Varieties of Minimal Rational Tangents and Congruences of Lines

A classical counterexample to a modern conjecture

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joint work with
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Outline

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V is **dominating** if $\overline{i(U)} = X$;

V is **minimal** if $\widetilde{V}_x := \pi(i^{-1}(x))$ is proper for a general x in $\mathrm{Locus}(V)$.

For a general $x \in X$ the normalization V_x of $\widetilde{V}_x$ is smooth.

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We define the **Variety of Minimal Rational Tangents** to be

$$\mathcal{C}_x = \overline{\tau_x(V_x)}$$

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The subvariety

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- *Thus* $\tau_x : V_x \to \mathcal{C}_x$ *is the normalization.*

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A prime Fano manifold is a Fano manifold such that $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle H_X\rangle$.
The index r_X of X is the largest integer such that

$$-K_X \sim r_X H_X$$

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$$f^* T_X \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus n-d-1}.$$

where $d = -K_X \cdot C$.

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3. *The VMRT cannot be an irreducible linear subspace.*

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$\mathbb{G}^{II}(k, 2m-1)$	$\mathbb{P}^k \times \mathbb{Q}^{2m-2k-3}$
$\mathbb{G}^{II}(m-1, 2m-1)$	$\mathbb{G}(1, m-1)$
$\mathbb{G}^{III}(k, 2m-1)$	$\mathbb{P}_{\mathbb{P}^m}(\mathcal{O}(2) \oplus \mathcal{O}(1)^{2m-2k-2})$
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The VRMT at every point of X is $\mathbb{P}^{s-1} \sqcup \mathbb{P}^{s-1}$.

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Theorem (Hwang-Mok 2004)

Let X and X' be prime Fano manifolds. Assume that at a general point $x \in X$ the VMRT $\mathcal{C}_x$ is non-linear and of positive dimension.

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Non linearity

Conjecture (Hwang-Mok 2004)

Let X be a prime Fano manifold.
Then the VMRT of X at a general point is not a union of positive dimensional linear subspaces.

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As a side effect, a counterexample to the non linearity conjecture has been found.

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2. Intersection theory combined with number theory provides $n = 2, 3$ or 5 if φ is of fiber type.

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  (D1) $\mathbb{P}^2 \subset \mathbb{G}^{II}(1,3)_{\mathbb{Q}^3} \subset \mathbb{G}(1,4)$ *(lines in $\mathbb{Q}^3$ meeting a fixed line),*
  (D2) $v_2(\mathbb{P}^2) \subset \mathbb{G}(1,3)$ *((bi)secant lines to $v_3(\mathbb{P}^1) \subset \mathbb{P}^3$),*
  (D3) $V_5^3 \subset \mathbb{G}(1,4)$ *(trisecant lines to the projection of $v_2(\mathbb{P}^2)$ into $\mathbb{P}^4$),*
  (C6) $\dots$

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- *ψ factorizes by a finite covering $\psi_1 : X \to X_1$ of one of the submanifolds of types* (P1)-(P5) *above and, either*

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- **fundamental locus**: set of the fundamental points.

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we get $\deg(Z) < \alpha^{m-z} \Longrightarrow Z$ cannot be a complete intersection.

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where σ and σ' denote, respectively, the blowing up of $\mathbb{P}^{3k+2}$ along S and the blowing up of $\mathbb{P}^{3k+2}$ along $S' \cong S$.

Example
Trisecants to projected Severi

Example

Let $X \subset \mathbb{G}(1, 3k+1)$ be the closure of the family of trisecant lines to a general isomorphic projection $Z \subset \mathbb{P}^{3k+1}$ of a $2k$-dimensional Severi variety $S \subset \mathbb{P}^{3k+2}$. Then X is a smooth congruence of order one with linear fibers.

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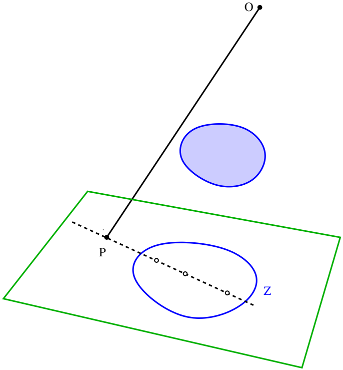

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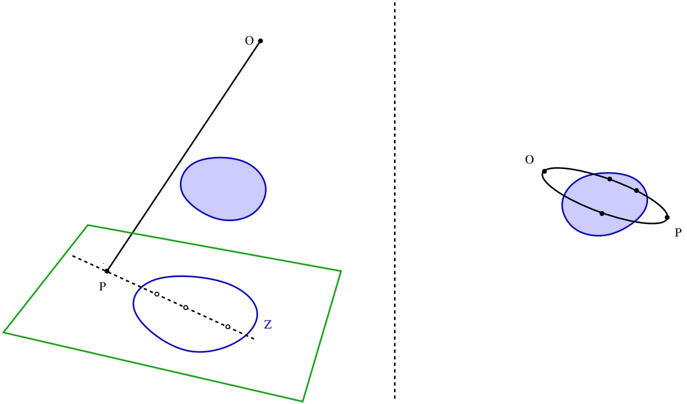

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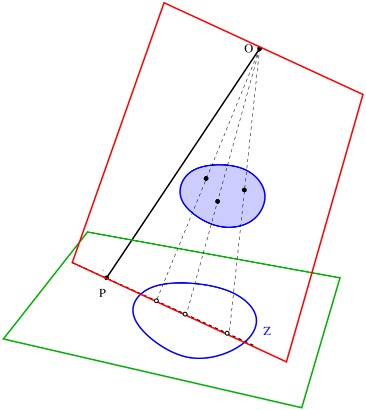

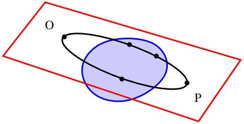

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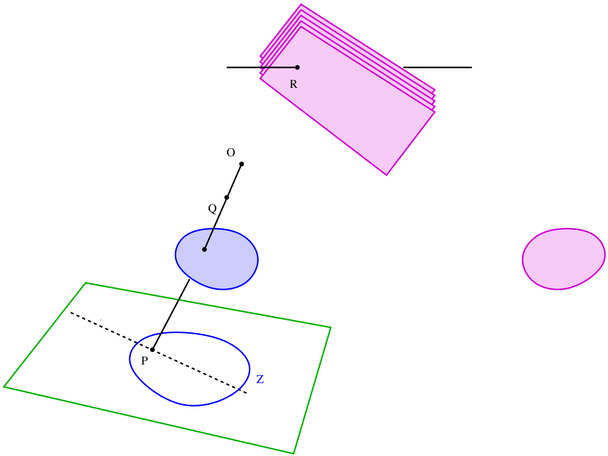

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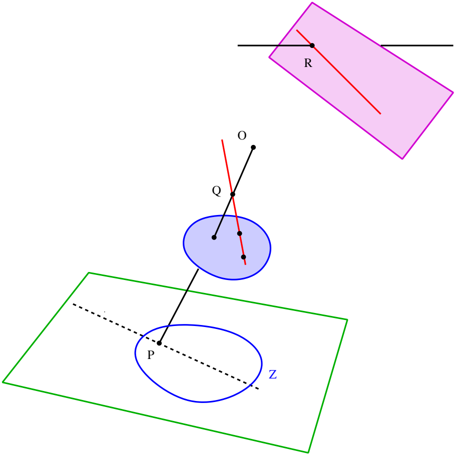

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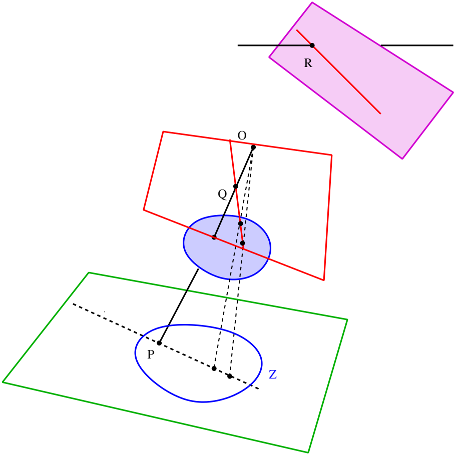

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$X \subset \mathbb{G}(1, 3k+1)$ is a congruence of order one, and $Y = \mathbb{P}_X(\mathcal{Q}|_X)$ is the blow-up of $\mathbb{P}^{3k+1}$ along Z.

Lines in X are the images of the fibers of $\pi' : Y \to \mathbb{P}^{3k+1}$.

Lines through a general $x \in X$ are the lines contained in three linear spaces passing through x and meeting only in x.

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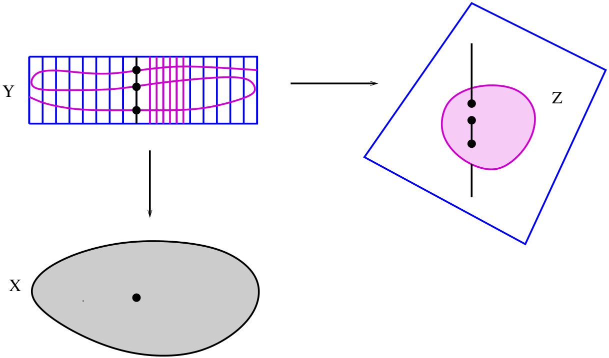

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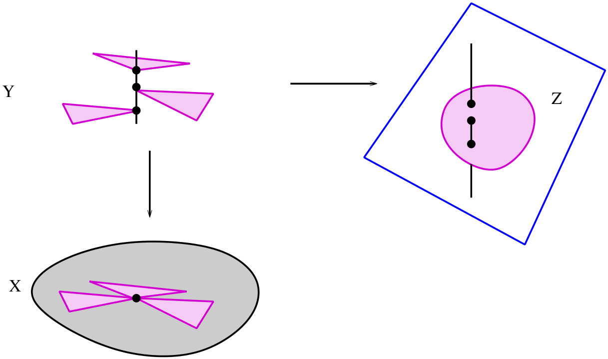

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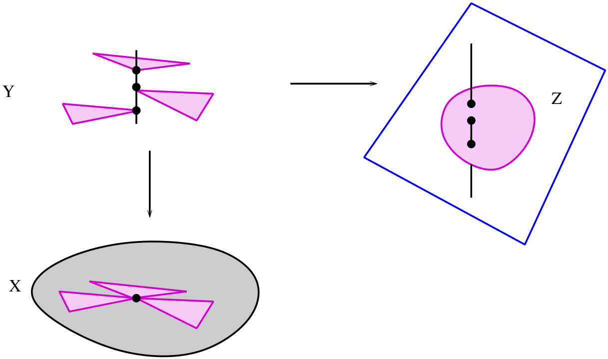

The VMRT at x is the union of three disjoint linear spaces!

THANK YOU!