The quaternionic Gauss-Lucas Theorem and some related results about quaternionic polynomials

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FTHD 2018 - Tampere, August 20-24, 2018

Outline

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Gauss-Lucas Theorem does not hold on $\mathbb{H}$

Let $p \in \mathbb{C}[z]$ be a complex polynomial of degree $d \geq 2$ and let p' be its derivative. The Gauss-Lucas Theorem asserts:

$$V(p') \subset \mathcal{K}(p)$$

where $\mathcal{K}(P)$ denotes the convex hull of the sero set $V(P)$ in $\mathbb{C}$

Gauss-Lucas Theorem does not hold for quaternionic polynomials (with right coefficients)

$$P(X) = \sum_{k=0}^{d} X^k a_k \in \mathbb{H}[X]$$

For example, the quadratic polynomial

$$P_2(X) = X^2 - X(i+j) + k$$

has zero set $V(P_2) = \{i\}$, while $V(P_2') = V(2X - i - j) = \{\frac{i+j}{2}\}$

Gauss-Lucas Theorem does not hold on $\mathbb{H}$

Given $P, Q \in \mathbb{H}[X]$, let $P \cdot Q$ denote the product obtained by imposing commutativity of X with the coefficients and set

$$P^c(X) = \sum_{k=0}^{d} X^k \bar{a}_k \quad \text{and} \quad N(P) = P \cdot P^c = P^c \cdot P \in \mathbb{R}[X]$$

($N(P)$ is the normal polynomial of P)

$$\text{If } \mathbb{S}_x = \{pxp^{-1} \in \mathbb{H} \mid p \in \mathbb{H}^*\} \quad \Rightarrow \quad V(N(P)) = \bigcup_{x \in V(P)} \mathbb{S}_x$$

Remark

$P_2(X) = (X - i) \cdot (X - j) \;\Rightarrow\; P_2^c(X) = (X + j) \cdot (X + i)$. Therefore
$V(P_2) = \{i\}$, $V(P_2^c) = \{-j\} \;\Rightarrow\; V(P_2') = \{\frac{i+j}{2}\} \subset \mathcal{K}(N(P_2))$

Since $N(P_2) = (X^2 + 1)^2$, $V(N(P_2)) = \mathbb{S}$ (the sphere of imaginary units) and $\mathcal{K}(N(P_2)) = \{x \in \mathbb{H} \mid \operatorname{Re}(x) = 0, \ |x| \leq 1\}$

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A conjectured Gauss-Lucas Theorem on $\mathbb{H}$

A natural reformulation in $\mathbb{H}[X]$ of the classic Gauss-Lucas Theorem is then the following:

$$V(P') \subset \mathcal{K}(N(P))$$

where $\mathcal{K}(N(P))$ denotes the convex hull of $V(N(P))$ in $\mathbb{H}$.
If $V(P') \subset \mathcal{K}(N(P))$ we say that P is a Gauss-Lucas polynomial

The inclusion holds when

- $d = 2$
- $P(X) = \sum_{k=0}^{d} X^k a_k$ is a complex polynomial, i.e. every $a_k \in \mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ for a fixed $I \in \mathbb{S}$ (P is a $\mathbb{C}_I$-polynomial)

Remark

Classic Gauss-Lucas Theorem holds for slice-preserving polynomials but not for $\mathbb{C}_I$-polynomials, i.e. one-slice-preserving polynomials

Gauss-Lucas Theorem for quadratic polynomials

Proposition

If P is a polynomial in $\mathbb{H}[X]$ of degree 2, then $V(N(P)) = \mathbb{S}_{x_1} \cup \mathbb{S}_{x_2}$ for some $x_1, x_2 \in \mathbb{H}$ (possibly with $\mathbb{S}_{x_1} = \mathbb{S}_{x_2}$) and

$$V(P') \subset \bigcup_{y_1 \in \mathbb{S}_{x_1},\, y_2 \in \mathbb{S}_{x_2}} \left\{ \frac{y_1 + y_2}{2} \right\} \subset \mathcal{K}(N(P))$$

Sketch of proof.

Assume P monic. Then $P(X) = (X - x_1) \cdot (X - x_2)$ with $x_1 \in V(P)$ and $\bar{x}_2 \in V(P^c)$. □

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The conjecture is false in its generality

$$P_3(X) = (X - i) \cdot (X - j) \cdot (X - k)$$

is not a Gauss-Lucas polynomial

Sketch of proof.

$$N(P_3) = (X^2 + 1)^3 \;\Rightarrow\; \mathcal{K}(N(P_3)) \subset \mathrm{Im}(\mathbb{H})$$

$$P_3'(X) = 3X^2 - 2X(i + j + k) + (i - j + k)$$

$$N(P_3') = 9X^4 + 12X^2 - 4X + 3 \;\Rightarrow\; V(N(P_3')) \cap \mathrm{Im}(\mathbb{H}) \subset \{0\}$$

but $V(N(P_3')) \not\subset \{0\}$ and hence $V(P_3') \not\subset \mathrm{Im}(\mathbb{H})$ □

$$P_d(X) = X^{d-3} \cdot (X - i) \cdot (X - j) \cdot (X - k)$$

is not a Gauss-Lucas polynomial for every $d \geq 3$

The Gauss-Lucas snail of a polynomial

We use the slice decomposition of $\mathbb{H}$ and projections on slices

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I$$

Definition

Let $I \in \mathbb{S}$ and let $P_I : \mathbb{C}_I \to \mathbb{H}$ be the restriction of P to $\mathbb{C}_I$.
If P_I is not constant, set $\mathcal{K}_{\mathbb{C}_I}(P) := \mathcal{K}(V(P) \cap \mathbb{C}_I)$.
If P_I is constant, we set $\mathcal{K}_{\mathbb{C}_I}(P) := \mathbb{C}_I$

Let $\pi_I : \mathbb{H} \to \mathbb{H}$ be the orthogonal projection onto $\mathbb{C}_I$.
Given $P(X) = \sum_{k=0}^{d} X^k a_k \in \mathbb{H}[X]$, let $P_+^I(X) = \pi_I \circ P = \sum_{k=1}^{d} X^k a_{k,I}$ be the $\mathbb{C}_I$-polynomial with coefficients $a_{k,I} := \pi_I(a_k)$.

Definition (Gauss-Lucas snail of P)

$$\mathfrak{sn}(P) := \bigcup_{I \in \mathbb{S}} \mathcal{K}_{\mathbb{C}_I}(P_+^I)$$

A quaternionic Gauss-Lucas Theorem

Theorem (Gauss-Lucas on $\mathbb{H}$)

For every polynomial $P \in \mathbb{H}[X]$ of degree ≥ 2,

$$V(P') \subset \mathfrak{sn}(P)$$

Sketch of proof.

Decompose P and P' as

$$P_I = \pi_I \circ P_I + \pi_I^\perp \circ P_I = P^I_{+|\mathbb{C}_I} + \pi_I^\perp \circ P_I$$

$P^I_+(\mathbb{C}_I) \subset \mathbb{C}_I$ and $(\pi_I^\perp \circ P_I)(\mathbb{C}_I) \subset \mathbb{C}_I^\perp \;\Rightarrow\; V(P') \cap \mathbb{C}_I \subset V((P^I_+)') \cap \mathbb{C}_I$
The classic Gauss-Lucas Theorem applied to P^I on $\mathbb{C}_I$ gives
$V(P') \cap \mathbb{C}_I \subset \mathcal{K}_{\mathbb{C}_I}(P^I_+)$ □

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Properties of the Gauss-Lucas snail $\mathfrak{sn}(P)$

It is not restrictive to consider *monic* polynomials.
If P is *monic* the theorem has the following equivalent formulation:

For every monic polynomial $P \in \mathbb{H}[X]$ of degree ≥ 2, it holds

$$\mathfrak{sn}(P') \subset \mathfrak{sn}(P)$$

Proposition

For every monic polynomial $P \in \mathbb{H}[X]$ of degree $d \geq 2$, the Gauss-Lucas snail $\mathfrak{sn}(P)$ is a compact subset of $\mathbb{H}$

Remark

$\mathfrak{sn}(P)$ can be strictly smaller than $\mathcal{K}(N(P))$. For example, consider the $\mathbb{C}_i$-polynomial $P(X) = X^3 + 3X + 2i$, with zero sets $V(P) = \{-i, 2i\}$ and $V(P') = \mathbb{S}$. The set $\mathcal{K}(N(P))$ is the closed three-dimensional disc in $\mathrm{Im}(\mathbb{H})$, with center at the origin and radius 2

$$P(X) = X^3 + 3X + 2i, \quad P'(X) = 3(X^2+1)$$

a Gauss-Lucas polynomial: $V(P') \subset \mathcal{K}(N(P))$ (but $V(P') \not\subset \mathcal{K}(P)$)

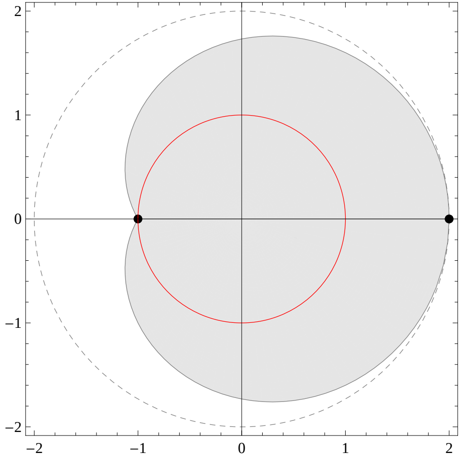

Figure: 2D-section (i, j-plane) of $\mathfrak{sn}(P)$ (gray), of $V(P')$ (red) and $\mathcal{K}(N(P))$ (dashed)

$$P(X) = P_3(X) = (X - i) \cdot (X - j) \cdot (X - k)$$

not a Gauss-Lucas polynomial: $V(P') \not\subset \mathcal{K}(N(P))$

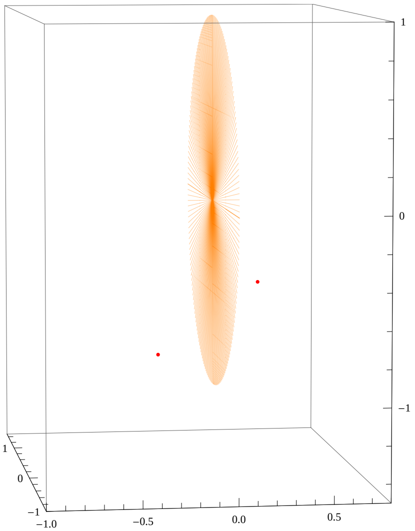

Figure: 3D-sections of $V(P')$ (red) and $\mathcal{K}(N(P))$ (orange)

$P(X) = P_3(X) = (X - i) \cdot (X - j) \cdot (X - k)$
not a Gauss-Lucas polynomial but $V(P') \subset \mathfrak{sn}(P)$

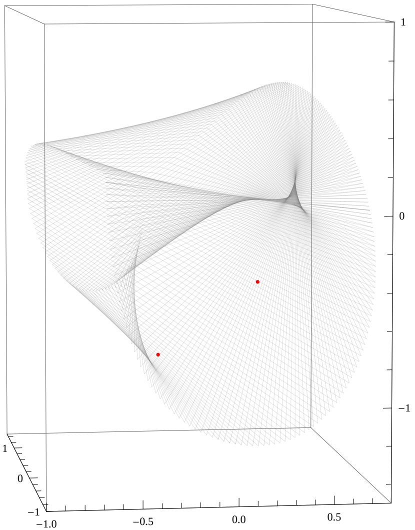

Figure: 3D-sections of $\mathfrak{sn}(P)$ (gray) and $V(P')$ (red)

$$P(X) = P_3(X) = (X - i) \cdot (X - j) \cdot (X - k)$$

not a Gauss-Lucas polynomial: but $\mathfrak{sn}(P') \subset \mathfrak{sn}(P)$

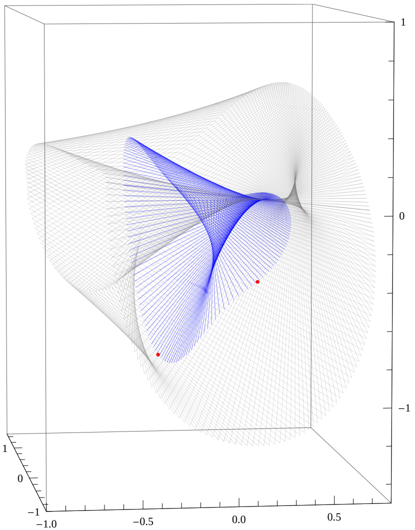

Figure: 3D-sections of $\mathfrak{sn}(P)$ (gray), of $\mathfrak{sn}(P')$ (blue) and $V(P')$ (red)

Estimates on the norm of the critical points

Let $p(z) = \sum_{k=0}^{d} a_k z^k$ be a complex polynomial. The norm of the roots of p can be estimated by means of the norm of the coefficients. A classic estimate is

$$\max_{z \in V(p)} |z| \leq |a_d|^{-1} \sqrt{\sum_{k=0}^{d} |a_k|^2}$$

The same proof as in $\mathbb{C}$ gives

For every polynomial $P(X) \in \mathbb{H}[X]$ of degree $d \geq 1$, it holds

$$\max_{x \in V(P)} |x| \leq |a_d|^{-1} \sqrt{\sum_{k=0}^{d} |a_k|^2}$$

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Estimates on the norm of the critical points

Given $P \in \mathbb{H}[X]$, define $C(P) := +\infty$ if P is a constant and

$$C(P) := |a_d|^{-1}\sqrt{\sum_{k=0}^{d} |a_k|^2} \quad \text{otherwise}$$

For critical points of $P \in \mathbb{H}[X]$ we then have:

$$\max_{x \in V(P')} |x| \leq C(P')$$

The quaternionic Gauss-Lucas Theorem allows to obtain a new estimate, which can be strictly better than classic estimate:

Proposition

Given any polynomial $P \in \mathbb{H}[X]$ of degree $d \geq 2$, it holds:

$$\max_{x \in V(P')} |x| \leq \sup_{I \in \mathbb{S}} \{C(P'_{+}{}^{I})\}$$

The singular set of a polynomial

The set of critical points of $P(X) = \sum_{k=0}^{d} X^k a_k \in \mathbb{H}[X]$ is the union of a finite set of points and a finite set of 2-spheres of the form $\mathbb{S}_x$. When P is considered as a mapping of $\mathbb{R}^4$, its singular set

$$Sing(P) := \{x \in \mathbb{H} \mid \det(J_P(x)) = 0\}$$

is a *real analytic, unbounded set of dimension 2 or 3*, which contains the critical points $V(P')$. It can be described slice-wise as

$$Sing(P) = \bigcup_{I \in \mathbb{S}} \{x \in \mathbb{C}_I \mid \pi_I(P'(x)\, \overline{\mathcal{D}_{CF} P(x)}) = 0\}$$

where $\mathcal{D}_{CF}$ is the Cauchy-Fueter operator on $\mathbb{H}$:

$$\mathcal{D}_{CF} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}$$

($\mathcal{D}_{CF} x^k = -2\sum_{m=0}^{k-1} x^{k-m-1}\overline{x}^m$. Up to a factor $-\frac{k}{2}$, it is the real zonal harmonic $\mathcal{Z}_{k-1}(x, 1)$ of degree $k-1$ and pole 1 in $\mathbb{R}^4$)

$$P(X) = P_3(X) = (X - i) \cdot (X - j) \cdot (X - k)$$

(the singular set $Sing(P) \not\subset \mathfrak{sn}(P)$ is unbounded, two-dimensional)

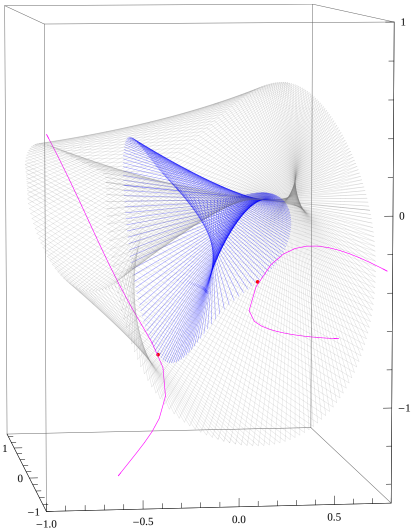

Figure: 3D-sections of $\mathfrak{sn}(P)$ (gray), of $\mathfrak{sn}(P')$ (blue), of $V(P')$ (red) and $Sing(P)$ (magenta)

$P(X) = X^3 + 3X + 2i, \quad P'(X) = 3(X^2 + 1)$
(the singular set $Sing(P)$ is the union of $\mathbb{S} = V(P')$ and a three-dimensional hyperboloid with axis $\mathbb{R}$)

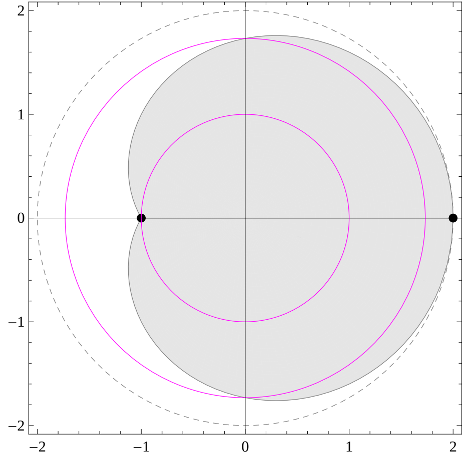

Figure: 2D-section (i, j-plane) of $\mathfrak{sn}(P)$ (gray), $\mathcal{K}(N(P))$ (dashed) and $Sing(P)$ (magenta)

References

- G. Gentili, C. Stoppato, and D. C. Struppa. *Regular Functions of a Quaternionic Variable*. Springer Monographs in Mathematics. Springer, 2013.
- Ghiloni, R., Perotti, A. *The quaternionic Gauss-Lucas Theorem*, Annali di Matematica Pura ed Applicata (2018) (arXiv:1712.00744).
- Perotti, A. *Slice regularity and harmonicity on Clifford algebras* (arXiv:1801.03045) (2018).
- Ghiloni, R., Perotti, A. *On a class of orientation-preserving maps of* $\mathbf{R}^4$. In preparation.

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