PROJECTIVE MANIFOLDS CONTAINING A LARGE LINEAR SUBSPACE WITH NEF NORMAL BUNDLE

Gianluca Occhetta (joint work with Carla Novelli)

Birational automorphism groups and birational geometry,
Pisa - October 2008
A theorem of Sato

One step further

Introduction

Theorem of Sato

One step further

Gianluca Occhetta

Manifolds containing a linear subspace

**Setup**

\[
\Lambda^s \subset X^n \subset \mathbb{P}^N
\]
**Setup**

\[ \Lambda^s \subset X^n \subset \mathbb{P}^N \]

- \( X \) smooth complex projective variety of dimension \( n \).
**Setup**

\[ \Lambda^s \subset X^n \subset \mathbb{P}^N \]

- \(X\) smooth complex projective variety of dimension \(n\).
- \(\Lambda\) linear space of dimension \(s\).
**Setup**

\[ \Lambda^s \subset X^n \subset \mathbb{P}^N \]

- $X$ smooth complex projective variety of dimension $n$.
- $\Lambda$ linear space of dimension $s$.
- $\mathcal{N} := N_{\Lambda/X}$ normal bundle.
**Setup**

\[ \Lambda^s \subset X^n \subset \mathbb{P}^N \]

- \( X \) smooth complex projective variety of dimension \( n \).
- \( \Lambda \) linear space of dimension \( s \).
- \( \mathcal{N} := N_{\Lambda/X} \) normal bundle.
- \( c = c_1(\mathcal{N}) \).
**Introduction**

A theorem of Sato

One step further

---

**Setup**

\[ \Lambda^s \subset X^n \subset \mathbb{P}^N \]

- $X$ smooth complex projective variety of dimension $n$.
- $\Lambda$ linear space of dimension $s$.
- $\mathcal{N} := N_{\Lambda/X}$ normal bundle.
- $c = c_1(\mathcal{N})$.

Problem: classify $X$ under suitable assumptions on $s$ and $\mathcal{N}$. 

---

**Gianluca Occhetta**

Manifolds containing a linear subspace
If one of the following occurs

1. $N$ is ample;
2. $N$ nef, $\rho_X = 1$ and $s > n^2$; then $X$ is a linear space.

In case (1) $X$ admits a dominating family of lines of anticanonical degree $n + 1$, hence $\rho_X = 1$ and the Kobayashi-Ochiai theorem applies.

In case (2) one can apply a consequence of Zak's Theorem on Tangencies.
K N O W N  r e s u l t s  I - L I N E A R  s p a c e s

If one of the following occurs

1. $\mathcal{N}$ is ample;
**Known results I - Linear spaces**

If one of the following occurs:

1. \( N \) is ample;
2. \( N \) nef, \( \rho_X = 1 \) and \( s > \frac{n}{2} \);
If one of the following occurs

1. $\mathcal{N}$ is ample;
2. $\mathcal{N}$ nef, $\rho_X = 1$ and $s > \frac{n}{2}$;

then $X$ is a linear space.
Known results I - Linear spaces

If one of the following occurs

1. $\mathcal{N}$ is ample;
2. $\mathcal{N}$ nef, $\rho_X = 1$ and $s > \frac{n}{2}$;

then $X$ is a linear space.

In case (1) $X$ admits a dominating family of lines of anticanonical degree $n+1$, hence $\rho_X = 1$ and Kobayashi-Ochiai theorem applies.
**Known results I - Linear spaces**

If one of the following occurs

1. \( \mathcal{N} \) is ample;
2. \( \mathcal{N} \) nef, \( \rho_X = 1 \) and \( s > \frac{n}{2} \);

then \( X \) is a linear space.

In case (1) \( X \) admits a dominating family of lines of anticanonical degree \( n + 1 \), hence \( \rho_X = 1 \) and Kobayashi-Ochiai theorem applies.

In case (2) one can apply a consequence of Zak’s Theorem on Tangencies.
**Known results II - Projective bundles**

Assume
Assume

1. $\mathcal{N}$ trivial and $n \leq 2s - 1$ (Ein);
**Known results II - Projective bundles**

Assume

1. $\mathcal{N}$ trivial and $n \leq 2s - 1$ (Ein);
2. $\mathcal{N}$ trivial and $n \leq 2s$ (Wiśniewski).
**Known results II - Projective bundles**

Assume

1. \( \mathcal{N} \) trivial and \( n \leq 2s - 1 \) (Ein);
2. \( \mathcal{N} \) trivial and \( n \leq 2s \) (Wiśniewski).

Then \( X \) has a projective bundle structure over a smooth variety \( Y \):
\[
X = \mathbb{P}_Y(\mathcal{E})
\]
and \( \Lambda \) is a fiber of the bundle projection.
Known results II - Projective bundles

Assume

1. $\mathcal{N}$ trivial and $n \leq 2s - 1$ (Ein);
2. $\mathcal{N}$ trivial and $n \leq 2s$ (Wiśniewski).

Then $X$ has a projective bundle structure over a smooth variety $Y$: $X = \mathbb{P}_Y(\mathcal{E})$ and $\Lambda$ is a fiber of the bundle projection.

In case (1) the result is obtained by studying the Hilbert scheme of $s$-planes in $X$. 
In case (2), besides the Hilbert scheme, a new ingredient appears:
In case (2), besides the Hilbert scheme, a new ingredient appears:

**Theorem (Beltrametti - Sommese - Wiśniewski)**

Assume that $X$ is covered by a family of lines of anticanonical degree $\geq \frac{n+3}{2}$. Then there is an extremal ray of $\text{NE}(X)$ generated by the numerical class of such a line.
By results of Mori theory, if $\varphi : X \to Y$ is a fiber type contraction associated to a (negative) extremal ray $R$, then $X$ is covered by rational curves whose numerical class is in $R$. 

**Gianluca Occhetta**

**Manifolds containing a linear subspace**
By results of Mori theory, if \( \varphi : X \to Y \) is a fiber type contraction associated to a (negative) extremal ray \( R \), then \( X \) is covered by rational curves whose numerical class is in \( R \).

By taking a minimal degree dominating family of rational curves in \( R \) one obtains a quasi-unsplit family.
By results of Mori theory, if $\varphi : X \to Y$ is a fiber type contraction associated to a (negative) extremal ray $R$, then $X$ is covered by rational curves whose numerical class is in $R$.

By taking a minimal degree dominating family of rational curves in $R$ one obtains a quasi-unsplit family.

Quasi-unsplit: curves in the family can degenerate to reducible cycles, but every irreducible component of such a cycle is numerically proportional to a curve in the family.
QUESTION

Assume that $X$ admits a quasi-unsplit dominating family of rational curves $V$. Do the numerical classes of the curves in the family generate an extremal ray of $\text{NE}(X)$?
**QUESTION**

Assume that $X$ admits a quasi-unsplit dominating family of rational curves $V$. Do the numerical classes of the curves in the family generate an extremal ray of $\text{NE}(X)$?

By results of Bonavero - Casagrande - Druel this is true if
Assume that $X$ admits a quasi-unsplit dominating family of rational curves $V$. Do the numerical classes of the curves in the family generate an extremal ray of $\text{NE}(X)$?

By results of Bonavero - Casagrande - Druel this is true if

1. $X$ is toric;
**Question**

Assume that $X$ admits a quasi-unsplit dominating family of rational curves $V$. Do the numerical classes of the curves in the family generate an extremal ray of $\text{NE}(X)$?

By results of Bonavero - Casagrande - Druel this is true if

1. $X$ is toric;
2. The dimension of a general $V$-equivalence class is $\geq \dim X - 3$. 

---

**Gianluca Occhetta**

**Manifolds containing a linear subspace**
**Recent results**

**Theorem (Novelli, _)**

Assume that $X$ is covered by a family of lines of anticanonical degree $\geq \frac{n-1}{2}$. Then there is an extremal ray generated by the numerical class of such a line.
RECENT RESULTS

THEOREM (NOVELLI, _)

Assume that $X$ is covered by a family of lines of anticanonical degree $\geq \frac{n-1}{2}$. Then there is an extremal ray generated by the numerical class of such a line.

The proof combines the main idea of B-S-W (studying the nefness of a suitable adjoint divisor) and the description of the indeterminacy locus of the rationally connected fibration associated to the family of lines given in B-C-D.
**Corollary**

Assume that $\mathcal{N}$ is nef and $s + c \geq \frac{n-3}{2}$. Then there exists an elementary contraction $\varphi : X \to Y$ which contracts $\Lambda$. 
**Corollary**

Assume that $\mathcal{N}$ is nef and $s + c \geq \frac{n-3}{2}$. Then there exists an elementary contraction $\varphi : X \to Y$ which contracts $\Lambda$.

/ line in $\Lambda$. 

Gianluca Occhetta

Manifolds containing a linear subspace
**Corollary**

Assume that $\mathcal{N}$ is nef and $s + c \geq \frac{n-3}{2}$. Then there exists an elementary contraction $\varphi : X \to Y$ which contracts $\Lambda$.

Let $l$ be a line in $\Lambda$.

By the sequence

$$0 \to N_{l/\Lambda} = \mathcal{O}_\Lambda(1)^{\oplus(s-1)} \to N_{l/X} \to \mathcal{N}|_l \to 0,$$


**Corollary**

Assume that $\mathcal{N}$ is nef and $s + c \geq \frac{n-3}{2}$. Then there exists an elementary contraction $\varphi : X \to Y$ which contracts $\Lambda$.

There is a line in $\Lambda$.

By the sequence

$$0 \to N_{l/\Lambda} = \mathcal{O}_\Lambda(1) \oplus (s-1) \to N_{l/X} \to \mathcal{N}|_l \to 0,$$

$N_{l/X}$ is nef; therefore $l$ is a free rational curve in $X$, hence it belongs to a dominating family of lines.
**Corollary**

Assume that $\mathcal{N}$ is nef and $s + c \geq \frac{n-3}{2}$. Then there exists an elementary contraction $\varphi : X \to Y$ which contracts $\Lambda$.

Let $l$ be a line in $\Lambda$.

By the sequence

$$0 \to N_{l/\Lambda} = \mathcal{O}_\Lambda(1)^{\oplus(s-1)} \to N_{l/X} \to N_{\mid l} \to 0,$$

$N_{l/X}$ is nef; therefore $l$ is a free rational curve in $X$, hence it belongs to a dominating family of lines.

By adjunction

$$-K_X \cdot l = s + 1 + c,$$

so we can apply the theorem.
A theorem of Sato

**Theorem (Sato)**

Assume that $X^{2s}$ is covered by $s$-dimensional linear spaces. Then the normal bundle of a general $s$-space is one of the following:

1. $\Omega^1 \oplus c \oplus O \oplus (s - c) \Lambda^2$; 
2. $\Omega^0 \Lambda^1$; 
3. $T \Lambda^1$.

And the corresponding varieties are

1. a projective bundle over a smooth variety; 
2. a smooth hyperquadric; 
3. If $s$ is even $G(1, s+1)$. 

**Gianluca Occhetta**

Manifolds containing a linear subspace
A theorem of Sato

**Theorem (Sato)**

Assume that $X^{2s}$ is covered by $s$-dimensional linear spaces. Then the normal bundle of a general $s$-space is one of the following:

1. $O^\wedge(1)^\oplus c \oplus O^\wedge(s-c)$,
A theorem of Sato

Theorem (Sato)

Assume that \( X^{2s} \) is covered by \( s \)-dimensional linear spaces. Then the normal bundle of a general \( s \)-space is one of the following:

1. \( \mathcal{O}(1)^{\oplus c} \oplus \mathcal{O}(s-c) \);
2. \( \Omega_{\Lambda}(2) \);
3. If \( s \) is even, \( G(1,s+1) \).
A theorem of Sato

Assume that $X^{2s}$ is covered by $s$-dimensional linear spaces. Then the normal bundle of a general $s$-space is one of the following:

1. $\mathcal{O}_\Lambda(1)^{\oplus c} \oplus O^{\oplus (s-c)}$;
2. $\Omega^{\wedge}(2)$;
3. $T^{\wedge}(-1)$.

And the corresponding varieties are:

1. a projective bundle over a smooth variety;
2. a smooth hyperquadric;
3. If $s$ is even $G^{(1,s+1)}$. 

Gianluca Occhetta

Manifolds containing a linear subspace
A theorem of Sato

Assume that $X^{2s}$ is covered by $s$-dimensional linear spaces. Then the normal bundle of a general $s$-space is one of the following:

1. $\mathcal{O}_\Lambda(1)^{\oplus c} \oplus \mathcal{O}_\Lambda^{\oplus (s-c)}$;
2. $\Omega_\Lambda(2)$;
3. $T_\Lambda(-1)$.

And the corresponding varieties are
A theorem of Sato

**Theorem (Sato)**

Assume that $X^{2s}$ is covered by $s$-dimensional linear spaces. Then the normal bundle of a general $s$-space is one of the following:

1. $\mathcal{O}_\Lambda(1)^{\oplus c} \oplus \mathcal{O}_\Lambda^{\oplus (s-c)}$;
2. $\Omega_\Lambda(2)$;
3. $T_\Lambda(-1)$.

And the corresponding varieties are

1. a projective bundle over a smooth variety;
THEOREM (Sato)

Assume that $X^{2s}$ is covered by $s$-dimensional linear spaces. Then the normal bundle of a general $s$-space is one of the following:

1. $\mathcal{O}_\Lambda(1)^{\oplus c} \oplus \mathcal{O}_\Lambda^{(s-c)}$;
2. $\Omega_\Lambda(2)$;
3. $T_\Lambda(-1)$.

And the corresponding varieties are

1. a projective bundle over a smooth variety;
2. a smooth hyperquadric;
A theorem of Sato

Assume that $X^{2s}$ is covered by $s$-dimensional linear spaces. Then the normal bundle of a general $s$-space is one of the following:

1. $\mathcal{O}_\Lambda(1)^\oplus c \oplus \mathcal{O}_\Lambda^{\oplus(s-c)}$;
2. $\Omega_\Lambda(2)$;
3. $T_\Lambda(-1)$.

And the corresponding varieties are:

1. a projective bundle over a smooth variety;
2. a smooth hyperquadric;
3. if $s$ is even $\mathbb{G}(1, s+1)$. 
**Remark**

$X^{2s} \subset \mathbb{P}^N$. There exists a linear $s$-space contained in $X$ with nef normal bundle iff $X$ is covered by linear spaces of dimension $s$. 
**Remark**

$X^{2s} \subset \mathbb{P}^N$. There exists a linear $s$-space contained in $X$ with nef normal bundle iff $X$ is covered by linear spaces of dimension $s$.

By the sequence

$$0 \to \mathcal{N} \to N_{\Lambda/\mathbb{P}^N} = \mathcal{O}_{\Lambda}(1)^{\oplus (N-s)} \to (N_{X/\mathbb{P}^N}|_X) \to 0$$
Remark

$X^{2s} \subset \mathbb{P}^N$. There exists a linear $s$-space contained in $X$ with nef normal bundle iff $X$ is covered by linear spaces of dimension $s$.

By the sequence

$$0 \to \mathcal{N} \to N_{\Lambda/\mathbb{P}^N} = O_{\Lambda}(1)^{\oplus (N-s)} \to \left( N_{X/\mathbb{P}^N} \right)_{|X} \to 0$$

and the nefness of $\mathcal{N}$ the splitting of $\mathcal{N}$ on lines in $\Lambda$ is of type $(0, \ldots, 0, 1, \ldots, 1)$, hence $\mathcal{N}$ is uniform.
Remark

$X^{2s} \subset \mathbb{P}^N$. There exists a linear $s$-space contained in $X$ with nef normal bundle iff $X$ is covered by linear spaces of dimension $s$.

By the sequence

$$0 \to \mathcal{N} \to N_{\Lambda/\mathbb{P}^N} = \mathcal{O}_{\Lambda}(1)^{\oplus(N-s)} \to (N_{X/\mathbb{P}^N})|_X \to 0$$

and the nefness of $\mathcal{N}$ the splitting of $\mathcal{N}$ on lines in $\Lambda$ is of type $(0,\ldots,0,1,\ldots,1)$, hence $\mathcal{N}$ is uniform.

By the classification of uniform bundles of rank $s$ on $\mathbb{P}^s$, $\mathcal{N}$ is one of the bundles in the list of Sato. These bundles are spanned and have trivial $h^1$. 
A theorem of Sato

One step further

**Remark**

$X^{2s} \subset \mathbb{P}^N$. There exists a linear $s$-space contained in $X$ with nef normal bundle iff $X$ is covered by linear spaces of dimension $s$.

By the sequence

$$0 \rightarrow \mathcal{N} \rightarrow N_{\Lambda/\mathbb{P}^N} = \mathcal{O}_{\Lambda}(1)^{\oplus (N-s)} \rightarrow (N_{X/\mathbb{P}^N}|_X) \rightarrow 0$$

and the nefness of $\mathcal{N}$ the splitting of $\mathcal{N}$ on lines in $\Lambda$ is of type $(0,\ldots,0,1,\ldots,1)$, hence $\mathcal{N}$ is uniform.

By the classification of uniform bundles of rank $s$ on $\mathbb{P}^s$, $\mathcal{N}$ is one of the bundles in the list of Sato. These bundles are spanned and have trivial $h^1$.

By standard arguments of Hilbert schemes it follows that $X$ is covered by linear spaces of dimension $s$. 

Gianluca Occhetta

Manifolds containing a linear subspace
\( X \) covered by linear spaces \( \Rightarrow \) the normal bundle of a general space is generically spanned.
\( X \) covered by linear spaces \( \Rightarrow \) the normal bundle of a general space is generically spanned.

Sato proved that such a bundle is uniform of type \((0, \ldots, 0, 1, \ldots, 1)\) at a general point, and then showed that such bundles are just the uniform ones:
$X$ covered by linear spaces $\implies$ the normal bundle of a general space is generically spanned.

Sato proved that such a bundle is uniform of type $(0, \ldots, 0, 1, \ldots, 1)$ at a general point, and then showed that such bundles are just the uniform ones:

$$O^\Lambda(1)^{\oplus c} \oplus O^\Lambda^{(s-c)},$$
X covered by linear spaces $\Rightarrow$ the normal bundle of a general space is generically spanned.

Sato proved that such a bundle is uniform of type $(0, \ldots, 0, 1, \ldots, 1)$ at a general point, and then showed that such bundles are just the uniform ones:

1. $\mathcal{O}_\Lambda(1)^\oplus c \oplus \mathcal{O}_\Lambda^{(s-c)}$;
2. $\Omega_\Lambda(2)$;
$X$ covered by linear spaces $\Rightarrow$ the normal bundle of a general space is generically spanned.

Sato proved that such a bundle is uniform of type $(0, \ldots, 0, 1, \ldots, 1)$ at a general point, and then showed that such bundles are just the uniform ones:

1. $\mathcal{O}_\Lambda(1)^{\oplus c} \oplus \mathcal{O}_\Lambda^{(s-c)}$;
2. $\Omega_\Lambda(2)$;
3. $T_\Lambda(-1)$. 

GIANLUCA OCCHETTA
MANIFOLDS CONTAINING A LINEAR SUBSPACE
$X$ covered by linear spaces $\Rightarrow$ the normal bundle of a general space is generically spanned.

Sato proved that such a bundle is uniform of type $(0,\ldots,0,1,\ldots,1)$ at a general point, and then showed that such bundles are just the uniform ones:

1. $\mathcal{O}_\Lambda(1)^{\oplus c} \oplus \mathcal{O}_\Lambda^{\oplus (s-c)}$;
2. $\Omega_\Lambda(2)$;
3. $T_\Lambda(-1)$.

Then he showed that
$X$ covered by linear spaces $\Rightarrow$ the normal bundle of a general space is generically spanned.

Sato proved that such a bundle is uniform of type $(0, \ldots, 0, 1, \ldots, 1)$ at a general point, and then showed that such bundles are just the uniform ones:

1. $\mathcal{O}_\Lambda(1)^{\oplus c} \oplus \mathcal{O}_\Lambda^{(s-c)}$;
2. $\Omega_\Lambda(2)$;
3. $T_\Lambda(-1)$.

Then he showed that

1. $X$ is a projective bundle over a smooth variety;
$X$ covered by linear spaces $\Rightarrow$ the normal bundle of a general space is generically spanned.

Sato proved that such a bundle is uniform of type $(0, \ldots, 0, 1, \ldots, 1)$ at a general point, and then showed that such bundles are just the uniform ones:

1. $\mathcal{O}_\Lambda(1)^{+c} \oplus \mathcal{O}_\Lambda^{+(s-c)}$;
2. $\Omega_\Lambda(2)$;
3. $T_\Lambda(-1)$.

Then he showed that

1. $X$ is a projective bundle over a smooth variety;
2. $X$ is a smooth hyperquadric;
$X$ covered by linear spaces $\Rightarrow$ the normal bundle of a general space is generically spanned.

Sato proved that such a bundle is uniform of type $(0, \ldots, 0, 1, \ldots, 1)$ at a general point, and then showed that such bundles are just the uniform ones:

1. $\mathcal{O}_\Lambda(1)^{\oplus c} \oplus \mathcal{O}_\Lambda^{(s-c)}$;
2. $\Omega_\Lambda(2)$;
3. $T_\Lambda(-1)$.

Then he showed that

1. $X$ is a projective bundle over a smooth variety;
2. $X$ is a smooth hyperquadric;
3. if $s$ is even, then $X$ is $\mathbb{G}(1, s+1)$.
Sato’s proof in case 3) is based on the study of the Hilbert scheme of $s$-planes and works if ALL the normal bundles are $T_{\Lambda}(-1)$. He proves that this is the case if $s$ is even.
Sato’s proof in case 3) is based on the study of the Hilbert scheme of $s$-planes and works if ALL the normal bundles are $T_{\Lambda}(-1)$. He proves that this is the case if $s$ is even.

To complete the classification in case $n = 2s$ we need to prove that...
Sato’s proof in case 3) is based on the study of the Hilbert scheme of $s$-planes and works if ALL the normal bundles are $T_\Lambda(-1)$. He proves that this is the case if $s$ is even.

To complete the classification in case $n = 2s$ we need to prove that

\[ \Lambda^s \subset X^n \subset \mathbb{P}^N \text{ linear subspace with } n = 2s \text{ and } \mathcal{N} = T_\Lambda(-1). \]

Then $X$ is the Grassmannian of lines $\mathbb{G}(1, s+1)$. 
Sato’s proof in case 3) is based on the study of the Hilbert scheme of $s$-planes and works if ALL the normal bundles are $T_\Lambda(-1)$. He proves that this is the case if $s$ is even.

To complete the classification in case $n = 2s$ we need to prove that

$$\Lambda^s \subset X^n \subset \mathbb{P}^N$$

linear subspace with $n = 2s$ and $\mathcal{N} = T_\Lambda(-1)$. Then $X$ is the Grassmannian of lines $\mathbb{G}(1, s+1)$.

**IDEA**

Study $\tilde{X}$, the blow-up of $X$ along $\Lambda$. 
Why the blow-up?

Motivation

- Nice structure of the exceptional divisor;
Why the blow-up?

Motivation

- Nice structure of the exceptional divisor;
- the blow-up is a resolution of the projection from $\Lambda$;
Why the blow-up?

Motivation

- Nice structure of the exceptional divisor;
- the blow-up is a resolution of the projection from $\Lambda$;
- $\tilde{X}$ is a Fano manifold of Picard number 2;
**Why the blow-up?**

**Motivation**

- Nice structure of the exceptional divisor;
- the blow-up is a resolution of the projection from $\Lambda$;
- $\tilde{X}$ is a Fano manifold of Picard number 2;
- it is possible to study $\tilde{X}$ by studying the “second contraction”.
Blowing-up Grassmannians

$\Lambda^s$ linear subspace of $\mathbb{G}(1, s+1)$ parametrizes the star of lines through a point $P$ and its normal bundle is $T_{\Lambda}(-1)$. We study the blow-up of $\mathbb{G}(1, s+1)$ along $\Lambda$. 
**Blowing-up Grassmannians**

\( \Lambda^s \) linear subspace of \( \mathbb{G}(1, s+1) \) parametrizes the star of lines through a point \( P \) and its normal bundle is \( T_{\Lambda}(-1) \).

We study the blow-up of \( \mathbb{G}(1, s+1) \) along \( \Lambda \).

\( \mathcal{H} \) hyperplane not containing \( P \),

\( \pi_P : \mathbb{P}^{s+1} \longrightarrow \mathcal{H} \) projection.
Blowing-up Grassmannians

\[ \Lambda^s \text{ linear subspace of } \mathbb{G}(1, s + 1) \text{ parametrizes the star of lines through a point } P \text{ and its normal bundle is } T_\Lambda(-1). \]
We study the blow-up of \( \mathbb{G}(1, s + 1) \) along \( \Lambda \).

\( \mathcal{H} \) hyperplane not containing \( P \),
\[ \pi_P : \mathbb{P}^{s+1} \longrightarrow \mathcal{H} \text{ projection.} \]

\( \mathbb{G}(1, s + 1) \) Grassmannian of lines in \( \mathbb{P}^{s+1} \),
\[ \mathbb{G}(1, \mathcal{H}) \text{ Grassmannian of lines in } \mathcal{H}, \]
\[ \pi_\mathbb{G} : \mathbb{G}(1, s + 1) \longrightarrow \mathbb{G}(1, \mathcal{H}) \text{ induced projection.} \]
BLOWING-UP GRASSMANNIANS

$\Lambda^s$ linear subspace of $G(1, s+1)$ parametrizes the star of lines through a point $P$ and its normal bundle is $T_{\Lambda}(-1)$. We study the blow-up of $G(1, s+1)$ along $\Lambda$.

$H$ hyperplane not containing $P$,
$\pi_P : \mathbb{P}^{s+1} \rightarrow H$ projection.

$G(1, s+1)$ Grassmannian of lines in $\mathbb{P}^{s+1}$, $G(1, H)$ Grassmannian of lines in $H$,
$\pi_G : G(1, s+1) \rightarrow G(1, H)$ induced projection.

$\pi_G$ is not defined on the lines passing through $P$, i.e., along $\Lambda$. 
**BLOWING-UP GRASSMANNIANS**

$\Lambda^s$ linear subspace of $G(1,s+1)$ parametrizes the star of lines through a point $P$ and its normal bundle is $T_\Lambda(-1)$.

We study the blow-up of $G(1,s+1)$ along $\Lambda$.

$\mathcal{H}$ hyperplane not containing $P$,

$\pi_P : \mathbb{P}^{s+1} \rightarrow \mathcal{H}$ projection.

$G(1,s+1)$ Grassmannian of lines in $\mathbb{P}^{s+1}$,

$G(1,\mathcal{H})$ Grassmannian of lines in $\mathcal{H}$,

$\pi_G : G(1,s+1) \rightarrow G(1,\mathcal{H})$ induced projection.

$\pi_G$ is not defined on the lines passing through $P$, i.e., along $\Lambda$. 

**Gianluca Occhetta**

**Manifolds containing a linear subspace**
Take the resolution of $\pi_G$:

\[ \tilde{G}(1, s+1) \xrightarrow{\tilde{\pi}_G} G(1, H) \]

\[ \sigma \]

\[ \tilde{\pi}_G \]

\[ G(1, s+1) \]
Take the resolution of $\pi_G$:

\[
\begin{align*}
\widetilde{G}(1, s+1) \xrightarrow{\tilde{\pi}_G} G(1, \mathcal{H}) \\
\sigma \downarrow \quad \downarrow \\
G(1, s+1)
\end{align*}
\]

It can be shown that $\widetilde{G}(1, s+1) = P_G(1, \mathcal{H}) \oplus \mathcal{O}(1)$, where $Q$ is the universal quotient bundle over $G(1, H)$. 

Gianluca Occhetta
Manifolds containing a linear subspace
Take the resolution of $\pi_G$:

$$
\tilde{\pi}_G(1, s + 1) \xrightarrow{\tilde{\pi}_G} \mathbb{G}(1, \mathcal{H})
$$

$\sigma$  

$$
\pi_G \\
\Rightarrow \\
\Rightarrow
\mathbb{G}(1, s + 1)
$$

$\tilde{\pi}_G$ is a $\mathbb{P}^2$-bundle whose fibers are the strict transforms of the 2-planes parametrizing lines contained in a 2-plane containing $P$. 
Take the resolution of $\pi_G$:

$$
\tilde{G}(1, s + 1) \xrightarrow{\tilde{\pi}_G} G(1, \mathcal{H})
$$

$\sigma$ $\downarrow$ $\pi_G$

$G(1, s + 1)$

$\tilde{\pi}_G$ is a $\mathbb{P}^2$-bundle whose fibers are the strict transforms of the 2-planes parametrizing lines contained in a 2-plane containing $P$.

It can be shown that $\tilde{G}(1, s + 1) = \mathbb{P}_{G(1, \mathcal{H})}(\mathcal{D} \oplus \mathcal{O}(1))$, where $\mathcal{D}$ is the universal quotient bundle over $G(1, \mathcal{H})$. 
**Theorem**

$\Lambda^s \subset X^n \subset \mathbb{P}^N$ linear subspace with $n = 2s$ and $\mathcal{N} = T_\Lambda(-1)$. Then $X$ is the Grassmannian of lines $\mathbb{G}(1, s+1)$.
**THEOREM**

$\Lambda^s \subset X^n \subset \mathbb{P}^N$ linear subspace with $n = 2s$ and $\mathcal{N} = T_{\Lambda}(-1)$. Then $X$ is the Grassmannian of lines $G(1, s+1)$.

**Picard number**

$X$ is covered by linear spaces of dimension $s$. 

*Gianluca Occhetta*

**Manifolds containing a linear subspace**
**Characterizing $\mathcal{G}(1, s+1)$**

**Theorem**

$\Lambda^s \subset X^n \subset \mathbb{P}^N$ linear subspace with $n = 2s$ and $\mathcal{N} = T_{\Lambda}(-1)$. Then $X$ is the Grassmannian of lines $\mathcal{G}(1, s+1)$.

**Picard number**

$X$ is covered by linear spaces of dimension $s$. If $\rho_X \geq 2$, there is an elementary contraction.
Characterizing $\mathbb{G}(1, s+1)$

**Theorem**

$\Lambda^s \subset X^n \subset \mathbb{P}^N$ linear subspace with $n = 2s$ and $\mathcal{N} = T_{\Lambda}(-1)$. Then $X$ is the Grassmannian of lines $\mathbb{G}(1, s+1)$.

**Picard number**

$X$ is covered by linear spaces of dimension $s$.
If $\rho_X \geq 2$, there is an elementary contraction. The general fiber is a projective space by Zak’s theorem.
CHARACTERIZING $G(1, s + 1)$

**Theorem**

$\Lambda^s \subset X^n \subset \mathbb{P}^N$ linear subspace with $n = 2s$ and $\mathcal{N} = T_\Lambda(-1)$. Then $X$ is the Grassmannian of lines $G(1, s + 1)$.

**Picard number**

$X$ is covered by linear spaces of dimension $s$.

If $\rho_X \geq 2$, there is an elementary contraction.

The general fiber is a projective space by Zak’s theorem.

$X$ is a projective bundle by Ein-Wiśniewski.
Characterizing $G(1, s + 1)$

**Theorem**

$\Lambda^s \subset X^n \subset \mathbb{P}^N$ linear subspace with $n = 2s$ and $\mathcal{N} = T_{\Lambda}(-1)$. Then $X$ is the Grassmannian of lines $G(1, s + 1)$.

**Picard number**

$X$ is covered by linear spaces of dimension $s$.
If $\rho_X \geq 2$, there is an elementary contraction.
The general fiber is a projective space by Zak’s theorem.
$X$ is a projective bundle by Ein-Wiśniewski.
Contradiction (no linear spaces with normal $T(-1)$).
THEOREM OF SATO

ONE STEP FURTHER

EXCEPTIONAL DIVISOR

\[ \text{Exceptional divisor} \]

**Exceptional Divisor**

**Grassmannian of lines in**

\[ \text{Grassmannian of lines in} \]

**Incidence Variety**

\[ \text{Incidence variety} \]

**Maps**

\[ p \]

\[ \varphi \]

\[ \text{Maps} p \]

\[ \text{Maps} \varphi \]

\[ \text{Maps} p \]

\[ \text{Maps} \varphi \]

\[ \text{Maps} p \]

\[ \text{Maps} \varphi \]

\[ \text{Maps} p \]

\[ \text{Maps} \varphi \]

**Universal Quotient Bundle**

\[ \text{Universal Quotient Bundle} \]

**Manifolds Containing a Linear Subspace**

**Manifolds Containing a Linear Subspace**
Exceptional divisor

\[ E = \mathbb{P}_\Lambda(N^*) = \mathbb{P}^s_\mathbb{P}^s(\Omega_{\mathbb{P}^s}(1)). \]

\( \mathbb{G}(1,s) \) Grassmannian of lines in \( \mathbb{P}^s \), \( \mathcal{I} \) incidence variety.
**Exceptional Divisor**

\[ E = \mathbb{P}_\Lambda(\mathcal{N}^*) = \mathbb{P}_{\mathbb{P}^s}(\Omega_{\mathbb{P}^s}(1)). \]

\( \mathbb{G}(1, s) \) Grassmannian of lines in \( \mathbb{P}^s \), \( \mathcal{I} \) incidence variety.

\[ \mathbb{P}^s \quad \mathcal{I} \quad \mathbb{G}(1, s). \]

The maps \( p \) and \( \varphi \) are projective bundles:
**EXCEPTIONAL DIVISOR**

\[ E = \mathbb{P}_\Lambda(\mathcal{N}^*) = \mathbb{P}_{\mathbb{P}^s}(\Omega_{\mathbb{P}^s}(1)). \]

\[ \mathcal{G}(1, s) \text{ Grassmannian of lines in } \mathbb{P}^s, \mathcal{I} \text{ incidence variety.} \]

![Diagram](attachment:image.png)

The maps \( p \) and \( \varphi \) are projective bundles:

- \( \mathcal{I} = \mathbb{P}_{\mathbb{P}^s}(\Omega_{\mathbb{P}^s}(1)) \);
**Exceptional divisor**

$$E = \mathbb{P}_\Lambda (\mathcal{N}^*) = \mathbb{P}_{\mathbb{P}^s} (\Omega_{\mathbb{P}^s}(1)).$$

$G(1,s)$ Grassmannian of lines in $\mathbb{P}^s$, $\mathcal{I}$ incidence variety.

The maps $p$ and $\varphi$ are projective bundles:

- $\mathcal{I} = \mathbb{P}_{\mathbb{P}^s} (\Omega_{\mathbb{P}^s}(1));$
- $\mathcal{I} = \mathbb{P}_{G(1,s)} (\mathcal{Q}).$

$\mathcal{Q}$ universal quotient bundle on $G(1,s)$. 
A theorem of Sato

One step further

Cone of curves

Gianluca Occhetta
Manifolds containing a linear subspace
CONE OF CURVES

\( \pi : X \rightarrow Y \) projection from \( \Lambda \)

\( \sigma : \tilde{X} \rightarrow X \) blow-up of \( X \) along \( \Lambda \)

\( \tilde{\pi} : \tilde{X} \rightarrow Y \) resolution of \( \pi \).
**Cone of curves**

\( \pi : X \to Y \) projection from \( \Lambda \)
\( \sigma : \tilde{X} \to X \) blow-up of \( X \) along \( \Lambda \)
\( \tilde{\pi} : \tilde{X} \to Y \) resolution of \( \pi \).

\( l \subset X \) line meeting \( \Lambda \) but not in it.
\( \tilde{\pi} \) contracts \( l \), the strict transform of \( l \).
### Cone of curves

\[ \pi : X \rightarrow Y \] projection from \( \Lambda \)

\[ \sigma : \tilde{X} \rightarrow X \] blow-up of \( X \) along \( \Lambda \)

\[ \tilde{\pi} : \tilde{X} \rightarrow Y \] resolution of \( \pi \).

\( l \subset X \) line meeting \( \Lambda \) but not in it.

\( \tilde{\pi} \) contracts \( \ell \), the strict transform of \( l \).

\[ \text{NE}(\tilde{X}) = \langle [\Gamma_\sigma], [\ell] \rangle \]

and the supporting divisors of the rays are \( H = \sigma^* \mathcal{O}(1) \) and \( H - E \).
STUDYING THE OTHER CONTRACTION
STUDYING THE OTHER CONTRACTION

By the description of the cone it is easy to check ampleness using Kleiman.
By the description of the cone it is easy to check ampleness using Kleiman.

By adjunction and the canonical bundle formula for blow-ups

\[-K_{\tilde{X}} = (s + 2)H - (s - 1)E\]

is positive on $\Gamma_{\sigma}$ and $\ell$, therefore $\tilde{X}$ is a Fano manifold.
By the description of the cone it is easy to check ampleness using Kleiman.

By adjunction and the canonical bundle formula for blow-ups

\[-K_{\tilde{X}} = (s + 2)H - (s - 1)E\]

is positive on $\Gamma_{\sigma}$ and $\ell$, therefore $\tilde{X}$ is a Fano manifold.

By Kleiman also $-K_{\tilde{X}} + H - 2E$ is ample.
STUDYING THE OTHER CONTRACTION

By the description of the cone it is easy to check ampleness using Kleiman.

By adjunction and the canonical bundle formula for blow-ups

\[-K_{\tilde{X}} = (s + 2)H - (s - 1)E\]

is positive on \(\Gamma_\sigma\) and \(\ell\), therefore \(\tilde{X}\) is a Fano manifold.

By Kleiman also \(-K_{\tilde{X}} + H - 2E\) is ample and so

\[H^0(\tilde{X}, H - E) \twoheadrightarrow H^0(E, (H - E)|_E)\]
STUDYING THE OTHER CONTRACTION

By the description of the cone it is easy to check ampleness using Kleiman.

By adjunction and the canonical bundle formula for blow-ups

\[-K_{\tilde{X}} = (s + 2)H - (s - 1)E\]

is positive on $\Gamma_\sigma$ and $\ell$, therefore $\tilde{X}$ is a Fano manifold.

By Kleiman also $-K_{\tilde{X}} + H - 2E$ is ample and so

\[H^0(\tilde{X}, H - E) \rightarrow H^0(E, (H - E)|_E)\]

Therefore the restriction of $\tilde{\pi}$ to $E$ is the $\mathbb{P}^1$-bundle $\varphi : E \rightarrow \mathbb{G}(1, s)$ coming from the incidence diagram.
As $\tilde{\pi}|_E$ is equidimensional with one-dimensional fibers, $\tilde{\pi}$ cannot have fibers of dimension greater than two.
As \( \tilde{\pi}|_E \) is equidimensional with one-dimensional fibers, \( \tilde{\pi} \) cannot have fibers of dimension greater than two.

\[
I(\mathbb{R}_+[\ell]) = -K_{\tilde{X}} \cdot \ell = 3,
\]

hence by Ionescu-Wiśniewski inequality \( \tilde{\pi} \) is of fiber type and every fiber of \( \tilde{\pi} \) has dimension 2.
As \( \tilde{\pi}|_E \) is equidimensional with one-dimensional fibers, \( \tilde{\pi} \) cannot have fibers of dimension greater than two.

\[
l(\mathbb{R}_+[\ell]) = -K_{\tilde{X}} \cdot \ell = 3,
\]

hence by Ionescu-Wiśniewski inequality \( \tilde{\pi} \) is of fiber type and every fiber of \( \tilde{\pi} \) has dimension 2.

It follows that \( \tilde{\pi}(\tilde{X}) = \tilde{\pi}(E) = G(1, s) \).
As $\tilde{\pi}|_E$ is equidimensional with one-dimensional fibers, $\tilde{\pi}$ cannot have fibers of dimension greater than two.

$$l(\mathbb{R}^+\lceil \ell \rceil) = -K_\tilde{X} \cdot \ell = 3,$$

hence by Ionescu-Wiśniewski inequality $\tilde{\pi}$ is of fiber type and every fiber of $\tilde{\pi}$ has dimension 2.

It follows that $\tilde{\pi}(\tilde{X}) = \tilde{\pi}(E) = \mathbb{G}(1, s)$.

A general fiber of $\tilde{\pi}$ is by adjunction a projective space of dimension two.
As $\tilde{\pi}|_E$ is equidimensional with one-dimensional fibers, $\tilde{\pi}$ cannot have fibers of dimension greater than two.

$$l(\mathbb{R}_+[\ell]) = -K_{\tilde{\mathcal{X}}} \cdot \ell = 3,$$

hence by Ionescu-Wiśniewski inequality $\tilde{\pi}$ is of fiber type and every fiber of $\tilde{\pi}$ has dimension 2.

It follows that $\tilde{\pi}(\tilde{\mathcal{X}}) = \tilde{\pi}(E) = \mathcal{G}(1, s)$.

A general fiber of $\tilde{\pi}$ is by adjunction a projective space of dimension two.

Moreover $2H - E$ is ample and $(2H - E)|_F \simeq \mathcal{O}_{\mathbb{P}^2}(1)$; therefore $\tilde{\pi}$ is a $\mathbb{P}^2$-bundle over $\mathcal{G}(1, s)$ by a result of Fujita.
**CONCLUSION**

\[ E := ϕ^∗H; \text{the inclusion} E = \mathcal{P}_{\mathcal{G}}(1,s)(Q) \hookrightarrow \tilde{X} = \mathcal{P}_{\mathcal{G}}(1,s)(E) \] gives an exact sequence

\[ 0 \to L \to E \to Q \to 0. \]

By the canonical bundle formula for projective bundles

\[ \det E = \mathcal{O}_{\mathcal{G}}(1,s)(2); \]

Recalling that \( \det Q = \mathcal{O}_{\mathcal{G}}(1,s)(1) \) we have

\[ L = \mathcal{O}_{\mathcal{G}}(1,s)(1). \]

Since \( h^1(Q^∗(1)) = h^1(Q) = 0 \), the sequence splits and

\[ \tilde{X} = \mathcal{P}_{\mathcal{G}}(1,s)(Q \oplus \mathcal{O}_{\mathcal{G}}(1,s)(1)) \]

**Gianluca Occhetta**

**Manifolds containing a linear subspace**
Conclusion

\( E := \varphi_* H \); the inclusion \( E = \mathbb{P}_{G(1,s)}(\mathcal{D}) \hookrightarrow \tilde{X} = \mathbb{P}_{G(1,s)}(E) \) gives an exact sequence

\[
0 \longrightarrow L \longrightarrow E \longrightarrow \mathcal{D} \longrightarrow 0.
\]
CONCLUSION

\[ \mathcal{E} := \varphi_* H; \]  
the inclusion \( E = \mathbb{P}_{G(1,s)}(\mathcal{Q}) \hookrightarrow \tilde{X} = \mathbb{P}_{G(1,s)}(\mathcal{E}) \) gives an exact sequence

\[ 0 \rightarrow L \rightarrow \mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0. \]

By the canonical bundle formula for projective bundles

\[ \det \mathcal{E} = \mathcal{O}_{G(1,s)}(2); \]

Recalling that \( \det \mathcal{Q} = \mathcal{O}_{G(1,s)}(1) \) we have \( L = \mathcal{O}_{G(1,s)}(1). \)
\( \mathcal{E} := \varphi_* H; \) the inclusion \( E = \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{D}) \hookrightarrow \tilde{X} = \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{E}) \) gives an exact sequence

\[
0 \longrightarrow L \longrightarrow \mathcal{E} \longrightarrow \mathcal{D} \longrightarrow 0.
\]

By the canonical bundle formula for projective bundles

\[
\det \mathcal{E} = \mathcal{O}_{\mathbb{G}(1,s)}(2);
\]

Recalling that \( \det \mathcal{D} = \mathcal{O}_{\mathbb{G}(1,s)}(1) \) we have \( L = \mathcal{O}_{\mathbb{G}(1,s)}(1) \).

Since \( h^1(\mathcal{D}^*(1)) = h^1(\mathcal{D}) = 0 \), the sequence splits and

\[
\tilde{X} = \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{D} \oplus \mathcal{O}_{\mathbb{G}(1,s)}(1)).
\]
**Main Theorem**

**Theorem**

Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s + 1$, containing a linear subspace $\Lambda$ of dimension $s$, such that its normal bundle $N_{\Lambda/\mathbb{P}^N}$ is numerically effective. If the Picard number of $X$ is one, then $X$ is one of the following:

- a linear space $\mathbb{P}^{2s + 1}$;
- a smooth hyperquadric $Q^{2s + 1}$;
- a cubic threefold in $\mathbb{P}^4$;
- the intersection of $G(1, 4) \subset \mathbb{P}^9$ with three general hyperplanes;
- a complete intersection of two hyperquadrics in $\mathbb{P}^5$;
- a hyperplane section of the Grassmannian of lines $G(1, s + 2)$ in its Plücker embedding.
**Main theorem**

**Theorem**

Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s + 1$, containing a linear subspace $\Lambda$ of dimension $s$, such that its normal bundle $N_{\Lambda/X}$ is numerically effective. If the Picard number of $X$ is one, then $X$ is one of the following:

- a linear space $\mathbb{P}^{2s+1}$;
MAIN THEOREM

THEOREM
Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s + 1$, containing a linear subspace $\Lambda$ of dimension $s$, such that its normal bundle $N_{\Lambda/X}$ is numerically effective. If the Picard number of $X$ is one, then $X$ is one of the following:

- a linear space $\mathbb{P}^{2s+1}$;
- a smooth hyperquadric $\mathbb{Q}^{2s+1}$;
MAIN THEOREM

THEOREM

Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s + 1$, containing a linear subspace $\Lambda$ of dimension $s$, such that its normal bundle $N_{\Lambda/X}$ is numerically effective. If the Picard number of $X$ is one, then $X$ is one of the following:

- a linear space $\mathbb{P}^{2s+1}$;
- a smooth hyperquadric $\mathbb{Q}^{2s+1}$;
- a cubic threefold in $\mathbb{P}^4$;
**Main theorem**

**Theorem**

Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s+1$, containing a linear subspace $\Lambda$ of dimension $s$, such that its normal bundle $N_{\Lambda/X}$ is numerically effective. If the Picard number of $X$ is one, then $X$ is one of the following:

- a linear space $\mathbb{P}^{2s+1}$;
- a smooth hyperquadric $\mathbb{Q}^{2s+1}$;
- a cubic threefold in $\mathbb{P}^4$;
- the intersection of $G(1,4) \subset \mathbb{P}^9$ with three general hyperplanes;
MAIN THEOREM

THEOREM

Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s + 1$, containing a linear subspace $\Lambda$ of dimension $s$, such that its normal bundle $N_{\Lambda/X}$ is numerically effective. If the Picard number of $X$ is one, then $X$ is one of the following:

- a linear space $\mathbb{P}^{2s+1}$;
- a smooth hyperquadric $Q^{2s+1}$;
- a cubic threefold in $\mathbb{P}^4$;
- the intersection of $G(1, 4) \subset \mathbb{P}^9$ with three general hyperplanes;
- a complete intersection of two hyperquadrics in $\mathbb{P}^5$. 
Main theorem

**Theorem**

Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s + 1$, containing a linear subspace $\Lambda$ of dimension $s$, such that its normal bundle $N_{\Lambda/X}$ is numerically effective. If the Picard number of $X$ is one, then $X$ is one of the following:

- a linear space $\mathbb{P}^{2s+1}$;
- a smooth hyperquadric $\mathbb{Q}^{2s+1}$;
- a cubic threefold in $\mathbb{P}^4$;
- the intersection of $G(1,4) \subset \mathbb{P}^9$ with three general hyperplanes;
- a complete intersection of two hyperquadrics in $\mathbb{P}^5$;
- a hyperplane section of the Grassmannian of lines $G(1,s+2)$ in its Plücker embedding.
**Main theorem**

**Theorem**

If the Picard number of $X$ is greater than one, then there is an elementary contraction $\varphi: X \to Y$ which contracts $\Lambda$ and one of the following occurs:

- $\varphi: X \to Y$ is a scroll;
- $Y$ is a curve, and the general fiber of $\varphi$ is the Grassmannian of lines $G(1,s)$;
- a smooth hyperquadric $Q^2_s$;
- a product of projective spaces $P^s_1 \times P^s_2$. 
Main theorem

Theorem

If the Picard number of $X$ is greater than one, then there is an elementary contraction $\varphi : X \to Y$ which contracts $\Lambda$ and one of the following occurs:

- $\varphi : X \to Y$ is a scroll;
MAIN THEOREM

THEOREM

If the Picard number of $X$ is greater than one, then there is an elementary contraction $\varphi: X \to Y$ which contracts $\Lambda$ and one of the following occurs:

- $\varphi: X \to Y$ is a scroll;
- $Y$ is a curve, and the general fiber of $\varphi$ is
**Main theorem**

If the Picard number of $X$ is greater than one, then there is an elementary contraction $\varphi: X \to Y$ which contracts $\Lambda$ and one of the following occurs:

- $\varphi: X \to Y$ is a scroll;
- $Y$ is a curve, and the general fiber of $\varphi$ is
  - the Grassmannian of lines $G(1,s)$;
Main theorem

If the Picard number of $X$ is greater than one, then there is an elementary contraction $\varphi: X \to Y$ which contracts $\Lambda$ and one of the following occurs:

- $\varphi: X \to Y$ is a scroll;
- $Y$ is a curve, and the general fiber of $\varphi$ is
  - the Grassmannian of lines $G(1, s)$;
  - a smooth hyperquadric $Q^{2s}$.
Main theorem

Theorem

If the Picard number of $X$ is greater than one, then there is an elementary contraction $\varphi : X \to Y$ which contracts $\Lambda$ and one of the following occurs:

- $\varphi : X \to Y$ is a scroll;
- $Y$ is a curve, and the general fiber of $\varphi$ is
  - the Grassmannian of lines $\mathbb{G}(1, s)$;
  - a smooth hyperquadric $\mathbb{Q}^{2s}$;
  - a product of projective spaces $\mathbb{P}^s \times \mathbb{P}^s$. 
NORMAL BUNDLES

\( \mathcal{N} \) is uniform of type \((0, \ldots, 0, 1, \ldots, 1)\), again by the sequence

\[
0 \to \mathcal{N} \to N_{\Lambda/P^N} = \mathcal{O}_\Lambda(1)^{\oplus(N-s)} \to (N_{X/P^N}|_X) \to 0
\]
NORMAL BUNDLES

$\mathcal{N}$ is uniform of type $(0, \ldots, 0, 1, \ldots, 1)$, again by the sequence

$$0 \to \mathcal{N} \to N_{\Lambda/P^N} = \mathcal{O}_{\Lambda}(1)^{\oplus (N-s)} \to (N_{X/P^N}|_X \to 0$$

By the classification (Ellia, Ballico)
NORMAL BUNDLES

\( \mathcal{N} \) is uniform of type \((0, \ldots, 0, 1, \ldots, 1)\), again by the sequence

\[
0 \to \mathcal{N} \to N_{\Lambda/\mathbb{P}^N} = \mathcal{O}_\Lambda(1)^{\oplus(N-s)} \to (N_{\mathbb{P}^N})|_X \to 0
\]

By the classification (Ellia, Ballico)

1. \( \Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda(1) \quad c = s \);
**NORMAL BUNDLES**

\( \mathcal{N} \) is uniform of type \((0, \ldots, 0, 1, \ldots, 1)\), again by the sequence

\[
0 \to \mathcal{N} \to N_{\Lambda/\mathbb{P}^N} = \mathcal{O}_\Lambda(1) \oplus (N-s) \to (N_{\mathbb{X}/\mathbb{P}^N}|_{\mathbb{X}}) \to 0
\]

By the classification (Ellia, Ballico)

1. \( \Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda(1) \quad c = s \);
2. \( \Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda \quad c = s - 1 \);
NORMAL BUNDLES

\( \mathcal{N} \) is uniform of type \((0, \ldots, 0, 1, \ldots, 1)\), again by the sequence

\[
0 \to \mathcal{N} \to N_{\Lambda/\mathbb{P}^N} = \mathcal{O}_\Lambda(1)^{\oplus (N-s)} \to (N_{X/\mathbb{P}^N}|_X) \to 0
\]

By the classification (Ellia, Ballico)

1. \( \Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda}(1) \quad c = s \);
2. \( \Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda} \quad c = s - 1 \);
3. \( T_{\Lambda}(-1) \oplus \mathcal{O}_{\Lambda}(1) \quad c = 2 \);
NORMAL BUNDLES

\( \mathcal{N} \) is uniform of type \((0, \ldots, 0, 1, \ldots, 1)\), again by the sequence

\[
0 \rightarrow \mathcal{N} \rightarrow N_{\Lambda/\mathbb{P}^N} = \mathcal{O}_{\Lambda}(1)^{\oplus(N-s)} \rightarrow (N_{X/\mathbb{P}^N})|_{X} \rightarrow 0
\]

By the classification (Ellia, Ballico)

1. \( \Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda}(1) \quad c = s; \)
2. \( \Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda} \quad c = s - 1; \)
3. \( T_{\Lambda}(-1) \oplus \mathcal{O}_{\Lambda}(1) \quad c = 2; \)
4. \( T_{\Lambda}(-1) \oplus \mathcal{O}_{\Lambda} \quad c = 1; \)
NORMAL BUNDLES

$\mathcal{N}$ is uniform of type $(0,\ldots,0,1,\ldots,1)$, again by the sequence

$$0 \to \mathcal{N} \to N_{\Lambda/\mathbb{P}^N} = \mathcal{O}_{\Lambda}(1)^{\oplus(N-s)} \to (N_{X/\mathbb{P}^N}|_X \to 0$$

By the classification (Ellia, Ballico)

1. $\Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda}(1) \quad c = s$
2. $\Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda} \quad c = s - 1$
3. $T_{\Lambda}(-1) \oplus \mathcal{O}_{\Lambda}(1) \quad c = 2$
4. $T_{\Lambda}(-1) \oplus \mathcal{O}_{\Lambda} \quad c = 1$
5. $\mathcal{O}_{\Lambda}^{\oplus c}(1) \oplus \mathcal{O}_{\Lambda}^{\oplus(s+1-c)}$. 

As before one proves that $X$ is covered by linear $s$-spaces.
NORMAL BUNDLES

\( \mathcal{N} \) is uniform of type \((0, \ldots, 0, 1, \ldots, 1)\), again by the sequence

\[
0 \to \mathcal{N} \to N_{\Lambda/\mathbb{P}^N} = \mathcal{O}_\Lambda(1)^{\oplus(N-s)} \to (N_{X/\mathbb{P}^N})|_X \to 0
\]

By the classification (Ellia, Ballico)

1. \( \Omega_{\Lambda}(2) \oplus \mathcal{O}_\Lambda(1) \quad c = s \);
2. \( \Omega_{\Lambda}(2) \oplus \mathcal{O}_\Lambda \quad c = s - 1 \);
3. \( T_{\Lambda}(-1) \oplus \mathcal{O}_\Lambda(1) \quad c = 2 \);
4. \( T_{\Lambda}(-1) \oplus \mathcal{O}_\Lambda \quad c = 1 \);
5. \( \mathcal{O}_{\Lambda}^{\oplus c}(1) \oplus \mathcal{O}_{\Lambda}^{\oplus(s+1-c)} \).

As before one proves that \( X \) is covered by linear \( s \)-spaces.
Picard number greater than one

If $\rho_X \geq 2$ there is an elementary contraction which contracts $\Lambda$. 
Picard number greater than one

If $\rho_X \geq 2$ there is an elementary contraction which contracts $\Lambda$.

By adjunction the general fiber $F$ of $\varphi$ is a Fano manifold of index

$$s + c + 1 \geq \frac{\dim F + 2}{2}.$$
PICARD NUMBER GREATER THAN ONE

If $\rho_X \geq 2$ there is an elementary contraction which contracts $\Lambda$.

By adjunction the general fiber $F$ of $\varphi$ is a Fano manifold of index

$$s + c + 1 \geq \frac{\dim F + 2}{2}.$$ 

By a theorem of Wiśniewski either $F \cong \mathbb{P}^s \times \mathbb{P}^s$ or $\rho_F = 1$. 

GIANLUCCA OCCHETTA
MANIFOLDS CONTAINING A LINEAR SUBSPACE
If $\rho_X \geq 2$ there is an elementary contraction which contracts $\Lambda$.

By adjunction the general fiber $F$ of $\varphi$ is a Fano manifold of index

$$s + c + 1 \geq \frac{\dim F + 2}{2}.$$

By a theorem of Wiśniewski either $F \cong \mathbb{P}^s \times \mathbb{P}^s$ or $\rho_F = 1$.

If $\rho_F = 1$, then $\dim F = 2s$ and $F$ is a projective space, a hyperquadric or a Grassmannian of lines by the previous theorems.
**Picard number one**

$X$ is a Fano manifold of Picard number one and index $s + 1 + c$

- $\Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda(1) \quad c = s \Rightarrow X$ is a hyperquadric.
**Picard Number One**

*$X$ is a Fano manifold of Picard number one and index $s + 1 + c$*

- $\Omega^\wedge(2) \oplus \mathcal{O}^\wedge(1)$  \hspace{1cm} $c = s \Rightarrow X$ is a hyperquadric.
- $\Omega^\wedge(2) \oplus \mathcal{O}^\wedge$  \hspace{1cm} $c = s - 1 \Rightarrow X$ is a del Pezzo manifold
**PICARD NUMBER ONE**

$X$ is a Fano manifold of Picard number one and index $s + 1 + c$

- $\Omega^2 \oplus O^1 \quad c = s \Rightarrow X$ is a hyperquadric.
- $\Omega^2 \oplus O \quad c = s - 1 \Rightarrow X$ is a del Pezzo manifold

**Lemma (Ein - Hironaka)**

$\Lambda \subset X \subset \mathbb{P}^N$ linear space contained in a smooth projective manifold s.t. $N_{\Lambda/X} \cong N' \oplus O(1)$. Then there exists a smooth hyperplane section $X'$ of $X$ which contains $\Lambda$ and such that $N_{\Lambda/X'} \cong N'$. 

**Gianluca Occhetta**

**Manifolds containing a linear subspace**
**Picard Number One**

$X$ is a Fano manifold of Picard number one and index $s + 1 + c$

- $\Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda}(1)$ \quad $c = s \Rightarrow X$ is a hyperquadric.
- $\Omega_{\Lambda}(2) \oplus \mathcal{O}_{\Lambda}$ \quad $c = s - 1 \Rightarrow X$ is a del Pezzo manifold

**Lemma (Ein-Hironaka)**

$\Lambda \subset X \subset \mathbb{P}^N$ linear space contained in a smooth projective manifold s.t. $N_{\Lambda/X} \simeq N' \oplus \mathcal{O}(1)$. Then there exists a smooth hyperplane section $X'$ of $X$ which contains $\Lambda$ and such that $N_{\Lambda/X'} \simeq N'$.

- $T_{\Lambda}(-1) \oplus \mathcal{O}_{\Lambda}(1)$
**Picard Number One**

$X$ is a Fano manifold of Picard number one and index $s + 1 + c$

- $\Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda(1)$ \quad $c = s \Rightarrow X$ is a hyperquadric.
- $\Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda$ \quad $c = s - 1 \Rightarrow X$ is a del Pezzo manifold

**Lemma (Ein-Hironaka)**

$\Lambda \subset X \subset \mathbb{P}^N$ linear space contained in a smooth projective manifold s.t. $N_{\Lambda/X} \cong N' \oplus \mathcal{O}(1)$. Then there exists a smooth hyperplane section $X'$ of $X$ which contains $\Lambda$ and such that $N_{\Lambda/X'} \cong N'$.

- $T_\Lambda(-1) \oplus \mathcal{O}_\Lambda(1) \Rightarrow X'$ is $\mathbb{G}(1, s + 1)$, but this is impossible
- $\mathcal{O}_\Lambda(1) \oplus c \oplus \mathcal{O}_\Lambda^{(s + 1 - c)}$
**PICARD NUMBER ONE**

$X$ is a Fano manifold of Picard number one and index $s + 1 + c$

- $\Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda(1)$  \quad  $c = s \Rightarrow X$ is a hyperquadric.
- $\Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda$  \quad  $c = s - 1 \Rightarrow X$ is a del Pezzo manifold

**Lemma (Ein-Hironaka)**

$\Lambda \subset X \subset \mathbb{P}^N$ linear space contained in a smooth projective manifold s.t. $N_\Lambda/X \cong N' \oplus \mathcal{O}(1)$. Then there exists a smooth hyperplane section $X'$ of $X$ which contains $\Lambda$ and such that $N_\Lambda/X' \cong N'$.

- $T_\Lambda(-1) \oplus \mathcal{O}_\Lambda(1) \Rightarrow X'$ is $\mathbb{G}(1, s + 1)$, but this is impossible
- $\mathcal{O}_\Lambda(1)^{\oplus c} \oplus \mathcal{O}_\Lambda^{\oplus(s+1-c)} \Rightarrow \mathcal{N}$ is trivial or $X'$ (and so $X$) is a linear space
Picard number one - Trivial normal bundle

(For $n \geq 4$). Use the blow-up construction.
Picard number one - Trivial normal bundle

(For $n \geq 4$). Use the blow-up construction. As before

$\pi : X \rightarrow Y$ projection from $\Lambda$

$\sigma : \tilde{X} \rightarrow X$ blow-up of $X$ along $\Lambda$

$\tilde{\pi} : \tilde{X} \rightarrow Y$ resolution of $\pi$. 

PICARD NUMBER ONE - TRIVIAL NORMAL BUNDLE

(For \( n \geq 4 \)). Use the blow-up construction. As before

\[
\pi : X \to Y \quad \text{projection from } \Lambda
\]

\[
\sigma : \tilde{X} \to X \quad \text{blow-up of } X \text{ along } \Lambda
\]

\[
\tilde{\pi} : \tilde{X} \to Y \quad \text{resolution of } \pi.
\]

\( l \subset X \) line meeting \( \Lambda \) but not in it.

\( \tilde{\pi} \) contracts \( \ell \), the strict transform of \( l \).
Picard number one - Trivial normal bundle

(For $n \geq 4$). Use the blow-up construction. As before

$\pi : X \dashrightarrow Y$ projection from $\Lambda$

$\sigma : \tilde{X} \to X$ blow-up of $X$ along $\Lambda$

$\tilde{\pi} : \tilde{X} \to Y$ resolution of $\pi$.

$l \subset X$ line meeting $\Lambda$ but not in it.

$\tilde{\pi}$ contracts $\ell$, the strict transform of $l$.

$$\text{NE}(\tilde{X}) = \langle [\Gamma_\sigma], [\ell] \rangle$$

and the supporting divisors of the rays are $H = \sigma^* \mathcal{O}(1)$ and $H - E$. 
Picard number one - Trivial normal bundle

(For $n \geq 4$). Use the blow-up construction. As before

$\pi : X \rightarrow Y$ projection from $\Lambda$

$\sigma : \tilde{X} \rightarrow X$ blow-up of $X$ along $\Lambda$

$\tilde{\pi} : \tilde{X} \rightarrow Y$ resolution of $\pi$.

$l \subset X$ line meeting $\Lambda$ but not in it.

$\tilde{\pi}$ contracts $\ell$, the strict transform of $l$.

$\text{NE}(\tilde{X}) = \langle [\Gamma_{\sigma}], [\ell] \rangle$

and the supporting divisors of the rays are $H = \sigma^* \mathcal{O}(1)$ and $H - E$.

Big difference: $(H - E)_E$ is ample!
It follows that $\tilde{\pi}$ has one dimensional non trivial fibers, and $\mathcal{Y}$ is smooth.
It follows that $\tilde{\pi}$ has one dimensional non trivial fibers, and $Y$ is smooth.

- $\tilde{\pi}$ of fiber type $\Rightarrow$ $Y$ dominated by $\mathbb{P}^s \times \mathbb{P}^s$. 

GIANLUCA OCCHETTA
MANIFOLDS CONTAINING A LINEAR SUBSPACE
It follows that $\tilde{\pi}$ has one dimensional non trivial fibers, and $Y$ is smooth.

- $\tilde{\pi}$ of fiber type $\Rightarrow$ $Y$ dominated by $\mathbb{P}^s \times \mathbb{P}^s$.
- $\tilde{\pi}$ birational $\Rightarrow$ $\mathbb{P}^s \times \mathbb{P}^s$ divisor in $Y$. 

But $\rho_Y = 1$, hence $Y$ is a projective space.

Since $n \geq 4$ we have a contradiction.
It follows that $\tilde{\pi}$ has one dimensional non trivial fibers, and $Y$ is smooth.

- $\tilde{\pi}$ of fiber type $\Rightarrow Y$ dominated by $\mathbb{P}^s \times \mathbb{P}^s$.
- $\tilde{\pi}$ birational $\Rightarrow \mathbb{P}^s \times \mathbb{P}^s$ divisor in $Y$.

But $\rho_Y = 1$, hence
It follows that $\tilde{\pi}$ has one dimensional non trivial fibers, and $Y$ is smooth.

- $\tilde{\pi}$ of fiber type $\Rightarrow$ $Y$ dominated by $\mathbb{P}^s \times \mathbb{P}^s$.
- $\tilde{\pi}$ birational $\Rightarrow$ $\mathbb{P}^s \times \mathbb{P}^s$ divisor in $Y$.

But $\rho_Y = 1$, hence

- $Y$ is a projective space.
It follows that $\tilde{\pi}$ has one dimensional non trivial fibers, and $Y$ is smooth.

- $\tilde{\pi}$ of fiber type $\Rightarrow$ $Y$ dominated by $\mathbb{P}^s \times \mathbb{P}^s$.
- $\tilde{\pi}$ birational $\Rightarrow$ $\mathbb{P}^s \times \mathbb{P}^s$ divisor in $Y$.

But $\rho_Y = 1$, hence

- $Y$ is a projective space.
- $\mathbb{P}^s \times \mathbb{P}^s$ ample in $Y$. 
It follows that $\tilde{\pi}$ has one dimensional non trivial fibers, and $Y$ is smooth.

- $\tilde{\pi}$ of fiber type $\Rightarrow$ $Y$ dominated by $\mathbb{P}^s \times \mathbb{P}^s$.
- $\tilde{\pi}$ birational $\Rightarrow$ $\mathbb{P}^s \times \mathbb{P}^s$ divisor in $Y$.

But $\rho_Y = 1$, hence

- $Y$ is a projective space.
- $\mathbb{P}^s \times \mathbb{P}^s$ ample in $Y$.

Since $n \geq 4$ we have a contradiction.
Last case - Sketch(es) of proof

The remaining case

\[ T_{\Lambda}(-1) \oplus \mathcal{O}_{\Lambda}. \]

requires a double blow-up construction.
**Last case - Sketch(es) of proof**

The remaining case

- $T_\Lambda(-1) \oplus \mathcal{O}_\Lambda$.

requires a double blow-up construction.

\[
\pi : X \to Y \quad \text{projection from } \Lambda \\
\sigma : \tilde{X} \to X \quad \text{blow-up of } X \text{ along } \Lambda \\
\tilde{\pi} : \tilde{X} \to Y \quad \text{resolution of } \pi.
\]
A theorem of Sato

One step further

GIANLUCA OCCHETTA

MANIFOLDS CONTAINING A LINEAR SUBSPACE
Exceptional Divisor

E

$\Lambda_0$  $\Lambda_1$

$E = P_{\Lambda}(\Omega(1) \oplus O)$

G(1,s)

Y = G(1,s+1)

Gianluca Occhetta

Manifolds containing a linear subspace
**Exceptional Divisor**

$E$ exceptional divisor of the blow-up.

$E = \mathbb{P}(\Omega(1) \oplus \mathcal{O})$.

$E$ is the blow-up of $G(1,s)$ along a subgrassmannian $G(1,s)$. $Y = G(1,s+1)$.
**Introduction**

A theorem of Sato

**One step further**

**Exceptional Divisor**

\[ E = \mathbb{P}_\Lambda(\Omega(1) \oplus \mathcal{O}) \]

\[ E \text{ exceptional divisor of the blow-up.} \]

\[ E = \mathbb{P}_\Lambda(\Omega(1) \oplus \mathcal{O}) \]

\[ E \text{ is the blow-up of } G(1,s+1) \text{ along a subgrassmannian } G(1,s). \]

\[ \Lambda \]

\[ \Lambda_0 \]

\[ \Lambda_1 \]

\[ G(1,s) \]

\[ Y = G(1,s+1) \]
An exceptional divisor of the blow-up.

\[ E = \mathbb{P}_\Lambda (\Omega(1) \oplus \mathcal{O}). \]

\( E \) is the blow-up of \( \mathbb{G}(1, s + 1) \) along a subgrassmannian \( \mathbb{G}(1, s) \).
OTHER CONTRACTION
OTHER CONTRACTION

\[ Y = G(1, s+1) \]

\[ \Lambda \]

\[ \tilde{\pi} \]

\[ \tilde{\pi} \text{ is a scroll over } G(1, s+1). \]

Not enough information to describe completely \( \tilde{\pi} \).

GIANLUCA OCCHETTA

MANIFOLDS CONTAINING A LINEAR SUBSPACE
\( \tilde{\pi} \) is a scroll over \( G(1, s+1) \).
Not enough information to describe completely \( \tilde{X} \).
**Blow it up again**

We find a subvariety $\Sigma \subset X$, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^s$, containing $\Lambda$. The blow-up of $X$ along $\Sigma$ is isomorphic to the blow-up of a hyperplane section of $G(1,s+2)$ along the subvariety parametrizing lines in $\mathbb{P}^{s+2}$ meeting a given line.
BLOW IT UP AGAIN

We find a subvariety $\Sigma \subset X$, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^s$, containing $\Lambda$. 
We find a subvariety \( \Sigma \subset X \), isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^s \), containing \( \Lambda \). The blow-up of \( X \) along \( \Sigma \) is isomorphic to the blow-up of a hyperplane section of \( G(1,s+2) \) along the subvariety parametrizing lines in \( \mathbb{P}^{s+2} \) meeting a given line.
**Example**

- $C'$ smooth curve with a free $\mathbb{Z}_2$-action.
**Example**

- $C'$ smooth curve with a free $\mathbb{Z}_2$-action.
- $\pi: C' \to C$ induced étale covering.
**Example**

- $C'$ smooth curve with a free $\mathbb{Z}_2$-action.
- $\pi: C' \to C$ induced étale covering.
- $G = \mathbb{P}^s \times \mathbb{P}^s$, with $\mathbb{Z}_2$-action exchanging the factors.
**Example**

- $C'$ smooth curve with a free $\mathbb{Z}_2$-action.
- $\pi: C' \to C$ induced étale covering.
- $G = \mathbb{P}^s \times \mathbb{P}^s$, with $\mathbb{Z}_2$-action exchanging the factors
- $X' = G \times C'$ and $X$ quotient by the product action of $\mathbb{Z}_2$. 
**Example**

- $C'$ smooth curve with a free $\mathbb{Z}_2$-action.
- $\pi: C' \to C$ induced étale covering.
- $G = \mathbb{P}^s \times \mathbb{P}^s$, with $\mathbb{Z}_2$-action exchanging the factors.
- $X' = G \times C'$ and $X$ quotient by the product action of $\mathbb{Z}_2$.

\[
\begin{array}{ccc}
X' & \longrightarrow & C' \\
\downarrow \pi' & & \downarrow \pi \\
X & \longrightarrow & C \\
\downarrow \varphi & & \downarrow \\
\end{array}
\]

$\varphi$ : $X \to C$ is an extremal contraction and every fiber is a product of projective spaces $\mathbb{P}^s \times \mathbb{P}^s$.
**Example**

- $C'$ smooth curve with a free $\mathbb{Z}_2$-action.
- $\pi: C' \to C$ induced étale covering.
- $G = \mathbb{P}^s \times \mathbb{P}^s$, with $\mathbb{Z}_2$-action exchanging the factors.
- $X' = G \times C'$ and $X$ quotient by the product action of $\mathbb{Z}_2$.

\[
\begin{array}{ccc}
X' & \longrightarrow & C' \\
\pi' \downarrow & & \pi \downarrow \\
X & \xrightarrow{\varphi} & C
\end{array}
\]

$\varphi: X \to C$ is an extremal contraction and every fiber is a product of projective spaces $\mathbb{P}^s \times \mathbb{P}^s$. 
\[ \text{line in } G; \]
\begin{itemize}
  \item $l$ line in $G$;
  \item $l \times C' \subset G \times C' = X'$;
\end{itemize}
I line in $G$;
$I \times C' \subset G \times C' = X$;
c point of $C$, $\{c'_1, c'_2\} = \pi^{-1}(c)$;
- $l$ line in $G$;
- $l \times C' \subset G \times C' = X'$;
- $c$ point of $C$, $\{c'_1, c'_2\} = \pi^{-1}(c)$;
- $l'_i = l \times \{c'_i\}$.
\begin{itemize}
\item $l$ line in $G$;
\item $l \times C' \subset G \times C' = X'$;
\item $c$ point of $C$, $\{c_1', c_2'\} = \pi^{-1}(c)$;
\item $l_i' = l \times \{c_i'\}$.
\end{itemize}

\[
\begin{array}{ccc}
G \times \{c_1', c_2'\} & \xrightarrow{\pi'} & \{c_1', c_2'\} \\
\downarrow \phi^{-1}(c) & \sim & \mathbb{P}^s \times \mathbb{P}^s \\
\downarrow & \phi & \downarrow \\
\{c_1', c_2'\} & \xrightarrow{\phi} & C
\end{array}
\]
A line in $G$;

A line $l \times C' \subset G \times C' = X'$;

A point $c$ of $C$, $\{c'_1, c'_2\} = \pi^{-1}(c)$;

A line $l'_i = l \times \{c'_i\}$.

$$G \times \{c'_1, c'_2\} \to \{c'_1, c'_2\}$$

$\phi^{-1}(c) \simeq \mathbb{P}^s \times \mathbb{P}^s \to c$

$l_1 = \pi'(l'_1)$ line in a fiber of the projection onto the first factor
A line $l$ in $G$;

$l \times C' \subset G \times C' = X'$;

$c$ point of $C$, $\{c'_1, c'_2\} = \pi^{-1}(c)$;

$l'_i = l \times \{c'_i\}$.

$$
\begin{array}{ccc}
G \times \{c'_1, c'_2\} & \xrightarrow{\pi'} & \{c'_1, c'_2\} \\
\downarrow \phi^{-1}(c) \simeq \mathbb{P}^s \times \mathbb{P}^s & & \downarrow \phi \\
& \quad & \quad \\
& \quad & \quad \\
& \quad & \quad \\
& \quad & \quad \\
& \quad & \quad \\
\end{array}
$$

$l_1 = \pi'(l'_1)$ line in a fiber of the projection onto the first factor

$l_2 = \pi'(l'_2)$ line in a fiber of the projection of onto the second factor.