Splitting criteria for rank two vector bundles on Fano manifolds

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joint work with
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- Discriminant: $\Delta(\mathcal{E}) = (c_1^2 - 4c_2/d)\Sigma := \Delta\Sigma;$

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In the second case we use that $S^2\mathcal{E}(1)$ is polystable (sum of stables).

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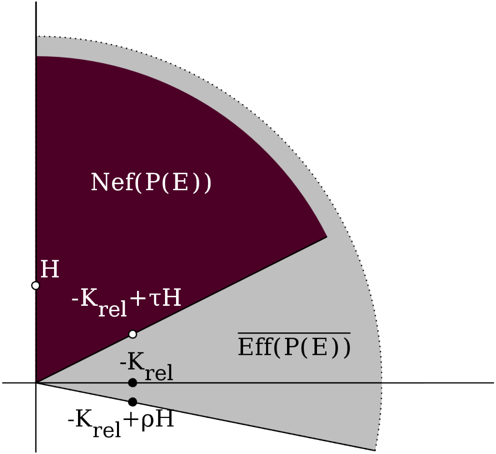

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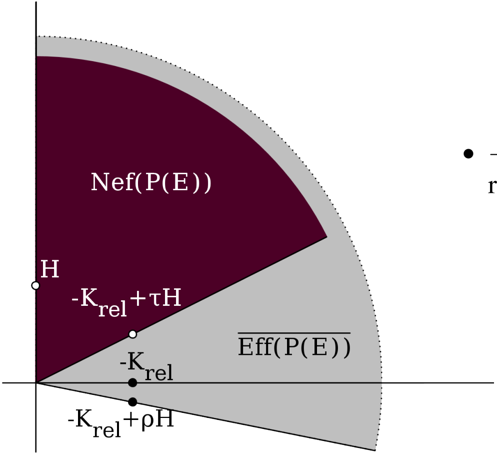

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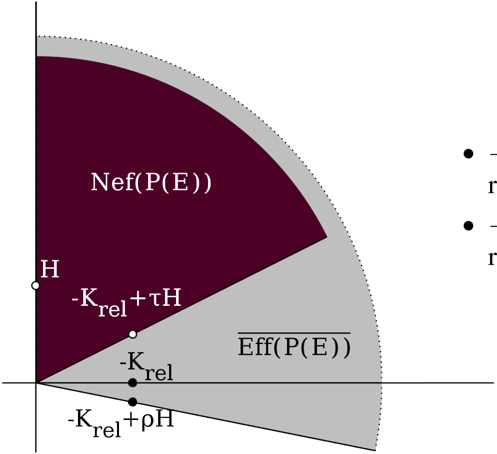

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Lemma (1)

Assume that, for some rational number q there is a surface $S \subset \mathbb{P}(\mathcal{E})$ such that $\pi_{|S}$ is finite and that $(L - qH) \cdot C = 0$ for every $C \subset S$. Then

$$c_2 = dq(c_1 - q).$$

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By the relative Euler sequence restricted to S

$$0 \rightarrow (\omega_{\mathbb{P}(\mathcal{E})/X}(1))|_S \longrightarrow (\pi^*\mathcal{E})|_S \longrightarrow (\mathcal{O}(1))|S \rightarrow 0.$$

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If there is a surface as in Lemma (1) containing a minimal section over a curve in $\mathcal{M}$ both conditions are satisfied.

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Proof.

Let S be the locus of curves parametrized by the complete curve T; in S every curve is numerically proportional to a curve of $\widetilde{\mathcal{M}}^{r\mu}_y$. ∎

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Proposition

Assume that $\mathcal{M}_x$ is proper for a general $x \in X$, that $\beta \leq 0$ and that

- $\tau < 2i_X - 2\beta - c_1 - 4/\mu$ *if* $(c_1, \beta) \neq (0,0)$;
- $\tau < 2i_X - 6/\mu$, *if* $(c_1, \beta) = (0,0)$.

Then $\mathcal{E}$ splits as a sum of line bundles.

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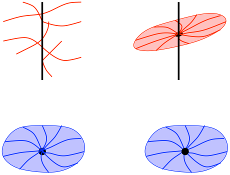

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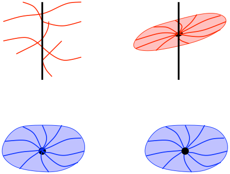

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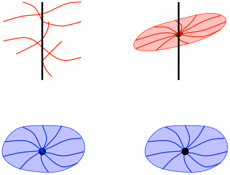

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We then compute the maximum number of possible splitting types and the dimension of $\mathcal{M}_x$ in terms of the invariants. ■

Splitting criteria
Applications

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Assume that $\Delta \geq 0$ and that $\tau < 2i_X + \sqrt{\Delta} - 4/\mu$. Then $\mathcal{E}$ splits.

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Remark

The bound in the above Corollary is better than the bound one gets from Castelnuovo-Mumford regularity.

Fano bundles

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we see that

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Fano bundles

Examples

Fano bundles

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X	Blow-ups	Conic bundles	$\mathbb{P}^1$-bundles
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In all cases Y is smooth and Fano.

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Notice that $\beta = 0$, then $c_2 > 0$, hence $\Delta < 0$.

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Fano bundles

Double $\mathbb{P}^1$-bundle structure

Fano bundles

Double $\mathbb{P}^1$-bundle structure

X	Y	bundle
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$$d_Y = H\overline{H}^n \qquad d_X = \overline{H}H^n$$

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Corollary (G-M for Fanos)

Let $\mathcal{M}$ be a covering family of rational curves on X such that $\mathcal{M}_x$ is irreducible for general $x \in X$. Let (a, b) with $a \leq b$ be the general splitting type of $\mathcal{E}$. If $\mathcal{E}$ is semistable, then $b - a \leq 1$.

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Uniform bundles

A condition for splitting

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Lemma (5)

Assume that $\mathcal{E}$ is uniform of type (a,b) with $a<b$ with respect to an unsplit covering family $\mathcal{M}$ of rational curves on X, and that $H^0(\mathcal{E}(-b)) \neq 0$. Then $\mathcal{E} = \mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$.

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The second contraction is a $\mathbb{P}^1$-bundle, and we apply the previous theorem, checking uniformity in the classification. ∎

THANK YOU!