A characterization of complete flag manifolds

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Outline

Introduction

Fano bundles

A vector bundle $\mathcal{E}$ on a smooth complex projective variety X is a Fano bundle iff $\mathbb{P}_X(\mathcal{E})$ is a Fano manifold.

If $\mathcal{E}$ is a Fano bundle on X then X is a Fano manifold.

Fano bundles of rank 2 on $\mathbb{P}^m$ and $\mathbb{Q}^m$ have been classified in the '90s (Ancona, Peternell, Sols, Szurek, Wiśniewski).

Generalization: Classification of Fano bundles of rank 2 on (Fano) manifolds with $b_2 = b_4 = 1$ (Muñoz, _ , Solá Conde, 2012).

As a special case we have the classification of Fano manifolds of Picard number two (and $b_4 = 2$) with two $\mathbb{P}^1$-bundle structures.

Later the assumption on b_4 was removed by Watanabe.

Varieties with two $\mathbb{P}^1$-bundle structures

Theorem (version with Bundles)

A Fano manifold with Picard number 2 and two $\mathbb{P}^1$-bundle structures is isomorphic to one of the following

- $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O})$
- $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$
- $\mathbb{P}_{\mathbb{P}^3}(\mathcal{N}) = \mathbb{P}_{\mathbb{Q}^3}(\mathcal{S})$ - $\mathcal{N}$ Null-correlation , $\mathcal{S}$ Spinor
- $\mathbb{P}_{\mathbb{Q}^5}(\mathcal{C}) = \mathbb{P}_{K(G_2)}(\mathcal{Q})$ - $\mathcal{C}$ Cayley, $\mathcal{Q}$ universal quotient.

This result can be reformulated as follows:

Theorem (version with Flags)

A Fano manifold with Picard number 2 and two $\mathbb{P}^1$-bundle structures is rational homogeneous and it is isomorphic to a complete flag manifold of type $A_1 \times A_1$, A_2, B_2 or G_2.

A generalization

- Classify Fano manifolds whose elementary contractions are $\mathbb{P}^1$-bundles - or just smooth $\mathbb{P}^1$-fibrations.
- The vector bundle approach seems difficult to apply to this more general situation.
- Is it possible to prove the homogeneity directly, or at least recover features of the complete flags using the $\mathbb{P}^1$-fibrations?

Theorem

X is a Fano manifold whose elementary contractions are smooth $\mathbb{P}^1$-fibrations (Flag Type manifold) if and only if X is a complete flag manifold.

Rational Homogeneous Manifolds

Lie Algebras
Root systems

G semisimple Lie group, $\mathfrak{g}$ Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra.

The action of $\mathfrak{h}$ on $\mathfrak{g}$ defines an eigenspace decomposition, called Cartan decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^\vee \setminus \{0\}} \mathfrak{g}_\alpha.$$

The spaces $\mathfrak{g}_\alpha$ are defined by

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h, g] = \alpha(h)g, \text{ for every } h \in \mathfrak{h}\};$$

$\alpha \neq 0$ such that $\mathfrak{g}_\alpha \neq 0$ is called a root of $\mathfrak{g}$.

The (finite) set Φ of such elements is called root system of $\mathfrak{g}$.

A set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi$ is a basis of $\mathfrak{h}^\vee$ such that the coordinates of root are integers, all ≥ 0 or all ≤ 0 .

Lie Algebras

Weyl group

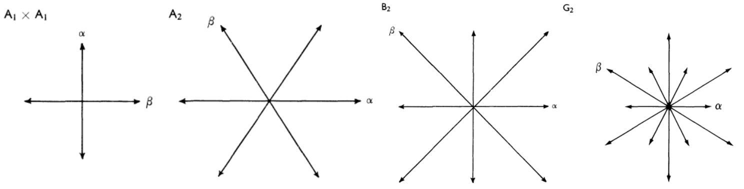
(E, κ) real vector space generated by the roots, with a symmetric bilinear positive form κ induced by the Killing form of $\mathfrak{g}$.

The reflections with respect to the roots:

$$\sigma_\alpha(x) = x - \langle x, \alpha\rangle \alpha, \quad \text{where} \quad \langle x, \alpha\rangle := 2\frac{\kappa(x,\alpha)}{\kappa(\alpha,\alpha)},$$

fix the root system and generate a finite group $W \subset \mathrm{Gl}(E)$, called the Weyl group of $\mathfrak{g}$.

Example (n=2)



Cartan matrix

Given a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ of $\mathfrak{g}$, the Cartan matrix A of $\mathfrak{g}$ is the $n \times n$ matrix whose entries are the Cartan integers

$$\langle \alpha_i, \alpha_j \rangle = 2\frac{\kappa(\alpha_i, \alpha_j)}{\kappa(\alpha_j, \alpha_j)}.$$

A and all its principal minors are positive definite and moreover

- $a_{ii} = 2$ for every i,
- $a_{ij} = 0$ iff $a_{ji} = 0$,
- if $a_{ij} \neq 0$, $i \neq j$, then $a_{ij}, a_{ji} \in \mathbb{Z}^-$ and $a_{ij}a_{ji} = 1, 2$ or 3.

Example (n=2)

The Cartan matrices of rank 2 Lie algebras are

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Dynkin diagrams

With the matrix A is associated a finite Dynkin diagram $\mathcal{D}$, in the following way

- $\mathcal{D}$ is a graph with n nodes,
- the nodes i and j are joined by $a_{ij}a_{ji}$ edges,
- if $|a_{ij}| > |a_{ji}|$ the edges are directed towards the node i.

Example (n=2)

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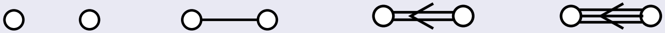
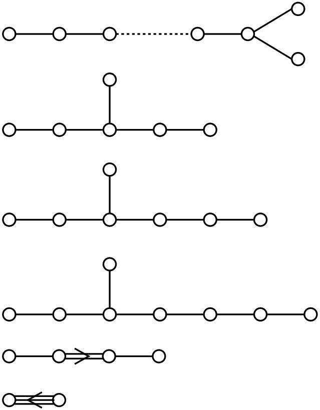


Dynkin diagrams



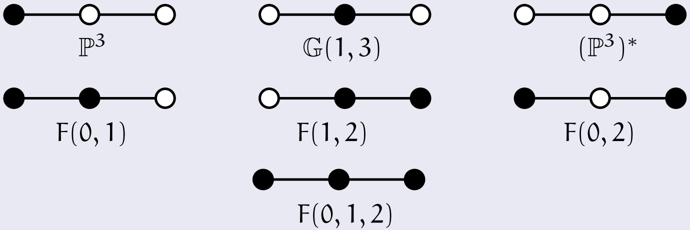
A_n	SL_{n+1}	CLASSICAL
B_n	SO_{2n+1}	
C_n	Sp_{2n}	
D_n	SO_{2n}	
E_6		EXCEPTIONAL
E_7		
E_8		
F_4		
G_2		

Rational homogeneous manifolds

Subgroups $P \subset G$ s.t. G/P is projective are called parabolic.

A parabolic subgroup is given by the choice of a set of simple roots, i.e. by $I \subset D$, and the variety G/P is denoted by marking the nodes of I.

$G = SL(4)$



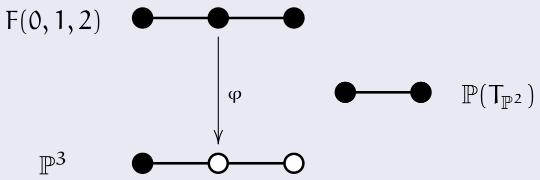
So a rational homogeneous (RH) manifold is given by a marked Dynkin diagram $(\mathcal{D}, \mathcal{I})$.

Cone and contractions

X RH given by $(\mathcal{D}, \mathcal{I})$.

1. X is a Fano manifold;
2. The Picard number ρ_X of X is $\#I$;
3. The cone $NE(X)$ is simplicial, and its faces correspond to proper subsets $J \subsetneq I$;
4. Every contraction $\pi : X \to Y$ is of fiber type and smooth.
5. Y is RH with marked Dynkin diagram $(\mathcal{D}, \mathcal{J})$,
6. Every fiber is RH with marked Dynkin diagram $(\mathcal{D} \setminus \mathcal{J}, \mathcal{I} \setminus \mathcal{J})$.

Example



Complete flag manifolds

A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. The corresponding parabolic subgroup B is called a Borel subgroup.

- Every RH manifold is dominated by a complete flag manifold.
- $p_i : G/B \to G/P^i$ contractions corresponding to the unmarking of one node are $\mathbb{P}^1$-bundles.
- If Γ_i is a fiber of p_i, and K_i the relative canonical, the intersection matrix $[-K_i \cdot \Gamma_j]$ is the Cartan matrix.

Example (A_n)

If $\mathcal{D} = A_n$, then G/B is the manifold parametrizing complete flags of linear subspaces in $\mathbb{P}^n$.

Fano manifolds whose elementary contractions are smooth $\mathbb{P}^1$-fibrations

Relative duality

$\pi : M \to Y$ smooth $\mathbb{P}^1$-fibration. Γ fiber, K relative canonical

Lemma

Let D *be a divisor on* M *and set* $l := D \cdot \Gamma + 1$. *Then,* $\forall i \in \mathbb{Z}$

$$
\begin{aligned}
H^i(M, D) &\cong H^{i-1}(M, D + lK) && \textit{if } l < 0 \\
H^i(M, D) &\cong \{0\} && \textit{if } l = 0 \\
H^i(M, D) &\cong H^{i+1}(M, D + lK) && \textit{if } l > 0
\end{aligned}
$$

In particular $\quad \chi(M, D) = -\chi(M, D + lK) \quad$ *for any* D.

Idea of Proof

I - Finding a homogeneous model

Theorem

X is Flag Type manifold if and only if X is a complete flag manifold.

- X Fano manifold with Picard number n.
- $\pi_i : X \to X_i$ elementary contration.
- K_i relative canonical, Γ_i fiber of π_i.
- $\chi_X : \mathrm{Pic}(X) \to \mathbb{Z}$ such that $\chi_X(L) = \chi(X, L)$.

Given $L_1, \dots, L_n$ basis of $\mathrm{Pic}(X)$,

$$\chi_X(m_1, \dots, m_n) = \chi(X, m_1 L_1 + \cdots + m_n L_n)$$

is a numerical polynomial of degree $\dim X$, so we can extend it to $\chi_X : N_1(X) \to \mathbb{R}$.

By the Lemma the affine involutions $r'_i : N^1(X) \to N^1(X)$

$$r'_i(D) := D + (D \cdot \Gamma_i + 1)K_i,$$

satisfy

$$\chi_X(D) = -\chi_X(r'_i(D)).$$

Since $K_X \cdot \Gamma_i = -2$ for every i, setting

$$\begin{aligned} T(D) &:= D + K_X/2 \\ r_i &:= T^{-1} \circ r'_i \circ T \\ \chi^T &:= \chi_X \circ T \end{aligned}$$

we have that the map r_i is a linear involution of $N^1(X)$ given by

$$r_i(D) = D + (D \cdot \Gamma_i)K_i,$$

which fixes pointwise the hyperplane $M_i := \{D \mid D \cdot \Gamma_i = 0\}$ and satisfies

$$r_i(K_i) = -K_i \qquad \chi^T(D) = -\chi^T(r_i(D));$$

in particular χ^T vanishes on M_i for every i.

Weyl group

Let $W \subset \mathrm{Gl}(\mathrm{N}^1(X))$ be the group generated by the r_i's.

Theorem

The group W is finite and the set

$$\Phi := \{w(-K_i) \mid w \in W,\ \ i = 1, \dots, n\} \subset \mathrm{N}^1(X),$$

is a root system, whose Weyl group is W

Idea of proof

For every divisor D and every $w \in W$

$$\chi^T(D) = \pm\chi^T(w(D)),$$

so χ^T_X vanishes on the hyperplanes $w(M_i)$; therefore the number of these hyperplanes is bounded by the dimension of X.

Then one proves that the isotropy subgroup of M_i is finite (by considering the induced action on $N_1(X)$, and writing the elements of W is a suitable basis).

By the finiteness there is a scalar product $(\ ,\ )$ on $N^1(X)$, which is W-invariant. In particular the r_i's are euclidean reflections.

Using that $r_i(K_i) = -K_i$ is then straightforward (but tedious) to prove that Φ is a root system with Weyl group W.

Homogeneous model

Since $(\ ,\ )$ is W-invariant, $(K_j, K_i) = (r_i(K_j), -K_i)$ which gives

$$-K_j \cdot \Gamma_i = 2\frac{(K_j, K_i)}{(K_i, K_i)} = \langle K_j, K_i \rangle,$$

so the intersection matrix $[-K_j \cdot \Gamma_i]$ is the Cartan matrix of Φ.

In particular the intersection matrix of X is the intersection matrix of a complete flag manifold G/B, the homogeneous model of X.

Define $\psi : N^1(X) \to N^1(G/B)$, by setting $\psi(K_i) = \overline{K}_i$.

Proposition

$\Lambda \subset \mathrm{Pic}(X)$ generated by the K_i's.

- $h^i(X, D) = h^i(G/B, \psi(D))$ for every $D \in \Lambda$, $i \in \mathbb{Z}$.
- $\dim X = \dim G/B$;

Idea of Proof

II - Proving the isomorphism

- X Flag Type manifold of Picard number n, $x \in X$ point;
- $\ell = (l_1, \dots, l_t)$, list of indices in $\{1, \dots, n\}$,
- $\ell[1] = (l_1, \dots, l_{t-1})$.

The Bott-Samelson variety Z_ℓ, with a morphism $f_\ell : Z_\ell \to X$, associated with the sequence ℓ, is constructed in the following way:

If $\ell = \emptyset$ we set $Z_\ell := \{x\}$ and $f_\ell : \{x\} \to X$ is the inclusion.

Inductively we build Z_ℓ on $Z_{\ell[1]}$:

$$\begin{tikzcd}
Z_\ell \arrow[r, "f_\ell"] \arrow[d, "p_{\ell[1]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\pi_{l_t}"] \\
Z_{\ell[1]} \arrow[ur, "f_{\ell[1]}"] \arrow[r, "g_{\ell[1]}"'] & X_{l_t}
\end{tikzcd}$$

The image of Z_ℓ in X is the set of points belonging to chains of rational curves $\Gamma_{l_1}, \Gamma_{l_2} \dots, \Gamma_{l_t}$ starting from x.

In the homogeneous case such loci are the Schubert varieties.

With a list ℓ it is associated an element $w(\ell)$ of the Weyl group:

$$w = r_{l_1} \circ \cdots \circ r_{l_t};$$

if there is no expression of $w(\ell)$ which contains less than t reflections, then $w(\ell)$ and ℓ are called reduced.

The length $\lambda(w(\ell))$ is the number of reflections appearing in a reduced expression of $w(\ell)$.

If $w(\ell)$ is reduced then $f_\ell : Z_\ell \to f_\ell(Z_\ell)$ is birational, hence

$$\dim f_\ell(Z_\ell) = \dim Z_\ell = \#(\ell) = \lambda(w(\ell)).$$

In W there exists a unique longest element w_0, of length $\dim X$.

If ℓ_0 is a reduced list such that $w(\ell_0) = w_0$ then $f_\ell : Z_{\ell_0} \to X$ is surjective and birational.

X Flag Type manifold, G/B homogeneus model of X

Find a list ℓ_0 such that $w(\ell_0) = w_0$ and prove that

$$Z_{\ell_0} \simeq \overline{Z}_{\ell_0} \qquad f_{\ell_0} = \overline{f}_{\ell_0}$$

The idea is to show inductively that Z_{ℓ_0} depends only on the list and on the intersection matrix.

Assume that $Z_{\ell[1]} \simeq \overline{Z}_{\ell[1]}$;

$$\begin{tikzcd}
Z_\ell \arrow[r, "f_\ell"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\pi_{l_r}"] \\
Z_{\ell[1]} \arrow[r] \arrow[ur, "f_{\ell[1]}"'] \arrow[u, bend left, "\sigma_{\ell[1]}"] & X_{l_r}
\end{tikzcd}$$

$f_{\ell[1]}$ factors via Z_ℓ, giving a section $\sigma_{\ell[1]}$, hence an extension

$$0 \to \mathcal{O}_{Z_{\ell[1]}}(f^*_{\ell[1]} K_{l_r}) \longrightarrow \mathcal{F}_\ell \longrightarrow \mathcal{O}_{Z_{\ell[1]}} \to 0.$$

One shows easily that the following are equivalent

- The extension is split;
- $h^1(Z_{\ell[1]}, f_\ell^*(K_{l_r})) = 0$;
- the index l_r does not appear in $\ell[1]$.

It is enough to show that if the index l_r appears in $\ell[1]$ then

$$h^1(Z_{\ell[1]}, f_\ell^*(K_{l_r})) \leq 1.$$

This can be done except for G_2, (already known from the $n = 2$ case) and F_4, for which an ad hoc argument is needed.

Campana-Peternell Conjecture

Positivity of the tangent bundle

X smooth complex projective variety.

Theorem [Mori (1979)]

T_X ample $\Leftrightarrow X = \mathbb{P}^m$.

- T_X nef $\Rightarrow$??
- Examples:

$$\begin{tikzcd} & \text{Abelian} \\ \text{Homogeneous manifolds} \arrow[ur] \arrow[dr] & \\ & \text{Rational} \end{tikzcd}$$

Theorem [Demailly, Peternell and Schneider (1994)]

$$T_X \text{ nef} \Rightarrow \begin{cases} X \xleftarrow{\text{étale}} X' \xrightarrow{F} A \\ A \text{ Abelian, } F \text{ Fano, } T_F \text{ nef} \end{cases}$$

Campana-Peternell Conjecture

Campana-Peternell Conjecture (1991)

Every Fano manifold with nef tangent bundle (CP manifold) is homogeneous.

Results:

- ☑ $\dim X = 3$ [Campana & Peternell(1991)]
- ☑ $\dim X = 4$ [CP (1993), Mok (2002), Hwang (2006)]
- ☑ $\dim X = 5$ and $\rho_X > 1$ [Watanabe (2012)]
- ☑ T_X big and 1-ample [Solá-Conde & Wiśniewski (2004)]

- The above results are obtained by classifying the manifolds satisfying the required properties;
- homogeneity is checked *a posteriori*.

An application

Homogeneity via families of rational curves

- X Fano of Picard number one;
- $\mathcal{M}$ dominating family of rational curves of minimal degree;
- $\mathcal{U}$ universal family.

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{M} & & X \end{tikzcd}$$

Theorem

Assume that $\mathcal{M}$ is unsplit, q is smooth and that $\mathcal{M}_x := q^{-1}(x)$ is RH for every $x \in X$. Then X is RH. Assume that T_X is nef and that $\mathcal{M}_x := q^{-1}(x)$ is RH for every $x \in X$. Then X is RH.

Remark

If T_X is nef then the assumptions on $\mathcal{M}$ and q hold.

Recognizing homogeneous spaces

- X Fano of Picard number one, T_X nef;
- $S = G/P$ RH space of Picard number one;
- $\mathcal{M}$, $\mathcal{L}$ minimal dominating families of rational curves;

Corollary

Assume $\mathcal{L}_0$ is RH. If $\mathcal{M}_x \simeq \mathcal{L}_0$ for every $x \in X$ then $X \simeq S$.

The following are equivalent:

- $\mathcal{L}_0$ is G-homogeneous.
- P is associated to a long simple root.
- There is no arrow in the Dynkin diagram pointing towards the node corresponding to P.

Recognizing homogeneous spaces

- X Fano of Picard number one;
- $S = G/P$ RH space of Picard number one;
- $\mathcal{M}$, $\mathcal{L}$ minimal dominating families of rational curves;
- $\mathcal{C}_0(S)$ VMRT of S;
- $\mathcal{C}_x(X)$ VMRT of X at a general point;

Theorem [Mok, Hong-Hwang]

If P is associated to a long simple root and $\mathcal{C}(X)_x$ is projectively equivalent to $\mathcal{C}(S)_0$, then $X \simeq S$.

Idea of the proof

Given the smooth fibration $q : \mathcal{U} \to X$, with RH fiber F, it is possible to construct the associated flag bundle over X, whose fibers over a point are complete flag manifolds.

The fibration q is defined by a cocycle $\vartheta \in H^1(X, G)$, where G is the identity component of $\mathrm{Aut}(F)$ - here we use that X is simply connected.

The cocycle ϑ defines a principal G-bundle $\mathcal{U}_G \to X$

Given a Borel subgroup $B \subset G$ we can define the G/B-bundle

$$\overline{\mathcal{U}} := \mathcal{U}_G \times^G G/B \to X$$

as a quotient of $\mathcal{U}_G \times G/B$ by $(x, gB) \sim (xg', g'^{-1}gB)$, and we have a commutative diagram

$$\begin{tikzcd} \overline{\mathcal{U}} \arrow[rr, "\pi"] \arrow[dr, "\overline{q}"'] & & \mathcal{U} \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

Idea of the proof

The flag bundle $\overline{\mathcal{U}}$ has Picard number $\rho(G/B)+1$, and has $\rho(G/B)$ contractions (over X) which are smooth $\mathbb{P}^1$-fibrations.

$$\begin{tikzcd}
\overline{\mathcal{M}} \arrow[d] & \overline{\mathcal{U}} \arrow[l, "\overline{p}"'] \arrow[d, "\pi"] \\
\mathcal{M} & \mathcal{U} \arrow[l, "p"] \arrow[d, "q"] \\
& X
\end{tikzcd}$$

Idea: show that the $\mathbb{P}^1$- fibration $p : \mathcal{U} \to \mathcal{M}$ can be lifted to $\overline{\mathcal{U}}$.

Idea of the proof

So $\overline{\mathcal{U}}$ has a number of $\mathbb{P}^1$-fibrations equal to its Picard number.

A priori it is not a Fano manifold; however we can prove a slightly stronger version of the main theorem

Theorem

Let X be a smooth projective variety of Picard number n, with n elementary contractions which are smooth $\mathbb{P}^1$-fibrations.
Then X is isomorphic to a complete flag manifold.

and get that $\overline{\mathcal{U}}$ is a complete flag manifold; hence X, being the image of a contraction of $\mathcal{U}$ is homogeneous.

Remark

A similar argument has been used to conclude the proof of CP conjecture in dimension 5 by Kanemitsu (2015).

Speculations

Given a CP-manifold X, we define:

$$\tau(X) := \sum_R (\ell(R) - 2)$$

where the sum is taken over the extremal rays of $\overline{\mathrm{NE}}(X)$.

In particular $\tau(X) = 0$ if and only if X is a Flag Type manifold.

CP conjecture will then follow from:

Conjecture

Given a CP-manifold satisfying $\tau(X) > 0$, there exists a contraction $f : X' \to X$ from a CP-manifold X' satisfying $\tau(X') < \tau(X)$.