Flag bundles on Fano manifolds

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Introduction

Positivity of the tangent bundle

Theorem [Mori (1979)]

X smooth complex projective, T_X ample $\Leftrightarrow X = \mathbb{P}^n$.

T_X nef $\Rightarrow$??

Theorem [Demailly, Peternell and Schneider (1994)]

$$T_X \text{ nef} \Rightarrow \begin{cases} X \xleftarrow{\text{étale}} X' \xrightarrow{F} A \\ A \text{ Abelian, } F \text{ Fano, } T_F \text{ nef} \end{cases}$$

Conjecture [Campana–Peternell (1991)]

X Fano with T_X nef is rational homogeneous.

Campana-Peternell Conjecture

Conjecture [Campana–Peternell (1991)]

X Fano with T_X nef is rational homogeneous.

Results:

- ☑ $\dim X = 3$ [Campana & Peternell (1991)]
- ☑ $\dim X = 4$ [CP (1993), Mok (2002), Hwang (2006)]
- ☑ $\dim X = 5$ and $\rho_X > 1$ [Watanabe (2012)]
- ☑ $\dim X = 5$ [Kanemitsu (2015)]
- ☑ T_X big and 1-ample [Solá Conde & Wiśniewski (2004)]
- ☑ X horospherical [Li (2015)]
- ☑ X is a Fano manifold and every elementary contraction of X is a $\mathbb{P}^1$-bundle (no assumptions on positivity of T_X) [_, Solá Conde, Watanabe & Wiśniewski (2014)]

Problem

Find geometric conditions on a Fano manifold X - which may or may not include T_X nef - which imply that X is rational homogeneous.

Main Theorem (Very vague formulation)

X Fano manifold of Picard number one, with a family of rational curves satisfying some conditions. Then X is rational homogeneous.

Rational Homogeneous Manifolds

Rational homogeneous manifolds

Definition

A Borel subgroup B of a semisimple Lie group G is a maximal closed, connected solvable algebraic subgroup. A subgroup $P \supseteq B$ is called a parabolic subgroup.

For example, if $G = PGL_{n+1}$, then the subgroup of (classes of) invertible upper triangular matrices is a Borel subgroup, while the parabolic subgroups correspond to $\emptyset \neq I \subseteq \{1, \dots, n\}$.

If $I = \{a_1, \dots, a_k\}$ and $a_{k+1} := n+1$, then $P(I)$ is the subgroup

$$\left[\begin{pmatrix} B_1 & * & * & * \\ 0 & B_2 & * & * \\ 0 & 0 & \dots & * \\ 0 & 0 & 0 & B_{k+1} \end{pmatrix}\right]$$

where the $B_j's$ are square matrices of order $a_j - a_{j-1}$.

Rational homogeneous manifolds

Definition

A rational homogeneous manifold is the quotient of a semisimple Lie group G by a parabolic subgroup P.

For example, if $G = PGL_{n+1}$, setting

- $E_i = (0 : \cdots : 1 : \cdots : 0)$;
- $I = \{a_1, \ldots, a_k\} \subseteq \{1, \ldots, n\}$;
- $\Lambda_{a_i} = \langle E_1, \ldots, E_i \rangle$

$P(I)$ is the stabilizer - w.r.t. the PGL_{n+1}-action - of the flag

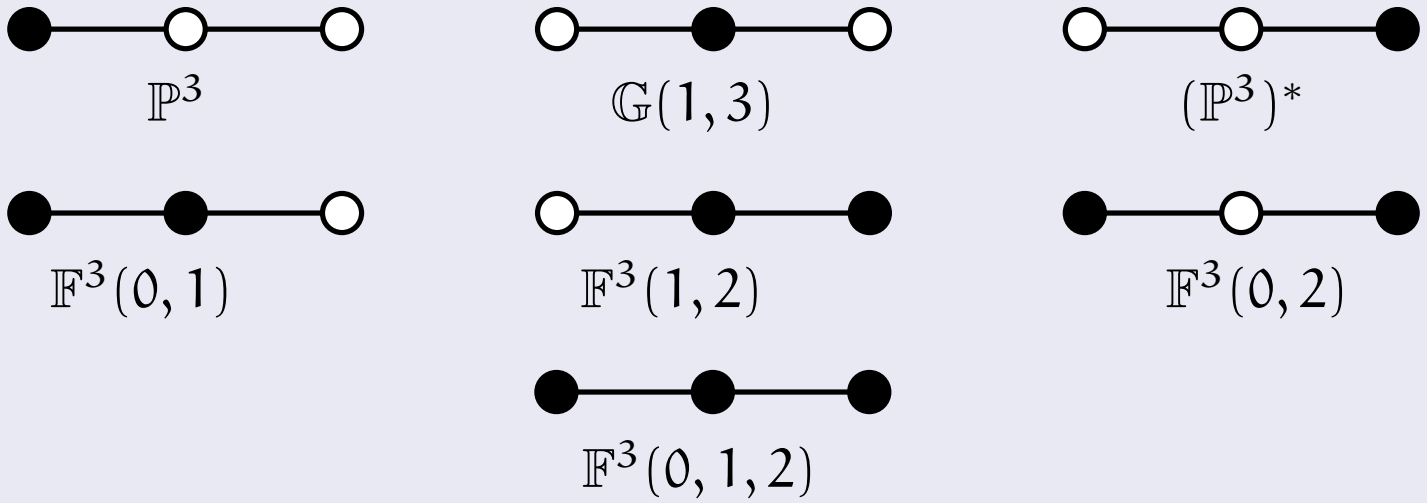
$$\Lambda_{a_1} \subset \Lambda_{a_2} \subset \cdots \subset \Lambda_{a_k}.$$

So $G/P(I)$ is the variety $\mathbb{F}^n(a_1 - 1, \ldots, a_k - 1)$ of flags of subspaces of dimensions $a_1 - 1, \ldots, a_k - 1$ of $\mathbb{P}^n$.

Rational homogeneous manifolds

We can denote the variety $\mathbb{F}^n(a_1 - 1, \ldots, a_k - 1)$ by a marked diagram.

$G = PGL_4$



The diagram used is the Dynkin diagram of the Lie algebra $\mathfrak{sl}_4$:

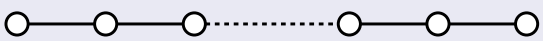
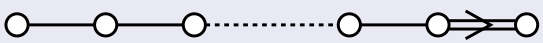
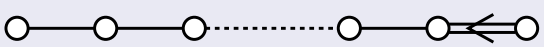
Dynkin diagrams

- G semisimple Lie group,
- $\mathfrak{g}$ associated Lie algebra,
- n rank of $\mathfrak{g}$.

Parabolic subgroups correspond to $\emptyset \neq I \subseteq D := \{1, \ldots, n\}$, and the variety $G/P(I)$ is denoted by marking the Dynkin diagram of $\mathfrak{g}$ along the nodes corresponding to I.

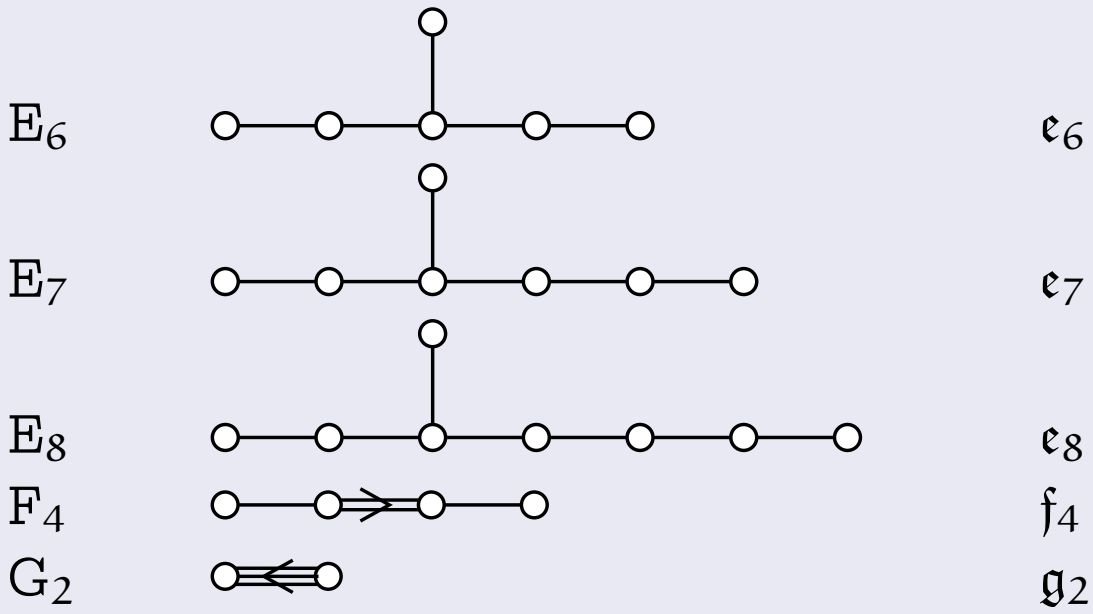
$$G/P(I) \quad \leftrightarrow \quad (\mathcal{D}, \mathcal{J})$$

Dynkin diagrams of the classical (simple) Lie algebras

A_n		$\mathfrak{sl}_{n+1}$
B_n		$\mathfrak{so}_{2n+1}$
C_n		$\mathfrak{sp}_{2n}$
D_n		$\mathfrak{so}_{2n}$

Dynkin diagrams

Dynkin diagrams of the exceptional (simple) Lie algebras

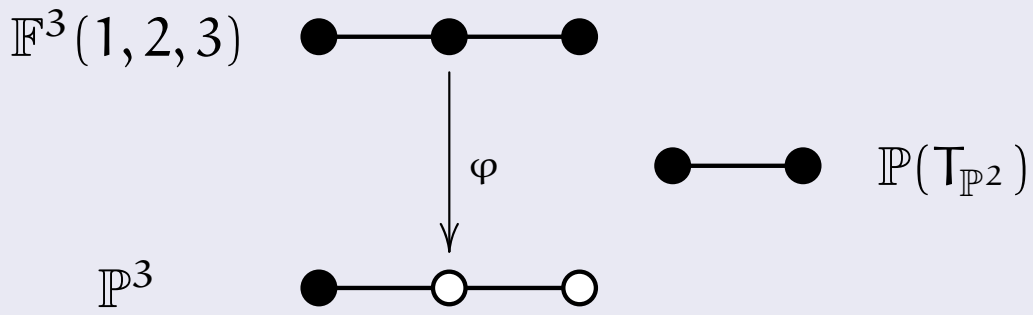


Cone and contractions

X Rational Homogeneous given by $(\mathcal{D}, \mathcal{I})$.

- X is a Fano manifold of Picard number $\rho_X = \#I$
- The cone $NE(X)$ is simplicial, and its faces correspond to proper subsets $J \subsetneq I$
- Every contraction $\pi : X \to Y$ is of fiber type and smooth
- Y is RH with marked Dynkin diagram $(\mathcal{D}, \mathcal{J})$
- Every fiber is RH with marked Dynkin diagram $(\mathcal{D} \backslash \mathcal{J}, \mathcal{I} \backslash \mathcal{J})$.

Example



Complete flag manifolds

Definition

A complete flag manifold is a RH manifold with a diagram in which all the nodes are marked. i.e. a quotient G/B by a Borel subgroup.

- Every RH manifold is dominated by a complete flag manifold.
- $\pi_i : G/B \to G/P(D \setminus \{i\})$ contractions corresponding to the unmarking of one node are $\mathbb{P}^1$-bundles.
- The fibers of any contraction of a complete flag manifold are complete flag manifolds.

Canonical bundle

The canonical bundle of a complete flag manifold can be written as a combination of the relative canonicals of its elementary contractions with integral positive coefficients:

$$K_{G/B} = \sum_{t=1}^{n} b_t K_t.$$

$\mathcal{D}$	$b_1, \ldots, b_n$
A_n	$n, (n-1)2, \ldots, 2(n-1), n$
B_n	$2n-1, 2(2n-2), \ldots, (n-1)(n+1), n^2$
C_n	$2n, 2(2n-1), \ldots, (n-1)(n+2), n(n+1)/2$
D_n	$2n-2, 2(2n-3), \ldots, (n-1)n/2, (n-1)n/2$
E_6	$16, 22, 30, 42, 30, 16$
E_7	$34, 49, 66, 96, 75, 52, 27$
E_8	$92, 136, 182, 270, 220, 168, 114, 58$
F_4	$16, 30, 42, 22$
G_2	$10, 6$

A geometric characterization

of complete Flag Manifolds

Theorem [_, Solá Conde, Watanabe & Wiśniewski (2014)]

A Fano manifold whose elementary contractions are $\mathbb{P}^1$-bundles is a complete flag manifold G/B, for some semisimple group G.

Theorem [_, Solá Conde & Wiśniewski (2015)]

X smooth projective variety of Picard number n, such that there exist n extremal rays, whose associated elementary contractions $\pi_i : X \to X_i$ are $\mathbb{P}^1$-bundles. Then X is isomorphic to a complete flag manifold G/B, for some semisimple group G.

Rational Homogeneous Bundles

Rational homogeneous bundles

Definition

A rational homogeneous bundle over X is a smooth, projective isotrivial morphism $q : E \to X$ with fibers isomorphic to a rational homogeneous manifold F.

By a theorem of Fischer and Grauert q is locally trivial in the analytic topology:

- $\{U_i\}$ trivializing cover, $U_{ij} := U_i \cap U_j$,
- $\{\phi_i : U_i \times F \to q^{-1}(U_i)\}$ trivializations,
- $\phi_{ij} = \phi_j^{-1} \circ \phi_i : U_{ij} \times F \to U_{ij} \times F$ transitions,
- $\theta_{ij} : U_{ij} \to \mathrm{Aut}(F)$ s.t. $\phi_{ij}(x, y) = (x, \theta_{ij}(x)(y))$.

The F-bundle is determined by a cocycle $\theta \in H^1(X, Aut(F))$.

Associated bundles

- G identity component of $\mathrm{Aut}(F)$.
- $F \simeq G/P(I)$ for some parabolic subgroup $P(I) \subset G$.
- If X is simply connected then $\theta \in H^1(X, G)$.

Then θ defines a principal G-bundle $q_G : E_G \to X$, given by the glueing of $U_i \times G$ by the transition functions:

$$\theta_{ij} : U_{ij} \to G \hookrightarrow \mathrm{Aut}(G),$$

For any parabolic subgroup $Q \subset G$, there is a standard way of constructing a G/Q-bundle $q_Q : E_Q \to X$

Definition

If B is a Borel subgroup then $\widetilde{E} := E_B$ is the flag bundle associated with the F-bundle $q : E \to X$.

Flag bundles

If $P(I) \supset P(J)$ there is a natural map $E_{P(J)} \to E_{P(I)}$ over X. In particular we have a morphism $\pi : \widetilde{E} \to E$

$$\begin{tikzcd}
 & & E = E_{P(I)} \arrow[dd, "q"] \\
E_G \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & \widetilde{E} \arrow[r] \arrow[ur, "\pi"] \arrow[dr, "\widetilde{q}"'] & E_{P(J)} \arrow[u] \arrow[d] \\
 & & X
\end{tikzcd}$$

Flag bundles

In particular $\widetilde{E}$ has $\rho(G/B)$ contractions which are $\mathbb{P}^1$-bundles.

$$\begin{tikzcd}
 & & \widetilde{E} \arrow[dll, "q_1"'] \arrow[dl, "q_2"'] \arrow[dd, "\widetilde{q}"] \arrow[dr, "q_n"] & \\
E_{P(D\setminus\{1\})} \arrow[drr] & E_{P(D\setminus\{2\})} \arrow[dr] & & E_{P(D\setminus\{n\})} \arrow[dl] \\
 & & X &
\end{tikzcd}$$

Lemma

The relative canonical class of $\widetilde{q}$ satisfies:

$$K_{\widetilde{q}} = \sum_{t=1}^{k} b_t K_t.$$

Flag bundles on $\mathbb{P}^1$

Grothendieck's Theorem

on principal bundles

Theorem [Grothendieck (1956)]

Let G be a reductive group, $H \subset G$ a Cartan subgroup, N the normalizer of H in G and $W = N/H$ the Weyl group of G. Then the natural map

$$H^1(\mathbb{P}^1, H)/W \to H^1(\mathbb{P}^1, G)$$

is an isomorphism.

For example, if $G = GL_{n+1}$, then $H \subset G$ is the subgroup of diagonal matrices, and the Weyl group is the permutation group Σ_{n+1}, so the theorem is saying that a vector bundle over $\mathbb{P}^1$ is a sum of line bundles, independent from the order in which we take the summands.

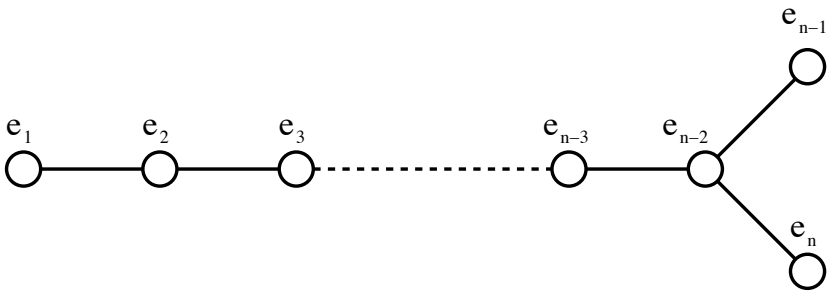
Grothendieck's Theorem

A geometric interpretation

$G = PGL_2$	G semisimple group
$q : \widetilde{E} \to \mathbb{P}^1$ $\mathbb{P}^1$-bundle	$q : \widetilde{E} \to \mathbb{P}^1$ G/B-bundle
$C_0 \subset \widetilde{E}$ minimal section K_q relative canonical	$C_0 \subset \widetilde{E}$ minimal section $K_1, \ldots, K_n$ relative canonicals
$e = -K_\pi \cdot C_0 \geq 0$	$e_i = -K_i \cdot C_0 \geq 0$
$\widetilde{E}$ is determined by e	$\widetilde{E}$ is determined by $e_1, \ldots, e_n$
$\widetilde{E} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ iff $e = 0$	$\widetilde{E} \simeq G/B \times \mathbb{P}^1$ iff $(e_1, \ldots, e_n) = (0, \ldots, 0)$

Tagged Dynkin diagrams

So a RH-bundle over $\mathbb{P}^1$ is determined by a Dynkin diagram whose nodes are tagged with nonnegative integers.



Example

If $q : E \to \mathbb{P}^1$ is a $\mathbb{P}^n$-bundle with tagged Dynkin diagram

then $E = \mathbb{P}(\oplus \mathcal{O}(a_i))$ and $a_i = a_{i-1} + e_i$, so the diagram gives us the usual splitting type (up to a twist, depending on a_0).

Families of rational curves

Basics

X Fano of Picard number one. X is uniruled (Mori), so there is a minimal dominating family of rational curves, i.e.

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{M} & & X \end{tikzcd}$$

- $\mathcal{M}$ is quasi-projective
- Fibers of p are $\mathbb{P}^1$'s
- q is dominant
- $\mathcal{M}_x = q^{-1}(x)$ is proper for a general x.

For a general x the tangent map $\tau_x : \mathcal{M}_x \to \mathbb{P}(T^\vee_{X,x})$ is defined, and it is the normalization of the image (Hwang-Mok, Kebekus)

Definition

The variety $\mathcal{C}_x = \tau_x(\mathcal{M}_x) \subset \mathbb{P}(T^\vee_{X,x})$ is called the VMRT at x.

The Rational Homogeneous case

Assume now that X is a rational homogeneous manifold of Picard number one. Then, in the diagram

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{M} & & X \end{tikzcd}$$

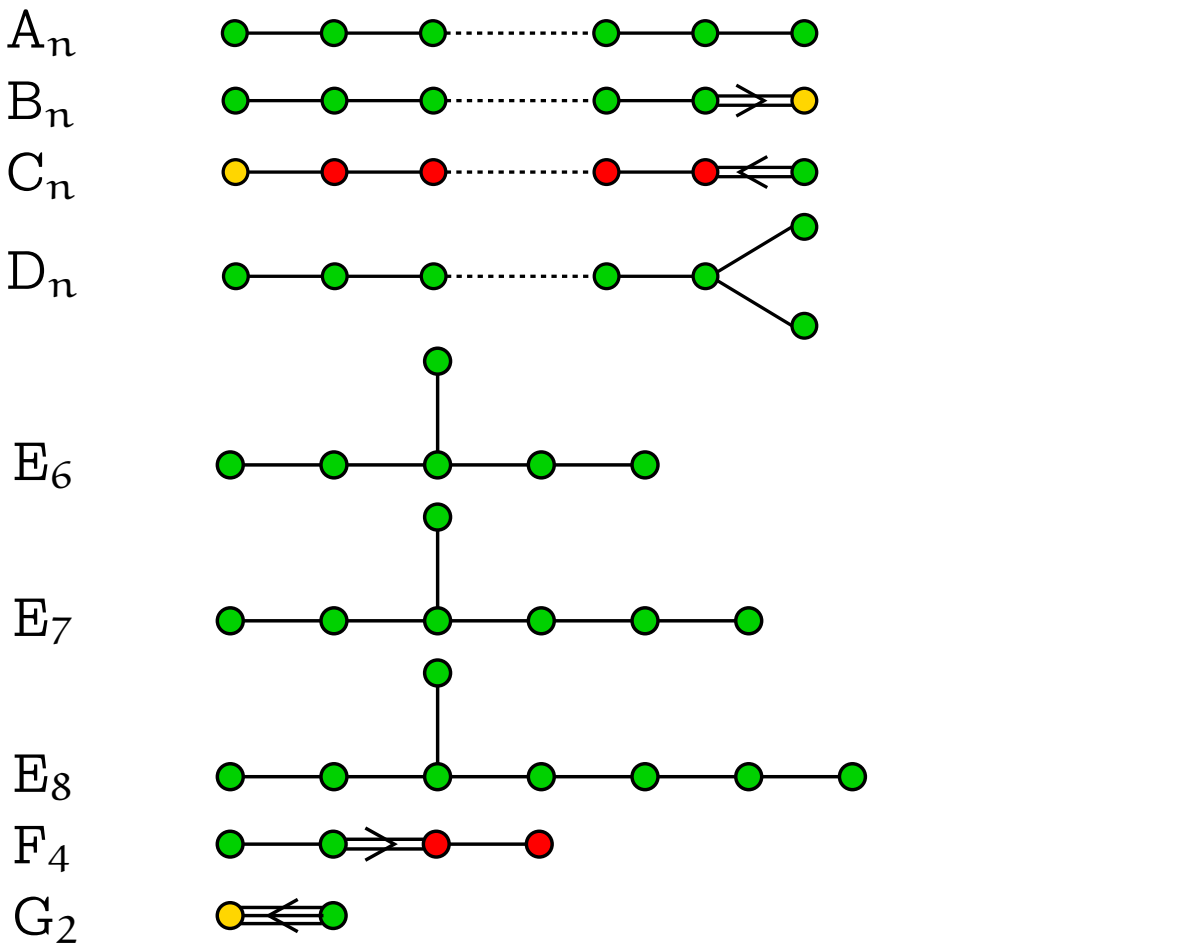
- $\mathcal{M}$ smooth and proper
- Fibers of p are $\mathbb{P}^1$'s
- q is smooth, surjective and isotrivial.

Definition

X rational homogeneous of Picard number one is a good guy if $\mathcal{U}$ is rational homogeneous.

If X is a good guy, also $\mathcal{M}$ and $\mathcal{M}_x$ are rational homogeneous.

Good guys and bad guys

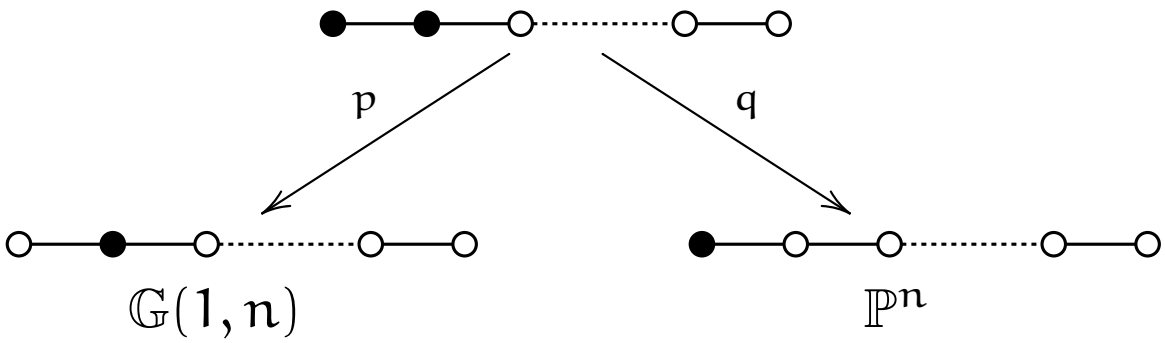


Rational curves on good guys

Assume now that X is a good guy (not under cover), corresponding to $(\mathcal{D}, \{i\})$. Then in the diagram

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{M} & & X \end{tikzcd}$$

- $\mathcal{U}$ is obtained by marking i and its neighbours.
- $\mathcal{M}$ is obtained from $\mathcal{U}$ by unmarking i.



- Fibers of q are obtained (as usual) from $\mathcal{U}$ by deleting i. In the example $q^{-1}(x) \simeq \mathbb{P}^{n-1}$ 

Recap

X good guy, $\mathcal{M}$ minimal dominating family of rational curves
Then, in the basic diagram

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{M} & & X \end{tikzcd}$$

- $\mathcal{M}$ is rational homogeneous;
- $\mathcal{U}$ is rational homogeneous;
- q is a rational homogeneous bundle.

Problem

If X is just a Fano manifold of Picard number one, what do we need to assume on the basic diagram to prove that X is rational homogeneous?

Main Result

Main result

Theorem

X Fano manifold of Picard number one, with a minimal dominating family of rational curves

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{M} & & X \end{tikzcd}$$

such that q is a rational homogeneous bundle. Then X is rational homogeneous.

- If T_X is nef, then q is smooth.
- If moreover a general fiber is rational homogeneous of Picard number one, then q is isotrivial.

Idea of Proof

We consider the flag bundle $\widetilde{q} : \widetilde{\mathcal{U}} \to X$ associated with $q : \mathcal{U} \to X$.

$$\begin{tikzcd} \widetilde{\mathcal{U}} \arrow[d, "\pi"] \arrow[dr, "\widetilde{q}"] & \\ \mathcal{U} \arrow[r, "q"] & X \end{tikzcd}$$

The idea is to show that $\widetilde{\mathcal{U}}$ is homogeneous by using the geometric characterization of complete flag manifolds.

The flag bundle $\widetilde{\mathcal{U}}$ has Picard number $\rho(\widetilde{\mathcal{U}}) = \rho(G/B) + 1$, and has $\rho(G/B)$ contractions (over X) which are $\mathbb{P}^1$-bundles.

So we need one more.

Idea of Proof

The idea is to lift the $\mathbb{P}^1$-bundle $p : \mathcal{U} \to \mathcal{M}$

$$\begin{tikzcd}
\widetilde{\mathcal{M}} \arrow[d, "\Downarrow"] & \widetilde{\mathcal{U}} \arrow[l, "\widetilde{p}"'] \arrow[d, "\pi"] \arrow[dr, "\widetilde{q}"] & \\
\mathcal{M} & \mathcal{U} \arrow[l, "p"'] \arrow[r, "q"] & X
\end{tikzcd}$$

To do this we consider the restriction of the flag bundle π to a fiber Γ of p and we compute its tag.

By deformation theory we know that $-K_q \cdot \Gamma = i_X - 2$

The same intersection number can be computed by using the relative canonical bundle formulas for π and $\widetilde{q}$.

Equality implies that the tag of π restricted to Γ is $(0, \dots 0)$, so fibers of $p \circ \pi$ are products with $\mathbb{P}^1$ as a factor and p lifts.

Related results

Theorem [Mok, Hong-Hwang]

X Fano of Picard number one, $S = G/P$ good guy; $\mathcal{C}_0(S)$ VMRT of S, $\mathcal{C}_x(X)$ VMRT of X at a general point.
If $\mathcal{C}_x(X)$ is projectively equivalent to $\mathcal{C}_0(S)$, then $X \simeq S$.

- One need to consider only the general point.
- No assumption on the smoothness of q.
- Asking for a projective isomorphism is very strong.

Going after the bad guys

In case C_n the bad guys are the Symplectic Grassmannians $L\mathbb{G}(k, 2n+1)$, i.e. they parametrize k dimensional linear spaces in $\mathbb{P}^{2n-1}$ isotropic with respect to a symplectic form.

Theorem [_, Solá Conde, Watanabe (2016)]

X Fano manifold of Picard number one, with a minimal dominating family of rational curves

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{M} & & X \end{tikzcd}$$

such that q is smooth and the VMRT of X at every point is isomorphic to the VMRT of the Symplectic Grassmannian $L\mathbb{G}(k, 2n+1)$. Then X is isomorphic to $L\mathbb{G}(k, 2n+1)$.