $E_1 \cdot \Gamma_1 < 0$, thus $e_1 := E_1 \cdot \Gamma_2 + 1 > 0$,

Let D_0 be a very ample line bundle, set $d_0 = D_0 \cdot \Gamma_1 + 1$ and apply relative duality with respect to π_1

$$h^0(D_0) = h^1(D_0 + d_0 K_1)$$

Set $E_1 := D_0 + d_0 K_1$.

By construction $E_1 \cdot \Gamma_1 < 0$, thus $e_1 := E_1 \cdot \Gamma_2 + 1 > 0$,

otherwise $-E_1$ would be ample, but its coefficients on K_1 and K_2 are negative.

Let D_0 be a very ample line bundle, set $d_0 = D_0 \cdot \Gamma_1 + 1$ and apply relative duality with respect to π_1

$$h^0(D_0) = h^1(D_0 + d_0 K_1)$$

Set $E_1 := D_0 + d_0 K_1$.

By construction $E_1 \cdot \Gamma_1 < 0$, thus $e_1 := E_1 \cdot \Gamma_2 + 1 > 0$,

otherwise $-E_1$ would be ample, but its coefficients on K_1 and K_2 are negative.

Apply relative duality with respect to π_2

$$h^1(E_1) = h^2(E_1 + e_1 K_2)$$

Let D_0 be a very ample line bundle, set $d_0 = D_0 \cdot \Gamma_1 + 1$ and apply relative duality with respect to π_1

$$h^0(D_0) = h^1(D_0 + d_0 K_1)$$

Set $E_1 := D_0 + d_0 K_1$.

By construction $E_1 \cdot \Gamma_1 < 0$, thus $e_1 := E_1 \cdot \Gamma_2 + 1 > 0$,

otherwise $-E_1$ would be ample, but its coefficients on K_1 and K_2 are negative.

Apply relative duality with respect to π_2

$$h^1(E_1) = h^2(E_1 + e_1 K_2)$$

Set $D_1 := E_1 + e_1 K_2$ and apply relative duality with respect to π_1.

So we have

$$0 \neq h^0(D_0) = h^1(E_1) = h^2(D_1) = \dots$$
$$\dots = h^{2k-1}(E_k) = h^{2k}(D_k) = \dots$$

So we have

$$0 \neq h^0(D_0) = h^1(E_1) = h^2(D_1) = \dots$$
$$\dots = h^{2k-1}(E_k) = h^{2k}(D_k) = \dots$$

getting a contradiction for $2k > \dim X$.

So we have

$$0 \neq h^0(D_0) = h^1(E_1) = h^2(D_1) = \dots$$
$$\dots = h^{2k-1}(E_k) = h^{2k}(D_k) = \dots$$

getting a contradiction for $2k > \dim X$.

The proof in the case $\det A = 0$ uses the same idea.

So we have

$$0 \neq h^0(D_0) = h^1(E_1) = h^2(D_1) = \dots$$
$$\dots = h^{2k-1}(E_k) = h^{2k}(D_k) = \dots$$

getting a contradiction for $2k > \dim X$.

The proof in the case $\det A = 0$ uses the same idea.

So the possible intersection matrices are (up to transposition):

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Idea of proof
Dimensions

Using the intersection matrices we write K_X (which has degree -2 on Γ_1 and Γ_2) as a combination of K_1 and K_2:

Idea of proof
Dimensions

Using the intersection matrices we write K_X (which has degree -2 on Γ_1 and Γ_2) as a combination of K_1 and K_2:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$K_1 + K_2 \qquad 2K_1 + 2K_2 \qquad 4K_1 + 3K_2 \qquad 10K_1 + 6K_2$$

Idea of proof
Dimensions

Using the intersection matrices we write K_X (which has degree -2 on Γ_1 and Γ_2) as a combination of K_1 and K_2:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$K_1 + K_2 \qquad 2K_1 + 2K_2 \qquad 4K_1 + 3K_2 \qquad 10K_1 + 6K_2$$

The dimension of X is the only positive integer m such that

$$h^m(K_X) \neq 0.$$

Idea of proof
Dimensions

Using the intersection matrices we write K_X (which has degree -2 on Γ_1 and Γ_2) as a combination of K_1 and K_2:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$K_1 + K_2 \qquad 2K_1 + 2K_2 \qquad 4K_1 + 3K_2 \qquad 10K_1 + 6K_2$$

The dimension of X is the only positive integer m such that

$$h^m(K_X) \neq 0.$$

The idea is to get to K_X starting from $\mathcal{O}_X$ using relative duality.

For example in the case

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

For example in the case

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$1 = h^0(\mathcal{O}_X) \overset{1}{=} \qquad h^1(K_1) \qquad\qquad \mathcal{O}_X \cdot \Gamma_1 + 1 = 1$$

For example in the case

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$1 = h^0(\mathcal{O}_X) \overset{1}{=} h^1(K_1) \qquad \mathcal{O}_X \cdot \Gamma_1 + 1 = 1$$

$$\overset{2}{=} h^2(K_1 + 2K_2) \qquad K_1 \cdot \Gamma_2 + 1 = 2$$

For example in the case

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{array}{lcc}
1 = h^0(\mathcal{O}_X) \overset{1}{=} & h^1(K_1) & \mathcal{O}_X \cdot \Gamma_1 + 1 = 1 \\
\overset{2}{=} & h^2(K_1 + 2K_2) & K_1 \cdot \Gamma_2 + 1 = 2 \\
\overset{1}{=} & h^3(2K_1 + 2K_2) & (K_1 + 2K_2) \cdot \Gamma_1 + 1 = 1
\end{array}$$

For example in the case

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{array}{rcccc} 1 = h^0(\mathcal{O}_X) & \overset{1}{=} & h^1(K_1) & & \mathcal{O}_X \cdot \Gamma_1 + 1 = 1 \\ & \overset{2}{=} & h^2(K_1 + 2K_2) & & K_1 \cdot \Gamma_2 + 1 = 2 \\ & \overset{1}{=} & h^3(2K_1 + 2K_2) & & (K_1 + 2K_2) \cdot \Gamma_1 + 1 = 1 \\ & = & h^3(K_X) & & \end{array}$$

For example in the case

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{array}{rcccc} 1 = h^0(\mathcal{O}_X) & \overset{1}{=} & h^1(K_1) & & \mathcal{O}_X \cdot \Gamma_1 + 1 = 1 \\ & \overset{2}{=} & h^2(K_1 + 2K_2) & & K_1 \cdot \Gamma_2 + 1 = 2 \\ & \overset{1}{=} & h^3(2K_1 + 2K_2) & & (K_1 + 2K_2) \cdot \Gamma_1 + 1 = 1 \\ & = & h^3(K_X) & & \end{array}$$

The dimension of X is three.

For example in the case

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{array}{rcll} 1 = h^0(\mathcal{O}_X) & \overset{1}{=} & h^1(K_1) & \mathcal{O}_X \cdot \Gamma_1 + 1 = 1 \\ & \overset{2}{=} & h^2(K_1 + 2K_2) & K_1 \cdot \Gamma_2 + 1 = 2 \\ & \overset{1}{=} & h^3(2K_1 + 2K_2) & (K_1 + 2K_2) \cdot \Gamma_1 + 1 = 1 \\ & = & h^3(K_X) & \end{array}$$

The dimension of X is three.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$2 \qquad\qquad 3 \qquad\qquad 4 \qquad\qquad 6$$

Idea of Proof

Conclusion

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

Idea of Proof

Conclusion

$$
\begin{tikzcd}
 & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X_1 & & X_2
\end{tikzcd}
$$

Let's do the case

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Idea of Proof

Conclusion

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

Let's do the case

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Set $L_2 = -2K_1 - K_2$. Then $L_2 \cdot \Gamma_2 = 0$.

Idea of Proof

Conclusion

$$
\begin{tikzcd}
 & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X_1 & & X_2
\end{tikzcd}
$$

Let's do the case

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Set $L_2 = -2K_1 - K_2$. Then $L_2 \cdot \Gamma_2 = 0$.

So there exists $H_2 \in \mathrm{Pic}(X_2)$ such that $L_2 = \pi_i^* H_2$.

Idea of Proof

Conclusion

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

Let's do the case

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Set $L_2 = -2K_1 - K_2$. Then $L_2 \cdot \Gamma_2 = 0$.

So there exists $H_2 \in \mathrm{Pic}(X_2)$ such that $L_2 = \pi_i^* H_2$.

We can write $-\pi_1^* K_{X_2} = -K_X + K_2 = 5L_2 = 5\pi_1^* H_2$.

Idea of Proof

Conclusion

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

Let's do the case

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Set $L_2 = -2K_1 - K_2$. Then $L_2 \cdot \Gamma_2 = 0$.

So there exists $H_2 \in \mathrm{Pic}(X_2)$ such that $L_2 = \pi_i^* H_2$.

We can write $-\pi_1^* K_{X_2} = -K_X + K_2 = 5L_2 = 5\pi_1^* H_2$.

So the index of X_2, which has dimension 5, is a multiple of 5; therefore the index is 5 and $X_2 \simeq \mathbb{Q}^5$.

Idea of Proof

Conclusion

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

Let's do the case

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Set $L_2 = -2K_1 - K_2$. Then $L_2 \cdot \Gamma_2 = 0$.

So there exists $H_2 \in \mathrm{Pic}(X_2)$ such that $L_2 = \pi_i^* H_2$.

We can write $-\pi_1^* K_{X_2} = -K_X + K_2 = 5L_2 = 5\pi_1^* H_2$.

So the index of X_2, which has dimension 5, is a multiple of 5; therefore the index is 5 and $X_2 \simeq \mathbb{Q}^5$.

Set $L_1 = -3K_1 - 2K_2$. Then $L_1 \cdot \Gamma_2 = 1$, hence π_2 is a projective bundle.

Idea of Proof

Conclusion

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

Let's do the case

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Set $L_2 = -2K_1 - K_2$. Then $L_2 \cdot \Gamma_2 = 0$.

So there exists $H_2 \in \mathrm{Pic}(X_2)$ such that $L_2 = \pi_i^* H_2$.

We can write $-\pi_1^* K_{X_2} = -K_X + K_2 = 5L_2 = 5\pi_1^* H_2$.

So the index of X_2, which has dimension 5, is a multiple of 5; therefore the index is 5 and $X_2 \simeq \mathbb{Q}^5$.

Set $L_1 = -3K_1 - 2K_2$. Then $L_1 \cdot \Gamma_2 = 1$, hence π_2 is a projective bundle.

The proof is finished by computing the Chern classes of the bundle.

The general case

The intersection matrix

X Fano of Picard number n with nef tangent bundle whose elementary contractions are $\mathbb{P}^1$-fibrations (FT-manifold for short).

The general case

The intersection matrix

X Fano of Picard number n with nef tangent bundle whose elementary contractions are $\mathbb{P}^1$-fibrations (FT-manifold for short).

- Every contraction $\pi : X \to Y$ is of fiber type, i.e. $\dim(Y) < \dim(X)$;

The general case

The intersection matrix

X Fano of Picard number n with nef tangent bundle whose elementary contractions are $\mathbb{P}^1$-fibrations (FT-manifold for short).

- Every contraction $\pi : X \to Y$ is of fiber type, i.e. $\dim(Y) < \dim(X)$;
- Fibers of every contraction $\pi : X \to Y$ are FT-manifolds;

The general case

The intersection matrix

X Fano of Picard number n with nef tangent bundle whose elementary contractions are $\mathbb{P}^1$-fibrations (FT-manifold for short).

- Every contraction $\pi : X \to Y$ is of fiber type, i.e. $\dim(Y) < \dim(X)$;
- Fibers of every contraction $\pi : X \to Y$ are FT-manifolds;
- The Picard number of $\pi^{-1}(y)$ is $\rho_X - \rho_Y$;

The general case

The intersection matrix

X Fano of Picard number n with nef tangent bundle whose elementary contractions are $\mathbb{P}^1$-fibrations (FT-manifold for short).

- Every contraction $\pi : X \to Y$ is of fiber type, i.e. $\dim(Y) < \dim(X)$;
- Fibers of every contraction $\pi : X \to Y$ are FT-manifolds;
- The Picard number of $\pi^{-1}(y)$ is $\rho_X - \rho_Y$;
- The cone $\mathrm{NE}(X)$ is simplicial.

The Cartan matrix of X is the $n \times n$ matrix $M(X)$ defined by $M(X)_{ij} = -K_i \cdot \Gamma_j$.

The Cartan matrix of X is the $n \times n$ matrix $M(X)$ defined by $M(X)_{ij} = -K_i \cdot \Gamma_j$.

Theorem (MOSWa 2013)

The Cartan matrix of an FT-manifold with nef tangent bundle is the Cartan matrix of a semi simple Lie algebra.

The Cartan matrix of X is the $n \times n$ matrix $M(X)$ defined by $M(X)_{ij} = -K_i \cdot \Gamma_j$.

Theorem (MOSWa 2013)

The Cartan matrix of an FT-manifold with nef tangent bundle is the Cartan matrix of a semi simple Lie algebra.

Let X be an FT-manifold, $I \subset \{1, \ldots, n\}$ any nonempty subset, and let $\pi_I : X \to X_I$ be the contraction of the corresponding face R_I.

The Cartan matrix of X is the $n \times n$ matrix $M(X)$ defined by $M(X)_{ij} = -K_i \cdot \Gamma_j$.

Theorem (MOSWa 2013)

The Cartan matrix of an FT-manifold with nef tangent bundle is the Cartan matrix of a semi simple Lie algebra.

Let X be an FT-manifold, $I \subset \{1, \dots, n\}$ any nonempty subset, and let $\pi_I : X \to X_I$ be the contraction of the corresponding face R_I.

Then every fiber of π_I is an FT-manifold whose Cartan matrix is the $|I| \times |I|$ principal submatrix of $M(X)$ obtained from $M(X)$ by subtracting rows and columns corresponding to indices which are not in I.

Idea of Proof

In particular any 2×2 principal submatrix is the Cartan matrix of an FT-manifold of Picard number 2. So

Idea of Proof

In particular any 2×2 principal submatrix is the Cartan matrix of an FT-manifold of Picard number 2. So

- $m_{ii} = 2$ for every i,
- $m_{ij} = 0$ iff $a_{ji} = 0$,
- if $m_{ij} \neq 0$, $i \neq j$, then $m_{ij}, m_{ji} \in \mathbb{Z}^-$ and $m_{ij}m_{ji} = 1, 2$ or 3.

Idea of Proof

In particular any 2×2 principal submatrix is the Cartan matrix of an FT-manifold of Picard number 2. So

- $m_{ii} = 2$ for every i,
- $m_{ij} = 0$ iff $a_{ji} = 0$,
- if $m_{ij} \neq 0$, $i \neq j$, then $m_{ij}, m_{ji} \in \mathbb{Z}^-$ and $m_{ij}m_{ji} = 1, 2$ or 3.

This implies that $M(X)$ is a generalized Cartan matrix (as in the theory of Kac-Moody algebras).

Idea of Proof

In particular any 2×2 principal submatrix is the Cartan matrix of an FT-manifold of Picard number 2. So

- $m_{ii} = 2$ for every i,
- $m_{ij} = 0$ iff $a_{ji} = 0$,
- if $m_{ij} \neq 0$, $i \neq j$, then $m_{ij}, m_{ji} \in \mathbb{Z}^-$ and $m_{ij}m_{ji} = 1, 2$ or 3.

This implies that $M(X)$ is a generalized Cartan matrix (as in the theory of Kac-Moody algebras).

By induction we may assume $M(X)$ is

- Of finite type (all the principal minors are positive definite) or
- Of affine type (all the proper principal minors are positive definite, but $\det A = 0$).

Idea of Proof

$M(X)$ affine implies that there exists a linear combination

$$\Gamma = \sum_1^n m_i \Gamma_i$$

with $m_i \in \mathbb{Z}_{>0}$, satisfying that $K_i \cdot \Gamma = 0$ for all i.

Idea of Proof

$M(X)$ affine implies that there exists a linear combination

$$\Gamma = \sum_1^n m_i \Gamma_i$$

with $m_i \in \mathbb{Z}_{>0}$, satisfying that $K_i \cdot \Gamma = 0$ for all i.

Since the Γ_i's are free curves the cycle $\sum_1^n m_i \Gamma_i$ is smoothable, and therefore numerically equivalent to an irreducible rational curve Γ.

Idea of Proof

$M(X)$ affine implies that there exists a linear combination

$$\Gamma = \sum_1^n m_i \Gamma_i$$

with $m_i \in \mathbb{Z}_{>0}$, satisfying that $K_i \cdot \Gamma = 0$ for all i.

Since the Γ_i's are free curves the cycle $\sum_1^n m_i \Gamma_i$ is smoothable, and therefore numerically equivalent to an irreducible rational curve Γ.

Using $K_i \cdot \Gamma = 0$ one can prove that, for every $\ell = (l_1, \ldots, l_r)$, if $x \in \Gamma$, then $Z_\ell(\Gamma) = Z_\ell \times \mathbb{P}^1$.

Idea of Proof

Take a reduced sequence $\ell = (l_1, \ldots, l_t)$ such that $\mathrm{Ch}(\ell) = X$

Idea of Proof

Take a reduced sequence $\ell = (l_1, \dots, l_t)$ such that $\mathrm{Ch}(\ell) = X$

$$\begin{tikzcd} Z_\ell \arrow[r, "f_\ell"] \arrow[d] & X \arrow[d, "\pi_{l_t}"] \\ Z_{\ell[1]} \arrow[r, "g_{\ell[1]}"] & X_t \end{tikzcd}$$

The map $g_{\ell[1]}$ is surjective, hence generically finite.

Idea of Proof

Take a reduced sequence $\ell = (l_1, \dots, l_t)$ such that $\mathrm{Ch}(\ell) = X$

$$\begin{tikzcd} Z_\ell \arrow[r, "f_\ell"] \arrow[d] & X \arrow[d, "\pi_{l_t}"] \\ Z_{\ell[1]} \arrow[r, "g_{\ell[1]}"] & X_t \end{tikzcd}$$

The map $g_{\ell[1]}$ is surjective, hence generically finite.

$$\begin{tikzcd} Z_{\ell[1]} \times \Gamma \arrow[r] \arrow[d] \arrow[dr] & X \arrow[d, "\pi_{l_r}"] \\ Z_{\ell[2]} \times \Gamma \arrow[r] & X_r \end{tikzcd}$$

The diagonal map $g_{\ell[1]} \times \pi_{l_r}$ is of fiber type, and this is a contradiction.

Main result

Theorem (OSWW 2014, OSWi 2015)

Let X be a smooth projective variety of Picard number n, such that there exist $\Gamma_i \in N_1(X)$, $i = 1, \ldots, n$, independent K_X-negative classes generating n extremal rays, whose associated elementary contractions $\pi_i : X \to X_i$ are smooth $\mathbb{P}^1$-fibrations.
Then X is isomorphic to a flag manifold G/B, for some semisimple group G.

Speculations

In the paper in which the conjecture is introduced, Campana and Peternell proposed the following approach:

Speculations

In the paper in which the conjecture is introduced, Campana and Peternell proposed the following approach:

1. Prove the conjecture for CP-manifolds of Picard number one.

Speculations

In the paper in which the conjecture is introduced, Campana and Peternell proposed the following approach:

1. Prove the conjecture for CP-manifolds of Picard number one.
2. Show that, given a CP-manifold X and a contraction $f : X \to Y$, from the homogeneity of Y and of the fibers of f one can reconstruct the homogeneity of X.

Speculations

In the paper in which the conjecture is introduced, Campana and Peternell proposed the following approach:

1. Prove the conjecture for CP-manifolds of Picard number one.
2. Show that, given a CP-manifold X and a contraction $f : X \to Y$, from the homogeneity of Y and of the fibers of f one can reconstruct the homogeneity of X.

The Picard number one case turned out to be very hard.

Speculations

In the paper in which the conjecture is introduced, Campana and Peternell proposed the following approach:

1. Prove the conjecture for CP-manifolds of Picard number one.
2. Show that, given a CP-manifold X and a contraction $f : X \to Y$, from the homogeneity of Y and of the fibers of f one can reconstruct the homogeneity of X.

The Picard number one case turned out to be very hard.

A possible different approach is the following:

Speculations

In the paper in which the conjecture is introduced, Campana and Peternell proposed the following approach:

1. Prove the conjecture for CP-manifolds of Picard number one.
2. Show that, given a CP-manifold X and a contraction $f : X \to Y$, from the homogeneity of Y and of the fibers of f one can reconstruct the homogeneity of X.

The Picard number one case turned out to be very hard.

A possible different approach is the following:

1. Prove the conjecture for a CP-manifold with "maximal" Picard number.

Speculations

In the paper in which the conjecture is introduced, Campana and Peternell proposed the following approach:

1. Prove the conjecture for CP-manifolds of Picard number one.
2. Show that, given a CP-manifold X and a contraction $f : X \to Y$, from the homogeneity of Y and of the fibers of f one can reconstruct the homogeneity of X.

The Picard number one case turned out to be very hard.

A possible different approach is the following:

1. Prove the conjecture for a CP-manifold with "maximal" Picard number.
2. Show that, given a CP-manifold X then X is dominated by a CP-manifold with "maximal" Picard number.

Speculations

Given a CP-manifold X, we define:

$$\tau(X) := \sum_R (\ell(R) - 2)$$

where the sum is taken over the extremal rays of $\overline{\mathrm{NE}}(X)$.

Speculations

Given a CP-manifold X, we define:

$$\tau(X) := \sum_R (\ell(R) - 2)$$

where the sum is taken over the extremal rays of $\overline{\mathrm{NE}}(X)$.

In particular $\tau(X) = 0$ if and only if X is a Flag Type manifold.

Speculations

Given a CP-manifold X, we define:

$$\tau(X) := \sum_R (\ell(R) - 2)$$

where the sum is taken over the extremal rays of $\overline{\mathrm{NE}}(X)$.

In particular $\tau(X) = 0$ if and only if X is a Flag Type manifold.

CP conjecture will then follow from:

Conjecture

Given a CP-manifold satisfying $\tau(X) > 0$, there exists a contraction $f : X' \to X$ from a CP-manifold X' satisfying $\tau(X') < \tau(X)$.

Speculations

Given a CP-manifold X, we define:

$$\tau(X) := \sum_R (\ell(R) - 2)$$

where the sum is taken over the extremal rays of $\overline{\mathrm{NE}}(X)$.

In particular $\tau(X) = 0$ if and only if X is a Flag Type manifold.

CP conjecture will then follow from:

Conjecture

Given a CP-manifold satisfying $\tau(X) > 0$, there exists a contraction $f : X' \to X$ from a CP-manifold X' satisfying $\tau(X') < \tau(X)$.