Vector bundles on Fano manifolds

Roberto Muñoz, Gianluca Occhetta and Luis Solá Conde
(work in progress)

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2. Find splitting criteria for rank two vector bundles on some Fano manifolds in terms of their "Fanitude".

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The first Chern class is identified with an integer c_1.

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X	$VMRT_x$
$\mathbb{P}^n$	$\mathbb{P}^{n-1}$
$\mathbb{Q}^n$	$\mathbb{Q}^{n-2}$
$\mathbb{G}(k,n)$	$\mathbb{P}^k \times \mathbb{P}^{n-k-1}$
$QG(m-1,2m-1)$	$\mathbb{G}(1,m-1)$

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We say that E is uniform with respect to $\mathscr{M}$ if the splitting type of E is the same on any curve parametrized by $\mathscr{M}$.

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we have

$$-2a + r_X - c_1 = r_X$$

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We thus get a nonzero morphism

$$\mathscr{O}_X(a) \longrightarrow E,$$

contradicting the semistability of E. $\square$

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Theorem

Let $r \leq \dim \mathscr{M}_x$ be a positive integer. If for any $x \in X$ the s-th Chow group $\mathrm{Ch}^s(\mathscr{M}_x)$ has rank one $\forall s \leq [r/2]$, then $r \leq u(X, \mathscr{M})$, i.e. every uniform vector bundle of rank r splits as sums of line bundles.

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- We get maps $\mathscr{M}_x \to \mathbb{G}(k-1, \mathbb{P}(E_x))$, $\forall x \in X$. If they are constant, we construct a uniform quotient E_0 of E of rank k.

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2. The equality of Chern classes implies $c_1(\phi^* \mathscr{S}^\vee) = 0$, i.e ϕ is constant. $\square$

Corollary

Let $X \subset \mathbb{P}^N$ be a Fano manifold of Picard number one covered by a family $\mathscr{L}$ of linear subspaces of dimension $d \geq 2$,

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Then E is a direct sum of line bundles.

Uniform vector bundles over Hermitian symmetric spaces

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$\mathbb{G}(k,n)$	$\mathbb{P}^k \times \mathbb{P}^{n-k-1}$	$\min\{k+1, n-k\}$
$\mathbb{Q}^{2k}$	$\mathbb{Q}^{2k-2}$	$\geq 2k-3$
$\mathbb{Q}^{2k-1}$	$\mathbb{Q}^{2k-3}$	$\geq 2k-3$
$QG(m-1,2m-1)$	$\mathbb{G}(1,m-1)$	$m-1$
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Remark

Notice that, since $\pi^* H$ is nef, $\mathbb{P}(E)$ is a Fano manifold if and only if $t(E) < 0$. In this case E is called a Fano bundle.

Fano threshold: examples

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The nefness of $E\left(\dfrac{-c_1 + r_X + t}{2}\right)$ easily implies that $d \leq r_X + t$.

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   In particular $t(F_{HM}) = 2$ and $d(F_{HM}) = 7$.

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The main idea we will use is that

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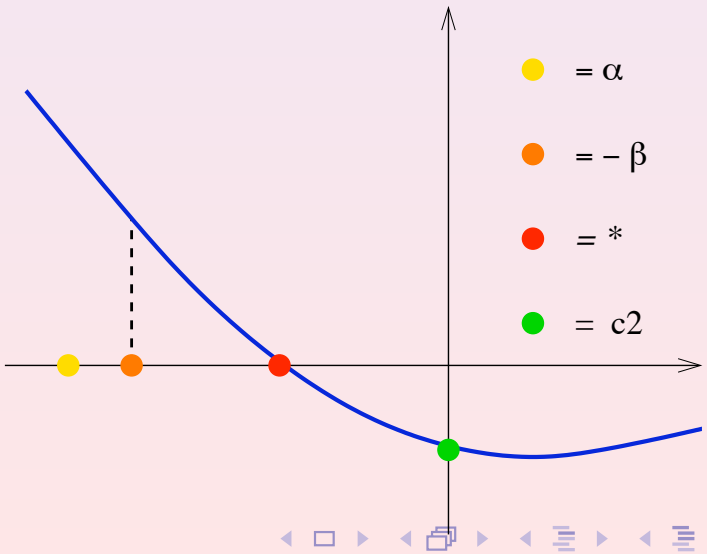

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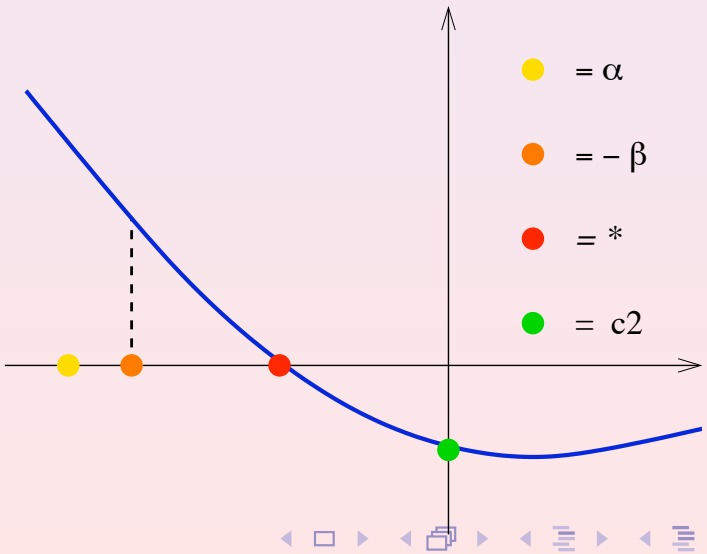

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In particular, if $-\frac{c_1+\sqrt{\Delta}}{2}$ is an integer and $\varepsilon < 1$ the bundle splits.

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Numerical splitting

A vector bundle E on X is called numerically split if it has the same Chern classes as a direct sum of line bundles.

In particular a rank two vector bundle E is called numerically split of type $(a,b) \in \mathbb{Z}^2$ with $a \leq b$ iff $c_1 = a+b$, $c_2 = ab$.

Notice that $\Delta = (a+b)^2 - 4ab = (a-b)^2$ and $\frac{-c_1-\sqrt{\Delta}}{2} = -b$.

Moreover, if $t \leq n-1$ then $\varepsilon < 1$, hence

Proposition

A numerically split bundle E such that $t(E) \leq n-1$ splits.

Rational curves and numerical splitting

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Lemma (A-P-W)

If, for some $y \in \mathbb{P}(E)$, $(\widetilde{\mathscr{M}^a})_y$ contains a complete curve, then E is numerically split.

How to get numerical splitting

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If $h^0(X, E(-c_1 - 1)) > 0$ and $t < r_X - 2$ then E splits numerically.

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Since $\#\{\mathscr{M}_x^a\} \leq a^{max} + 1$ and $2a^{max} \leq d(E) \leq r_X + t$ we get the bound.

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$$-\Delta \leq (r_X + t)^2 \tan^2\left(\frac{\pi}{n+1}\right).$$

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Concluding remarks

Under which conditions...

1. is t rational?
2. is there a rational curve $C \in \mathbb{P}(E)$ s. t. $K_{\mathbb{P}(E)} \cdot C = tH \cdot C$?
3. is $-K_{\mathbb{P}(E)} + tH$ semiample?

Given $P_1, \ldots, P_k \in \mathbb{P}^2$ there exists E whose splitting type on a rational curve C:

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Concluding remarks

Assume that $t < 0$

1. is t an integer?
2. is there a minimal section C_a over a line such that $K_{\mathbb{P}(E)} \cdot C_a = tH \cdot C_a$?