Directional quaternionic Hilbert operators

Alessandro Perotti

Abstract. The paper discusses harmonic conjugate functions and Hilbert operators in the space of Fueter regular functions of one quaternionic variable. We consider left–regular functions in the kernel of the Cauchy–Riemann operator

$$\mathcal{D} = 2\left(\frac{\partial}{\partial \bar z_1} + j\frac{\partial}{\partial \bar z_2}\right) = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}.$$

Let J_1, J_2 be the complex structures on the tangent bundle of $\mathbb{H} \simeq \mathbb{C}^2$ defined by left multiplication by i and j. Let J_1^*, J_2^* be the dual structures on the cotangent bundle and set $J_3^* = J_1^* J_2^*$. For every complex structure $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ ($p \in \mathbb{S}^2$ an imaginary unit), let $\overline{\partial}_p = \frac{1}{2}\left(d + pJ_p^* \circ d\right)$ be the Cauchy–Riemann operator w.r.t. the structure J_p. Let $\mathbb{C}_p = \langle 1, p\rangle \simeq \mathbb{C}$. If Ω satisfies a geometric condition, for every $\mathbb{C}_p$–valued function f_1 in a Sobolev space on the boundary $\partial\Omega$, we obtain a function $H_p(f_1) : \partial\Omega \to \mathbb{C}_p^\perp$, such that $f = f_1 + H_p(f_1)$ is the trace of a regular function on Ω. The function $H_p(f_1)$ is uniquely characterized by $L^2(\partial\Omega)$–orthogonality to the space of CR–functions w.r.t. the structure J_p. In this way we get, for every direction $p \in \mathbb{S}^2$, a bounded linear Hilbert operator H_p, with the property that $H_p^2 = id - S_p$, where S_p is the Szegö projection w.r.t. the structure J_p.

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1. Introduction

The aim of this paper is to obtain some generalizations of the classical Hilbert transform used in complex analysis. We define a range of harmonic conjugate functions and Hilbert operators in the space of regular functions of one quaternionic variable.

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Let Ω be a smooth bounded domain in $\mathbb{C}^2$. Let $\mathbb{H}$ be the space of real quaternions $q = x_0+ix_1+jx_2+kx_3$, where i, j, k denote the basic quaternions. We identify $\mathbb{H}$ with $\mathbb{C}^2$ by means of the mapping that associates the quaternion $q = z_1 + z_2 j$ with the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$.

We consider the class $\mathcal{R}(\Omega)$ of *left–regular* (also called *hyperholomorphic*) functions $f : \Omega \to \mathbb{H}$ in the kernel of the Cauchy–Riemann operator

$$\mathcal{D} = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}.$$

This differential operator is a variant of the original Cauchy–Riemann–Fueter operator (cf. for example [37] and [18, 19])

$$\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

Hyperholomorphic functions have been studied by many authors (see for instance [1, 21, 24, 28, 34, 35]). Many of their properties can be easily deduced from known properties satisfied by Fueter–regular functions. However, regular functions in the space $\mathcal{R}(\Omega)$ have some characteristics that are more intimately related to the theory of holomorphic functions of two complex variables.

This space contains the identity mapping and any holomorphic map (f_1, f_2) on Ω defines a regular function $f = f_1 + f_2 j$. This is no longer true if we adopt the original definition of Fueter regularity. This definition of regularity is also equivalent to that of q-holomorphicity given by Joyce in [20], in the setting of hypercomplex manifolds.

The space $\mathcal{R}(\Omega)$ exhibits other interesting links with the theory of two complex variables. In particular, it contains the spaces of holomorphic maps with respect to any constant complex structure, not only the standard one.

Let J_1, J_2 be the complex structures on the tangent bundle $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j. Let J_1^*, J_2^* be the dual structures on the cotangent bundle $T^*\mathbb{H} \simeq \mathbb{H}$ and set $J_3^* = J_1^* J_2^*$. For every complex structure $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ (p a imaginary unit in the unit sphere $\mathbb{S}^2$), let d be the exterior derivative and

$$\overline{\partial}_p = \frac{1}{2}\left(d + pJ_p^* \circ d\right)$$

the Cauchy–Riemann operator with respect to the structure J_p. Let $\mathrm{Hol}_p(\Omega, \mathbf{H}) = \mathrm{Ker}\,\overline{\partial}_p$ be the space of holomorphic maps from (Ω, J_p) to $(\mathbf{H}, L_p)$, where L_p is the complex structure defined by left multiplication by p. Then every element of $\mathrm{Hol}_p(\Omega, \mathbf{H})$ is regular.

These subspaces do not fill the whole space of regular functions: it was proved in [27] that there exist regular functions that are not holomorphic for any p. This result is a consequence of an applicable criterion of J_p-holomorphicity, based on the energy–minimizing property of regular functions.

Other regular functions can be constructed by means of holomorphic maps with respect to non–constant almost complex structures on Ω (cf. [30]).

The classical Hilbert transform expresses one of the real components of the boundary values of a holomorphic function in terms of the other. We are interested in a quaternionic analogue of this relation, which links the boundary values of one of the complex components of a regular function $f = f_1 + f_2 j$ (f_1, f_2 complex functions) to those of the other.

In [21] and [32] some generalizations of the Hilbert transform to hyperholomorphic functions were proposed. In these papers the functions considered are defined on plane or spatial domains, while we are interested in domains of two complex variables. In the latter case, pseudoconvexity becomes relevant, since a domain in $\mathbb{C}^2$ is pseudoconvex if and only if every complex harmonic function on it is a complex component of a regular function (cf. [23] and [25]).

In the complex variable case, there is a close connection between harmonic conjugates and the Hilbert transform (see for example the monograph [6, §21]), given by harmonic extension and boundary restriction. Several generalizations of this relation to higher dimensional spaces have been given (cf. e.g. [7, 8, 9, 12]), mainly in the framework of Clifford analysis, which can be considered as a generalization of quaternionic (and complex) analysis.

Our aim is to propose another variant of the quaternionic Hilbert operator, in which the complex structures J_p play a decisive role. Since these structures depend on a "direction" p in the unit sphere $\mathbb{S}^2$, we call it a *directional Hilbert operator* H_p.

The construction of H_p makes use of the rotational properties of regular functions (see §2.3), which were firstly studied in [37] in the context of Fueter-regularity. This allows to reduce the problem to the standard complex structure.

Let $\mathbb{C}_p = \langle 1, p \rangle$ be the copy of $\mathbb{C}$ in $\mathbb{H}$ generated by 1 and p and consider $\mathbb{C}_p$–valued function on the boundary $\partial\Omega$.

Assume that Ω satisfies a p–dependent geometric condition (see §3.1 for precise definitions), which is related to the pseudoconvexity property of Ω.

In Theorems 5 and 6 we show that for every $\mathbb{C}_p$–valued function f_1 in a Sobolev–type space $W^1_{\overline{\partial}_p}(\partial\Omega)$ and every fixed $q \in \mathbb{S}^2$ orthogonal to p, there exists a function $H_{p,q}(f_1) : \partial\Omega \to \mathbb{C}_p$ in the same space as f_1, such that $f = f_1 + H_{p,q}(f_1)q$ is the boundary value of a regular function on Ω. The function $H_{p,q}(f_1)$ is uniquely characterized by $L^2(\partial\Omega)$–orthogonality to the space of CR–functions with respect to the structure J_p. Moreover, $H_{p,q}$ is a bounded operator on the space $W^1_{\overline{\partial}_p}(\partial\Omega)$.

In Section 7 we prove our main result. We show how it is possible, for every fixed direction p, to choose a quaternionic regular harmonic conjugate of a $\mathbb{C}_p$–valued harmonic function in a way independent of the chosen orthogonal direction q. Taking restrictions to the boundary $\partial\Omega$, this construction permits to define the directional, p-dependent, Hilbert operator H_p.

In Theorem 10 we prove that even if the function $H_{p,q}(f_1)$ given by Theorem 6 depends on q, the product $H_{p,q}(f_1)q$ does not. Therefore we get a $\mathbb{C}_p$–antilinear, bounded operator

$$H_p : W^1_{\overline{\partial}_p}(\partial\Omega) \to W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p^{\perp}),$$

which exactly vanishes on the subspace $CR_p(\partial\Omega)$. Observe how the orthogonal decomposition of the codomain $\mathbb{H} = \mathbb{C}_p \oplus \mathbb{C}_p^\perp$ resembles the decomposition $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ which appears in the classical Hilbert transform.

The Hilbert operator H_p can be extended by right $\mathbb{H}$–linearity to the space $W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{H})$. The "*regular signal*" $R_p(f) := f + H_p(f)$ associated with f is always the trace of a regular function on Ω (Corollary 11). Moreover we show (Corollary 12) that $R_p(f)$ has a property similar to the one satisfied by analytic signals (cf. [31, Theorem 1.1]): f is the trace of a regular function on Ω if and only if $R_p(f) = 2f$ (modulo CR_p–functions).

The Hilbert operator H_p is also linked to the Szegö projection S_p with respect to J_p. In Theorem 13 we prove that $H_p^2 = id - S_p$ is the $L^2(\partial\Omega)$–orthogonal projection on the orthogonal complement of $CR_p(\partial\Omega)$.

When Ω is the unit ball B of $\mathbb{C}^2$, many of the stated results have a more precise formulation (see Theorem 7). The geometric condition is satisfied on the unit sphere $S = \partial B$ for every $p \in \mathbb{S}^2$. On S we are able to prove optimality of the boundary estimates satisfied by H_p.

In Section 6, we recall some applications of the harmonic conjugate construction to the characterization of the boundary values of pluriholomorphic functions. These functions are solutions of the PDE system

$$\frac{\partial^2 g}{\partial \bar{z}_i \partial \bar{z}_j} = 0 \quad \text{on } \Omega \quad (1 \le i, j \le 2)$$

(see for example [2, 3, 13, 14, 15] for properties of pluriholomorphic functions of two or more variables). The key point is that if $f = f_1 + f_2 j$ is regular, then f_1 is pluriholomorphic (and harmonic) if and only if f_2 is pluriharmonic, i.e. $\frac{\partial^2 f_2}{\partial z_i \partial \bar{z}_j} = 0$ on Ω $(1 \le i, j \le 2)$. Then known results about the boundary values of pluriharmonic functions (cf. [26]) can be applied to obtain a characterization of the traces of pluriholomorphic functions (Theorem 8).

2. Notations and definitions

2.1. Fueter regular functions

We identify the space $\mathbb{C}^2$ with the set $\mathbb{H}$ of quaternions by means of the mapping that associates the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ with the quaternion $q = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$. A quaternionic function $f = f_1 + f_2 j \in C^1(\Omega)$ is *(left) regular* (or *hyperholomorphic*) on Ω if

$$\mathcal{D}f = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega.$$

We will denote by $\mathcal{R}(\Omega)$ the space of regular functions on Ω.

With respect to this definition of regularity, the space $\mathcal{R}(\Omega)$ contains the identity mapping and every holomorphic mapping (f_1, f_2) on Ω (with respect to the standard complex structure) defines a regular function $f = f_1 + f_2 j$. We recall some properties of regular functions, for which we refer to the papers of Sudbery[37], Shapiro and Vasilevski[34] and Nōno[24]:

1. The complex components are both holomorphic or both non–holomorphic.
2. Every regular function is harmonic.
3. If Ω is pseudoconvex, every complex harmonic function is the complex component of a regular function on Ω.
4. The space $\mathcal{R}(\Omega)$ of regular functions on Ω is a *right* $\mathbb{H}$–module with integral representation formulas.

A definition equivalent to regularity has been given by Joyce[20] in the setting of hypercomplex manifolds. Joyce introduced the module of *q–holomorphic* functions on a hypercomplex manifold.

A hypercomplex structure on the manifold $\mathbb{H}$ is given by the complex structures J_1, J_2 on $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j. Let J_1^*, J_2^* be the dual structures on $T^*\mathbb{H} \simeq \mathbb{H}$. In complex coordinates

$$\begin{cases} J_1^* dz_1 = i\, dz_1, & J_1^* dz_2 = i\, dz_2 \\ J_2^* dz_1 = -d\bar{z}_2, & J_2^* dz_2 = d\bar{z}_1 \\ J_3^* dz_1 = i\, d\bar{z}_2, & J_3^* dz_2 = -i\, d\bar{z}_1 \end{cases}$$

where we make the choice $J_3^* = J_1^* J_2^*$, which is equivalent to $J_3 = -J_1 J_2$.

A function f is regular if and only if f is *q–holomorphic*, i.e.

$$df + iJ_1^*(df) + jJ_2^*(df) + kJ_3^*(df) = 0.$$

In complex components $f = f_1 + f_2 j$, we can rewrite the equations of regularity as

$$\overline{\partial} f_1 = J_2^*(\partial \overline{f}_2).$$

The original definition of regularity given by Fueter (cf. [37] or [18]) differs from that adopted here by a real co-ordinate reflection. Let γ be the transformation of $\mathbb{C}^2$ defined by $\gamma(z_1, z_2) = (z_1, \bar{z}_2)$. Then a C^1 function f is regular on the domain Ω if and only if $f \circ \gamma$ is Fueter–regular on $\gamma^{-1}(\Omega)$, i.e. it satisfies the differential equation

$$\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3} \right)(f \circ \gamma) = 0 \quad \text{on } \gamma^{-1}(\Omega).$$

2.2. Holomorphic functions with respect to a complex structure J_p

Let $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ be the orthogonal complex structure on $\mathbb{H}$ defined by a unit imaginary quaternion $p = p_1 i + p_2 j + p_3 k$ in the sphere $\mathbb{S}^2 = \{p \in \mathbb{H} \mid p^2 = -1\}$. In particular, J_1 is the standard complex structure of $\mathbb{C}^2 \simeq \mathbb{H}$.

Let $\mathbb{C}_p = \langle 1, p \rangle$ be the complex plane spanned by 1 and p and let L_p be the complex structure defined on $T^*\mathbb{C}_p \simeq \mathbb{C}_p$ by left multiplication by p. If $f = f^0 + if^1 : \Omega \to \mathbb{C}$ is a J_p*–holomorphic* function, i.e. $df^0 = J_p^*(df^1)$ or, equivalently,

$df + iJ_p^*(df) = 0$, then f defines a regular function $\tilde{f} = f^0 + pf^1$ on Ω. We can identify $\tilde{f}$ with a holomorphic function

$$\tilde{f} : (\Omega, J_p) \to (\mathbb{C}_p, L_p).$$

We have $L_p = J_{\gamma(p)}$, where $\gamma(p) = p_1 i + p_2 j - p_3 k$. More generally, we can consider the space of holomorphic maps from (Ω, J_p) to $(\mathbb{H}, L_p)$

$$\mathrm{Hol}_p(\Omega, \mathbb{H}) = \{f : \Omega \to \mathbb{H} \text{ of class } C^1 \mid \overline{\partial}_p f = 0 \text{ on } \Omega\} = \mathrm{Ker}\, \overline{\partial}_p$$

where $\overline{\partial}_p$ is the Cauchy–Riemann operator with respect to the structure J_p

$$\overline{\partial}_p = \frac{1}{2}\left(d + pJ_p^* \circ d\right).$$

These functions will be called J_p*–holomorphic maps* on Ω.

For any positive orthonormal basis $\{1, p, q, pq\}$ of $\mathbb{H}$ ($p, q \in \mathbb{S}^2$), let $f = f_1 + f_2 q$ be the decomposition of f with respect to the orthogonal sum

$$\mathbb{H} = \mathbb{C}_p \oplus (\mathbb{C}_p)q.$$

Let $f_1 = f^0 + pf^1$, $f_2 = f^2 + pf^3$, with f^0, f^1, f^2, f^3 the real components of f w.r.t. the basis $\{1, p, q, pq\}$. Then the equations of regularity can be rewritten in complex form as

$$\overline{\partial}_p f_1 = J_q^*(\partial_p \overline{f}_2),$$

where $\overline{f}_2 = f^2 - pf^3$ and $\partial_p = \frac{1}{2}\left(d - pJ_p^* \circ d\right)$. Therefore every $f \in \mathrm{Hol}_p(\Omega, \mathbb{H})$ is a regular function on Ω.

Remark 1.
1. The identity map belongs to the spaces $\mathrm{Hol}_i(\Omega, \mathbb{H}) \cap \mathrm{Hol}_j(\Omega, \mathbb{H})$, but not to $\mathrm{Hol}_k(\Omega, \mathbb{H})$.
2. For every $p \in \mathbb{S}^2$, $\mathrm{Hol}_{-p}(\Omega, \mathbb{H}) = \mathrm{Hol}_p(\Omega, \mathbb{H})$.
3. Every $\mathbb{C}_p$–valued regular function is a J_p–holomorphic function.

Proposition 1. *If $f \in \mathrm{Hol}_p(\Omega, \mathbb{H}) \cap \mathrm{Hol}_q(\Omega, \mathbb{H})$, with $p \neq \pm q$, then $f \in \mathrm{Hol}_r(\Omega, \mathbb{H})$ for every $r = \frac{\alpha p + \beta q}{\|\alpha p + \beta q\|}$ ($\alpha, \beta \in \mathbb{R}$) in the circle of $\mathbb{S}^2$ generated by p and q.*

Proof. Let $a = \|\alpha p + \beta q\|$. Then $a^2 = \alpha^2 + \beta^2 + 2\alpha\beta(p \cdot q)$, where $p \cdot q$ is the scalar product of the vectors p and q in $\mathbb{S}^2$. An easy computation shows that

$$pJ_q^* + qJ_p^* = -2(p \cdot q)Id.$$

From these identities we get that

$$\begin{aligned} rJ_r^*(df) &= a^{-2}(\alpha p + \beta q)(\alpha J_p^* + \beta J_q^*) \\ &= a^{-2}(\alpha^2 pJ_p^*(df) + \beta^2 qJ_q^*(df) + \alpha\beta(pJ_q^* + qJ_p^*)(df)) \\ &= a^{-2}(\alpha^2 pJ_p^*(df) + \beta^2 qJ_q^*(df) - 2\alpha\beta(p \cdot q)(df)) \\ &= a^{-2}(\alpha^2(-df) + \beta^2(-df) + 2\alpha\beta(p \cdot q)(-df)) = -df. \end{aligned}$$

Therefore $f \in \mathrm{Hol}_r(\Omega, \mathbb{H})$. □

In [27] it was proved that on every domain Ω there exist regular functions that are not J_p-holomorphic for any p. A similar result was obtained by Chen and Li[10] for the larger class of *q-maps* between hyperkähler manifolds.

This result is a consequence of a criterion of J_p-holomorphicity, which is obtained using the energy–minimizing property of regular functions.

2.3. Rotated regular functions

In [37] Proposition 5, Sudbery studied the action of rotations on Fueter–regular functions. Using that result and the reflection γ introduced in §2.1, we can obtain new regular functions by rotation.

Proposition 2. *Let $f \in \mathcal{R}(\Omega)$ and let $a \in \mathbb{H}$, $a \neq 0$. Let $r_a(z) = aza^{-1}$ be the three–dimensional rotation of $\mathbb{H}$ defined by a. Then the function*

$$f^a = r_{\gamma(a)} \circ f \circ r_a$$

is regular on $\Omega^a = r_a^{-1}(\Omega) = a^{-1}\Omega a$. Moreover, if $\gamma(r_a(i)) = p$, then $f \in \mathrm{Hol}_p(\Omega)$ if and only if $f^a \in \mathrm{Hol}_i(\Omega^a)$.

Proof. The first assertion is an immediate application of the cited result of Sudbery. Now let $p = \gamma(r_a(i))$, $p' = \gamma(p) = r_a(i)$ and $q = r_a(j)$ in $\mathbb{S}^2$. We first show that

$$r_a : (\mathbb{H}, J_1) \to (\mathbb{H}, L_{p'})$$

is holomorphic. Let $r_a(z) = aza^{-1} = x_0 + p'x_1 + qx_2 + p'qx_3 = (x_0 + p'x_1) + (x_2 + p'x_3)q = g_1 + g_2 q$, where g_1, g_2 are the $\mathbb{C}_{p'}$–valued $J_{p'}$–holomorphic functions induced by z_1 and z_2. Then

$$p' J_1^*(dr_a) = p' J_1^*(dg_1) + p' J_1^*(dg_2)q = -dg_1 - dg_2\, q = -dr_a.$$

From this we get that also the map

$$r_{\gamma(a)}^{-1} = r_{\gamma(a)^{-1}} : (\mathbb{H}, J_1) \to (\mathbb{H}, L_p)$$

is holomorphic, since $r_{\gamma(a)^{-1}}(i) = \gamma(a)^{-1} i \gamma(a) = \gamma(aia^{-1}) = \gamma(r_a(i)) = p$. Now the commutative diagram

$$\begin{array}{ccc} (\Omega, L_{\gamma(p)}) & \xrightarrow{\ f\ } & (\mathbb{H}, L_p) \\ {\scriptstyle r_a}\uparrow & & \downarrow{\scriptstyle r_{\gamma(a)}} \\ (\Omega^a, J_1) & \xrightarrow[\ f^a\ ]{} & (\mathbb{H}, J_1) \end{array}$$

gives the stated equivalence, since $J_p = L_{\gamma(p)}$. □

Remark 2. The rotated function f^a has the following properties:

1. $(f^a)^{a^{-1}} = f$.
2. $f^{-a} = f^a$.
3. If $a \in \mathbb{S}^2$, then $(f^a)^a = f$.
4. If f is $\mathbb{C}_p$–valued on Ω, for $p = \gamma(r_a(i))$, then f^a is $\mathbb{C}$–valued on Ω^a.

2.4. Cauchy–Riemann operators

Let $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ be a bounded domain with C^∞–smooth boundary in $\mathbb{C}^2$. We assume ρ of class C^∞ on $\mathbb{C}^2$ and $d\rho \neq 0$ on $\partial\Omega$. For every complex valued function $g \in C^1(\overline{\Omega})$, we can define on a neighborhood of $\partial\Omega$ the normal components of ∂g and $\overline{\partial} g$

$$\partial_n g = \sum_k \frac{\partial g}{\partial z_k}\frac{\partial \rho}{\partial \bar z_k}\frac{1}{|\partial\rho|} \quad \text{and} \quad \overline{\partial}_n g = \sum_k \frac{\partial g}{\partial \bar z_k}\frac{\partial \rho}{\partial z_k}\frac{1}{|\partial\rho|},$$

where $|\partial\rho|^2 = \sum_{k=1}^2 \left|\frac{\partial\rho}{\partial z_k}\right|^2$. By means of the Hodge $*$–operator and the Lebesgue surface measure $d\sigma$, we can also write

$$\overline{\partial}_n g\, d\sigma = *\,\overline{\partial} g_{|\partial\Omega}.$$

In a neighbourhood of $\partial\Omega$ we have the decomposition of $\overline{\partial} g$ in the tangential and the normal parts

$$\overline{\partial} g = \overline{\partial}_t g + \overline{\partial}_n g \frac{\overline{\partial}\rho}{|\overline{\partial}\rho|}.$$

Let $\mathcal{L}$ be the tangential Cauchy–Riemann operator

$$\mathcal{L} = \frac{1}{|\partial\rho|}\left(\frac{\partial\rho}{\partial\bar z_2}\frac{\partial}{\partial\bar z_1} - \frac{\partial\rho}{\partial\bar z_1}\frac{\partial}{\partial\bar z_2}\right).$$

The tangential part of $\overline{\partial} g$ is related to $\mathcal{L}g$ by the following formula

$$\overline{\partial}_t g \wedge d\zeta_{|\partial\Omega} = 2\mathcal{L}g\, d\sigma.$$

A complex function $g \in C^1(\partial\Omega)$ is a CR–function if and only if $\mathcal{L}g = 0$ on $\partial\Omega$. Notice that $\overline{\partial} g$ has coefficients of class $L^2(\partial\Omega)$ if and only if both $\overline{\partial}_n g$ and $\mathcal{L}g$ are of class $L^2(\partial\Omega)$.

If $g = g_1 + g_2 j$ is a regular function of class C^1 on Ω, then the equations $\overline{\partial}_n g_1 = -\overline{\mathcal{L}(g_2)}$, $\overline{\partial}_n g_2 = \overline{\mathcal{L}(g_1)}$ hold on $\partial\Omega$. Conversely, a harmonic function f of class $C^1(\Omega)$ is regular if it satisfies these equations on $\partial\Omega$ (cf. [28]). If Ω has connected boundary, it is sufficient that one of the equations is satisfied.

In place of the standard complex structure J_1, we can take on $\mathbb{C}^2$ a different complex structure J_p and consider the corresponding Cauchy–Riemann operators. We will denote by $\partial_{p,n}$ and $\overline{\partial}_{p,n}$ the normal components of ∂_p and $\overline{\partial}_p$ respectively, by $\overline{\partial}_{p,t}$ the tangential component of $\overline{\partial}_p$ and by $\mathcal{L}_p$ the tangential Cauchy–Riemann operator with respect to the structure J_p. Then we have the relations

$$\overline{\partial}_p g = \overline{\partial}_{p,t}\, g + \overline{\partial}_{p,n} g \frac{\overline{\partial}_p\rho}{|\overline{\partial}_p\rho|},$$
$$\overline{\partial}_{p,t}\, g \wedge d\zeta_{|\partial\Omega} = 2\mathcal{L}_p g\, d\sigma,$$
$$\overline{\partial}_{p,n} g\, d\sigma = *\,\overline{\partial}_p g_{|\partial\Omega}.$$

The space

$$CR_p(\partial\Omega) = \operatorname{Ker}\mathcal{L}_p = \{g : \partial\Omega \to \mathbb{C}_p \mid \mathcal{L}_p g = 0\}$$

has elements the CR–functions on $\partial\Omega$ with respect to the operator $\overline{\partial}_p$.

Remark 3. The operators $\overline{\partial}_p$, $\partial_{p,n}$, $\overline{\partial}_{p,n}$ and $\mathcal{L}_p$ are $\mathbb{C}_p$–linear and they map $\mathbb{C}_p$–valued functions of class C^1 to continuous $\mathbb{C}_p$–valued functions.

The relation between the Cauchy–Riemann operators $\overline{\partial}$ and $\overline{\partial}_p$ can be expressed by means of the rotations introduced in Proposition 2.

Proposition 3. *Let $a \in \mathbb{H}$, $a \neq 0$. If $p = \gamma(r_a(i))$ and $g : \overline{\Omega} \to \mathbb{C}_p$ is of class $C^1(\overline{\Omega})$, then $\overline{\partial} g^a = (\overline{\partial}_p g)^a$. Moreover $\overline{\partial}_n g^a = (\overline{\partial}_{p,n} g)^a$ and $\mathcal{L} g^a = (\mathcal{L}_p g)^a$ on $\partial\Omega^a$. In particular, $g \in CR_p(\partial\Omega)$ if and only if $g^a \in CR(\partial\Omega^a)$.*

Proof. Let $p' = \gamma(p)$, $a' = \gamma(a)$. We have

$$2(\overline{\partial}_p g)^a = dr_{a'} \circ (dg + pJ_p^*(dg)) \circ dr_a = dg^a + dr_{a'} \circ L_p \circ J_p^*(dg) \circ dr_a,$$

while

$$2\overline{\partial} g^a = dg^a + L_i \circ J_1^*(dg^a) = dg^a + L_i \circ dg^a \circ J_1.$$

The last term is

$$L_i \circ dg^a \circ J_1 = J_1^*(dr_{a'}) \circ dg \circ (dr_a \circ J_1) = (dr_{a'} \circ L_p) \circ dg \circ (L_{p'} \circ dr_a),$$

since $r_a : (\mathbb{H}, J_1) \to (\mathbb{H}, L_{p'})$ and $r_{a'} : (\mathbb{H}, L_p) \to (\mathbb{H}, J_1)$ are holomorphic, as seen in the proof of Proposition 2. Therefore it suffices to notice that $J_p^*(dg) = dg \circ L_{p'}$ and this is true because $J_p = L_{p'}$. For the second statement, we have

$$* \, \overline{\partial} g^a{}_{|\partial\Omega^a} = \overline{\partial}_n g^a d\sigma^a$$

where $d\sigma^a$ is the Lebesgue measure on $\partial\Omega^a$. On the other hand,

$$* \, (\overline{\partial}_p g)^a{}_{|\partial\Omega} = (* \, \overline{\partial}_p g_{|\partial\Omega})^a = (\overline{\partial}_{p,n} g \, d\sigma)^a = (\overline{\partial}_{p,n} g)^a d\sigma^a.$$

From the first part it follows that $\overline{\partial}_n g^a = (\overline{\partial}_{p,n} g)^a$. Then also the tangential parts are in the same relation and this implies that $\mathcal{L} g^a = (\mathcal{L}_p g)^a$ on $\partial\Omega^a$. □

3. Quaternionic harmonic conjugation

3.1. L^2 boundary estimates

Let $p \in \mathbb{S}^2$. Given a $\mathbb{C}_p$–valued function $f = f^0 + pf^1$, with f^0, f^1 real functions of class $L^2(\partial\Omega)$, we define the $L^2(\partial\Omega)$–norm of f as

$$\|f\| = (\|f^0\|^2 + \|f^1\|^2)^{1/2},$$

and the $L^2(\partial\Omega)$–product of f and $g = g^0 + pg^1$ as

$$(f, g) = (f^0, g^0)_{L^2(\partial\Omega)} + (f^1, g^1)_{L^2(\partial\Omega)}.$$

We will denote by $L^2(\partial\Omega, \mathbb{C}_p)$ the space of functions $f = f^0 + pf^1$, $f^0, f^1 \in L^2(\partial\Omega)$ real–valued functions.

In the following we shall assume that Ω satisfies a $L^2(\partial\Omega)$–estimate for some $p \in \mathbb{S}^2$: there exists a positive constant C_p such that

$$|(f, \mathcal{L}_p g)| \le C_p \|\partial_{p,n} f\| \, \|\overline{\partial}_{p,n} g\| \tag{1}$$

for every $\mathbb{C}_p$–valued harmonic functions f, g on Ω, of class C^1 on $\overline{\Omega}$.

From Proposition 3 and the invariance of the laplacian w.r.t. rotations, it follows that Ω satisfies (1) if and only if the rotated domain $\Omega^a = r_a^{-1}(\Omega)$, with $p = \gamma(r_a(i))$, satisfies the estimate with $p = i$:

$$|(f, \mathcal{L}g)| \leq C_p \|\partial_n f\| \, \|\overline{\partial}_n g\| \tag{2}$$

for all complex–valued harmonic functions f, g on Ω^a, of class C^1 on $\overline{\Omega^a}$.

Proposition 4. *On the unit ball B of $\mathbb{C}^2$, the estimate* (1) *is satisfied with constant $C_p = 1$ for every $p \in \mathbb{S}^2$.*

Proof. From rotational symmetry of B, it is sufficient to prove the estimate for the case $p = i$, the standard complex structure. In this case, the proof was given in [29]. For convenience of the reader, we repeat here the proof.

We denote $\mathcal{L}_i$ by $\mathcal{L}$, $\partial_{i,n}$ by ∂_n and $\overline{\partial}_{i,n}$ by $\overline{\partial}_n$. Let $S = \partial B$. The space $L^2(S)$ is the sum of the pairwise orthogonal spaces $\mathcal{H}_{s,t}$, whose elements are the harmonic homogeneous polynomials of degree s in z_1, z_2 and t in $\bar{z}_1, \bar{z}_2$ (cf. for example Rudin[33, §12.2]). The spaces $\mathcal{H}_{s,t}$ can be identified with the spaces of the restrictions of their elements to S (*spherical harmonics*).

It suffices to prove the estimate for a pair of polynomials $f \in \mathcal{H}_{s,t}$, $g \in \mathcal{H}_{l,m}$, since the orthogonal subspaces $\mathcal{H}_{s,t}$ are eigenspaces of the operators ∂_n and $\overline{\partial}_n$. We can restrict ourselves to the case $s = l + 1 > 0$ and $m = t + 1 > 0$, since otherwise the product $(f, \mathcal{L}g)$ is zero. We have

$$|(f, \mathcal{L}g)|^2 \leq \|f\|^2 \|\mathcal{L}g\|^2 = \|f\|^2 (\mathcal{L}^* \mathcal{L}g, g) = \|f\|^2 (-\overline{\mathcal{L}} \mathcal{L}g, g) = \|f\|^2 (l+1) m \|g\|^2$$

since the $L^2(S)$-adjoint $\mathcal{L}^*$ is equal to $-\overline{\mathcal{L}}$ (cf. [33, §18.2.2]) and $\overline{\mathcal{L}}\mathcal{L} = -(l+1) m \, Id$ when $m > 0$. On the other hand,

$$\|\partial_n f\| \|\overline{\partial}_n g\| = (l+1) m \|f\| \|g\|.$$

and the estimate is proved. □

Remark 4. It was proved in [29] that the estimate (2) implies the pseudoconvexity of Ω with respect to the standard structure. It can be shown that the same holds for a complex structure J_p. We conjecture that in turn the estimate (1) is always valid on a (strongly) pseudoconvex domain in $\mathbb{C}^2$ (w.r.t. J_p).

A domain Ω biholomorphic to B in the standard structure (e.g. an ellipsoid with defining function $\rho = c_1^2 |z_1|^2 + c_2^2 |z_2|^2 - 1$) satisfies estimate (2) but it does not necessarily satisfies estimate (1) for $p \neq i$, since the domain Ω^a can be not pseudoconvex.

3.2. Harmonic conjugate

We now prove some results about the existence of quaternionic harmonic conjugates in the space of $\mathbb{C}_p$–valued functions of class $L^2(\partial\Omega)$. We consider the following

Sobolev–type Hilbert subspace of $L^2(\partial\Omega, \mathbb{C}_p)$:

$$\begin{aligned} W^1_{\overline{\partial}_p}(\partial\Omega) &= \{f \in L^2(\partial\Omega, \mathbb{C}_p) \mid \overline{\partial}_p f \in L^2(\partial\Omega, \mathbb{C}_p)\} \\ &= \{f \in L^2(\partial\Omega, \mathbb{C}_p) \mid \overline{\partial}_{p,n} f \text{ and } \mathcal{L}_p f \in L^2(\partial\Omega, \mathbb{C}_p)\} \end{aligned}$$

with product

$$(f, g)_{W^1_{\overline{\partial}_p}} = (f, g) + (\overline{\partial}_{p,n} f, \overline{\partial}_{p,n} g) + (\mathcal{L}_p f, \mathcal{L}_p g).$$

Here and in the following we always identify $f \in L^2(\partial\Omega)$ with its harmonic extension on Ω. We will use also the space

$$\overline{W}^1_{p,n}(\partial\Omega) = \{f \in L^2(\partial\Omega, \mathbb{C}_p) \mid \overline{\partial}_{p,n} f \in L^2(\partial\Omega, \mathbb{C}_p)\} \supset W^1_{\overline{\partial}_p}(\partial\Omega)$$

with product

$$(f, g)_{\overline{W}^1_{p,n}} = (f, g) + (\overline{\partial}_{p,n} f, \overline{\partial}_{p,n} g),$$

and the conjugate space

$$W^1_{p,n}(\partial\Omega) = \{f \in L^2(\partial\Omega, \mathbb{C}_p) \mid \partial_{p,n} f \in L^2(\partial\Omega)\}$$

with product

$$(f, g)_{W^1_{p,n}} = (f, g) + (\partial_{p,n} f, \partial_{p,n} g).$$

These spaces are vector spaces over $\mathbb{R}$ and over $\mathbb{C}_p$.

For every $\alpha > 0$, the spaces $W^1_{\overline{\partial}_p}(\partial\Omega)$, $\overline{W}^1_{p,n}(\partial\Omega)$ and $W^1_{p,n}(\partial\Omega)$ contain, in particular, every $\mathbb{C}_p$–valued function f of class $C^{1+\alpha}(\partial\Omega)$. Indeed, under this regularity condition f has an harmonic extension of class (at least) C^1 on $\overline{\Omega}$.

Let S_p be the Szegö projection from $L^2(\partial\Omega, \mathbb{C}_p)$ onto the (closure of the) subspace of holomorphic functions with respect to the structure J_p, continuous up to the boundary. We have the following orthogonal decomposition

$$W^1_{\overline{\partial}_p}(\partial\Omega) = CR_p(\partial\Omega) \oplus CR_p(\partial\Omega)^\perp = \operatorname{Ker} S_p^\perp \oplus \operatorname{Ker} S_p,$$

where $S_p^\perp = Id - S_p$.

In the case of the standard complex structure ($p = i$), we will denote the space $W^1_{\overline{\partial}_i}(\partial\Omega)$ simply by $W^1_{\overline{\partial}}(\partial\Omega)$ and the same for the spaces $\overline{W}^1_{i,n}(\partial\Omega) = \overline{W}^1_n(\partial\Omega)$ and $W^1_{i,n}(\partial\Omega) = W^1_n(\partial\Omega)$.

Remark 5. From Proposition 3 it follows that if $p = \gamma(r_a(i))$, then

$$W^1_{\overline{\partial}_p}(\partial\Omega)^a := \{f^a \mid f \in W^1_{\overline{\partial}_p}(\partial\Omega)\} = W^1_{\overline{\partial}}(\partial\Omega^a).$$

Similar relations hold for the other function spaces and the correspondence $f \mapsto f^a$ is an isometry between these spaces. The Szegö projection S_p on Ω is related to the standard Szegö projection S on Ω^a by $S_p(f)^a = S(f^a)$.

Theorem 5. *Assume that the boundary $\partial\Omega$ is connected and that the domain Ω satisfies estimate* (1). *Given $f_1 \in \overline{W}^1_{p,n}(\partial\Omega)$, for every $q \in \mathbb{S}^2$ orthogonal to p, there exists $f_2 \in L^2(\partial\Omega, \mathbb{C}_p)$, unique up to a CR_p–function, such that $f = f_1 + f_2 q$ is the trace of a regular function on Ω. Moreover, f_2 satisfies the estimate*

$$\inf_{f_0} \|f_2 + f_0\|_{L^2(\partial\Omega)} \leq C_p \|f_1\|_{\overline{W}^1_{p,n}(\partial\Omega)},$$

where the infimum is taken among the CR_p–functions $f_0 \in L^2(\partial\Omega, \mathbb{C}_p)$. The constant C_p is the same occurring in the estimate (1).

Theorem 6. *Assume that $\partial\Omega$ is connected and that Ω satisfies estimate* (1). *Given $f_1 \in W^1_{\overline{\partial}_p}(\partial\Omega)$, for every $q \in \mathbb{S}^2$ orthogonal to p, there exists $H_{p,q}(f_1) \in W^1_{\overline{\partial}_p}(\partial\Omega)$ such that $f = f_1 + H_{p,q}(f_1)q$ is the trace of a regular function on Ω. Moreover, $H_{p,q}(f_1)$ satisfies the estimate*

$$\|H_{p,q}(f_1)\|_{W^1_{\overline{\partial}_p}} \leq \sqrt{C_p^2 + 1}\, \|f_1\|_{W^1_{\overline{\partial}_p}}$$

with the same constant C_p given in (1). *The operator $H_{p,q}$ is a $\mathbb{C}_p$–antilinear bounded operator of the space $W^1_{\overline{\partial}_p}(\partial\Omega)$, with kernel the subspace $CR_p(\partial\Omega)$.*

We will show in section 5 that when $\Omega = B$, the unit ball, then a sharper estimate can be proved.

4. Proof of Theorems 5 and 6

4.1. An existence principle

We recall a powerful existence principle in Functional Analysis proved by Fichera in the 50's (cf. [16, 17] and [11, §12]).

Let M_1 and M_2 be linear homomorphisms from a vector space V over the real (or complex) numbers into the Banach spaces B_1 and B_2, respectively.

Let us consider the following problem: given a linear functional Ψ_1 defined on B_1, find a linear functional Ψ_2 defined on B_2 such that

$$\Psi_1(M_1(v)) = \Psi_2(M_2(v)) \quad \forall\, v \in V.$$

Fichera's existence principle is the following:

Theorem (Fichera). *A necessary and sufficient condition for the existence, for any $\Psi_1 \in B_1^*$, of a linear functional Ψ_2 defined on B_2 such that*

$$\Psi_1(M_1(v)) = \Psi_2(M_2(v)) \quad \forall\, v \in V$$

is that there exists a positive constant C such that, for all $v \in V$,

$$\|M_1(v)\| \leq C\|M_2(v)\|.$$

Moreover, we have the following dual estimate with the same constant C:

$$\inf_{\Psi_0 \in \mathcal{N}} \|\Psi_2 + \Psi_0\| \leq C\|\Psi_1\|,$$

where $\mathcal{N}$ is the subspace of B_2^* composed of the functionals Ψ_0 that are orthogonal to the range of M_2, i.e. $\mathcal{N} = \{\Psi_0 \in B_2^* \mid \Psi_0(M_2(v)) = 0 \ \forall v \in V\}$.

The theorem can be applied only if the kernel of M_2 is contained in the kernel of M_1. If this condition is not satisfied, the vector Ψ_1 has to satisfy the compatibility conditions:

$$\Psi_1(M_1(v)) = 0 \quad \forall\ v \in \mathrm{Ker}(M_2).$$

As mentioned in [11], this result includes important existence theorems, like e.g. the Hahn–Banach theorem and the Lax–Milgram lemma.

4.2. Proof of Theorem 5

Given two orthogonal imaginary units p, q, there exists a unique rotation $r_{a'}$ that fixes the reals and maps p to i and q to j. Let $a = \gamma(a')$. Then $p = \gamma(r_a(i))$ and the domain Ω^a satisfies the estimate (2) of §3.1. The rotated function f_1^a belongs to the space $\overline{W}_n^1(\partial\Omega^a)$.

Now we state and prove the theorem for the standard structure $p = i$ (cf.[26, Theorem 3]) and then we will show how this is sufficient to get the general result.

Theorem. *Suppose that the estimate* (2) *is satisfied. For every $f_1 \in \overline{W}_n^1(\partial\Omega)$, there exists $f_2 \in L^2(\partial\Omega)$, unique up to a CR–function, such that $f = f_1 + f_2 j$ is the trace of a regular function on Ω. Moreover, f_2 satisfies the estimate*

$$\inf_{f_0} \|f_2 + f_0\|_{L^2(\partial\Omega)} \le C\|f_1\|_{\overline{W}_n^1(\partial\Omega)},$$

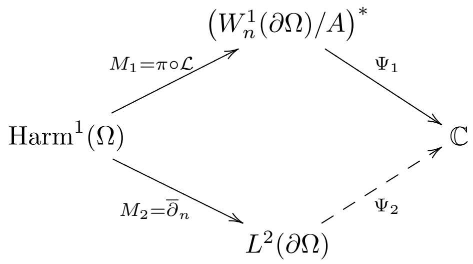
where the infimum is taken among the CR–functions $f_0 \in L^2(\partial\Omega)$.

Proof. We apply the existence principle to the following setting. Let $V = \mathrm{Harm}^1(\Omega)$ be the space of complex valued harmonic functions on Ω, of class C^1 on $\overline{\Omega}$.

By means of the identification of $L^2(\partial\Omega)$ with its dual, we get dense, continuous injections $W_n^1(\partial\Omega) \subset L^2(\partial\Omega) = L^2(\partial\Omega)^* \subset W_n^1(\partial\Omega)^*$.

Let $A = \overline{CR(\partial\Omega)}$ be the closed subspace of $L^2(\partial\Omega)$ whose elements are conjugate CR–functions. It was shown by Kytmanov in [22, §17.1] that the set of the harmonic extensions of elements of A is the kernel of ∂_n.

Let $B_1 = \left(W_n^1(\partial\Omega)/A\right)^*$ and $B_2 = L^2(\partial\Omega)$. Let $M_1 = \pi \circ \mathcal{L}$, $M_2 = \overline{\partial}_n$, where π is the quotient projection $\pi : L^2 \to L^2/A = \left(L^2/A\right)^* \subset B_1$.



For every $g \in L^2(\partial\Omega)$, let $g^\perp$ denote the component of g in $A^\perp \subset L^2(\partial\Omega)$. A function $h_1 \in W^1_n(\partial\Omega)$ defines a linear functional $\Psi_1 \in B_1^* = W^1_n(\partial\Omega)/A$ such that

$$\Psi_1(\pi(g)) = (g^\perp, h_1)_{L^2} \quad \text{for every } g \in L^2(\partial\Omega).$$

If h is a CR–function on $\partial\Omega$,

$$(\mathcal{L}\phi, \bar{h}) = \frac{1}{2}\int_{\partial\Omega} h\overline{\partial}(\phi dz) = 0 \quad \text{and then} \quad (\mathcal{L}\phi)^\perp = \mathcal{L}\phi.$$

This implies that $\Psi_1(M_1(\phi)) = (\mathcal{L}\phi, h_1)$.

By the previous principle of Fichera, the existence of $h_2 \in L^2(\partial\Omega)$ such that

$$\Psi_1(M_1(\phi)) = (\mathcal{L}\phi, h_1)_{L^2} = \Psi_2(M_2(\phi)) = (\overline{\partial}_n\phi, h_2)_{L^2} \quad \forall\ \phi \in \mathrm{Harm}^1(\Omega)$$

is equivalent to the existence of $C > 0$ such that

$$\|\pi(\mathcal{L}\phi)\|_{(W^1_n(\partial\Omega)/A)^*} \leq C\|\overline{\partial}_n\phi\|_{L^2(\partial\Omega)} \quad \forall\ \phi \in \mathrm{Harm}^1(\Omega). \tag{**}$$

The functional $\pi(\mathcal{L}\phi) \in L^2/A = \left(L^2/A\right)^* \subset B_1$ acts on $\pi(g) \in L^2/A$ in the following way:

$$\pi(\mathcal{L}\phi)(\pi(g)) = (g^\perp, \mathcal{L}\phi)_{L^2} = (g, \mathcal{L}\phi)_{L^2}$$

since $\mathcal{L}\phi \in A^\perp$. From the estimate (2) we imposed on Ω we get

$$\sup_{\|\pi(g)\|_{W^1_n(\partial\Omega)/A} \leq 1} |(g, \mathcal{L}\phi)| \leq C\|\overline{\partial}_n\phi\|_{L^2(\partial\Omega)} \quad \forall\ \phi \in \mathrm{Harm}^1(\Omega)$$

which is the same as estimate $(**)$. From the existence principle applied to $h_1 = \bar{f}_1 \in W^1_n(\partial\Omega)$, we get $f_2 = -h_2 \in L^2(\partial\Omega)$ such that

$$(\mathcal{L}\phi, \bar{f}_1)_{L^2} = -(\overline{\partial}_n\phi, f_2)_{L^2} \quad \forall\ \phi \in \mathrm{Harm}^1(\Omega).$$

Therefore

$$\frac{1}{2}\int_{\partial\Omega} f_1\overline{\partial}\phi \wedge d\zeta = -\int_{\partial\Omega} \bar{f}_2 * \overline{\partial}\phi \quad \forall\ \phi \in \mathrm{Harm}^1(\Omega)$$

and the result follows from the $L^2(\partial\Omega)$–version of Theorem 5 in [28], that can be proved as in [28] using the results given in [34, §3.7]. The estimate given by the existence principle is

$$\inf_{f_0 \in \mathcal{N}} \|f_2 + f_0\|_{L^2(\partial\Omega)} \leq C\|\Psi_1\|_{W^1_n/A} \leq C\|h_1\|_{W^1_n(\partial\Omega)} = C\|f_1\|_{\overline{W}^1_n(\partial\Omega)},$$

where $\mathcal{N} = \{f_0 \in L^2(\partial\Omega) \mid (\overline{\partial}_n\phi, f_0)_{L^2(\partial\Omega)} = 0\ \forall\phi \in \mathrm{Harm}^1(\Omega)\}$ is the subspace of CR–functions in $L^2(\partial\Omega)$ (cf. [22, §17.1] and [11, §23]). □

We can now complete the proof of Theorem 5.

Proof of Theorem 5. Applying the preceding theorem to $f_1^a \in \overline{W}^1_n(\partial\Omega^a)$, we obtain a complex–valued function $g_2 \in L^2(\partial\Omega^a)$ such that $g = f_1^a + g_2 j$ is the trace of a regular function on Ω^a. We denote by the same symbols the extensions on the domains. Let $f_2 = (g_2)^{1/a}$ and $f = f_1 + f_2 q$. Then $f^a = r_{a'} \circ f \circ r_a = f_1^a + g_2 r_{a'}(q) = g$. Therefore $f \in \mathcal{R}(\Omega)$.

Given two functions $f_2, f_2' \in L^2(\Omega, \mathbb{C}_p)$ such that $f = f_1 + f_2 q$ and $f' = f_1 + f_2' q$ are regular on Ω, then $(f' - f)q = f_2 - f_2'$ is a $\mathbb{C}_p$–valued regular function and then it is J_p–holomorphic. Therefore f_2 is unique up to a CR_p–function.

The estimate for f on $\partial\Omega$ is a direct consequence of that satisfied by g on $\partial\Omega^a$. $\square$

4.3. Proof of Theorem 6

Let $q \in \mathbb{S}^2$ be orthogonal to p and let f_2 be any function given by Theorem 5. Let $H_{p,q}(f_1)$ be the uniquely defined function $S_p^{\perp}(f_2) = f_2 - S_p(f_2)$. Notice that $f_1 \in CR_p(\partial\Omega)$ if and only if $f_2 \in CR_p(\partial\Omega)$ and therefore $H_{p,q}(f_1) = 0$ if and only if f_1 is a CR_p–function. Besides, for every f_1, we have

$$\|H_{p,q}(f_1)\|_{L^2(\partial\Omega)} = \|f_2 - S_p(f_2)\|_{L^2(\partial\Omega)} \le \|f_2 + f_0\|_{L^2(\partial\Omega)}$$

for every CR_p–function f_0 on $\partial\Omega$. From Theorem 5 we get

$$\|H_{p,q}(f_1)\|_{L^2(\partial\Omega)} \le C_p(\|f_1\|^2_{L^2(\partial\Omega)} + \|\overline{\partial}_{p,n} f_1\|^2_{L^2(\partial\Omega)})^{1/2}.$$

If $g = g_1 + g_2 j$ is a regular function of class C^1 on Ω, then the equations $\overline{\partial}_n g_1 = -\overline{\mathcal{L}(g_2)}$, $\overline{\partial}_n g_2 = \overline{\mathcal{L}(g_1)}$ hold on $\partial\Omega$ (cf. [28]). Then

$$\|\mathcal{L} g_2\|_{L^2(\partial\Omega)} = \|\overline{\partial}_n g_1\|_{L^2(\partial\Omega)}, \; \|\overline{\partial}_n g_2\|_{L^2(\partial\Omega)} = \|\mathcal{L} g_1\|_{L^2(\partial\Omega)}.$$

If g is regular, with trace of class $L^2(\partial\Omega)$, but not necessarily smooth up to the boundary, by taking its restriction to the boundary of $\Omega_\epsilon \subset \Omega$ and passing to the limit as ϵ goes to zero, we get the same norm equalities. Using rotations as in the proof of Theorem 5, we get

$$\|\mathcal{L}_p f_2\|_{L^2(\partial\Omega)} = \|\overline{\partial}_{p,n} f_1\|_{L^2(\partial\Omega)}, \; \|\overline{\partial}_{p,n} f_2\|_{L^2(\partial\Omega)} = \|\mathcal{L}_p f_1\|_{L^2(\partial\Omega)},$$

and then also

$$\|\mathcal{L}_p H_{p,q}(f_1)\|_{L^2(\partial\Omega)} = \|\overline{\partial}_{p,n} f_1\|_{L^2(\partial\Omega)}, \; \|\overline{\partial}_{p,n} H_{p,q}(f_1)\|_{L^2(\partial\Omega)} = \|\mathcal{L}_p f_1\|_{L^2(\partial\Omega)}.$$

Putting all together, we obtain

$$\begin{aligned}\|H_{p,q}(f_1)\|_{\overline{W}^1_{p,n}} &\le C_p(\|f_1\|^2_{L^2} + \|\overline{\partial}_{p,n} f_1\|^2_{L^2})^{1/2} + \|\mathcal{L}_p f_1\|_{L^2} \\ &\le \max\{1, C_p\} \|f_1\|_{W^1_{\overline{\partial}_p}}\end{aligned}$$

and finally the desired estimate

$$\begin{aligned}\|H_{p,q}(f_1)\|^2_{W^1_{\overline{\partial}_p}} &\le C_p^2(\|f_1\|^2_{L^2} + \|\overline{\partial}_{p,n} f_1\|^2_{L^2}) + \|\mathcal{L}_p f_1\|^2_{L^2} + \|\overline{\partial}_{p,n} f_1\|^2_{L^2} \\ &\le (C_p^2 + 1)\|f_1\|^2_{W^1_{\overline{\partial}_p}}.\end{aligned}$$

5. The case of the unit ball

On the unit ball B, an estimate sharper than the one given in Theorem 5 can be proved.

Theorem 7. *Given $f_1 \in \overline{W}^1_{p,n}(S)$, for every $q \in \mathbb{S}^2$ orthogonal to p, there exists $f_2 \in L^2(S, \mathbb{C}_p)$, unique up to a CR_p–function, such that $f = f_1 + f_2 q$ is the trace of a regular function on B. It satisfies the estimate*

$$\inf_{f_0 \in CR_p(S)} \|f_2 + f_0\|_{L^2(S)} \leq \|\overline{\partial}_{p,n} f_1\|_{L^2(S)}.$$

If $f_1 \in W^1_{\overline{\partial}_p}(S)$, for every $q \in \mathbb{S}^2$ orthogonal to p, there exists $H_{p,q}(f_1) \in W^1_{\overline{\partial}_p}(S)$ such that $f = f_1 + H_{p,q}(f_1) q$ is the trace of a regular function on B. Moreover, $H_{p,q}(f_1)$ satisfies the estimate

$$\|H_{p,q}(f_1)\|_{W^1_{\overline{\partial}_p}} \leq \left(2\|\overline{\partial}_{p,n} f_1\|^2_{L^2(S)} + \|\mathcal{L}_p f_1\|^2_{L^2(S)}\right)^{1/2}.$$

Proof. As in the proof of Theorem 5, it is sufficient to prove the thesis in the case of the standard complex structure. We use the same notation of section §4.2. The space $W^1_n(S)/A$ is a Hilbert space also w.r.t. the product

$$(\pi(f), \pi(g))_{W^1_n/A} = (\partial_n f, \partial_n g).$$

This follows from the estimate $\|g^\perp\|_{L^2(S)} \leq \|\partial_n g\|_{L^2(S)}$, which holds for every $g \in W^1_n(S)$: if $g = \sum_{p \geq 0, q \geq 0} g_{p,q}$ is the orthogonal decomposition of g in $L^2(S)$, then

$$\|\partial_n g\|^2 = \sum_{p>0,q\geq 0} \|p g_{p,q}\|^2 \geq \sum_{p>0,q\geq 0} \|g_{p,q}\|^2 = \|g^\perp\|^2.$$

Then

$$\|\pi(g)\|^2_{W^1_n/A} = \|g^\perp\|^2_{L^2} + \|\partial_n g\|^2_{L^2} \leq 2\|\partial_n g\|^2_{L^2}$$

and therefore $\|\pi(g)\|_{W^1_n/A}$ and $\|\partial_n g\|_{L^2}$ are equivalent norms on $W^1_n(S)/A$. Now we can repeat the arguments of the proof of section §4.2 and get the first estimate. The second estimate can be obtained in the same way as in the proof of Theorem 6. □

Remark 6. The last estimate in the statement of the previous Theorem is optimal: for example, if $f_1 = \bar{z}_1$, then $\overline{\partial}_n f_1 = \bar{z}_1$, $\mathcal{L} f_1 = -z_1$, $H_{i,j}(f_1) = \bar{z}_2$ and

$$\|H_{i,j}(f_1)\|^2_{W^1_{\overline{\partial}}} = \frac{3}{2} = 2\|\overline{\partial}_n f_1\|^2_{L^2(S)} + \|\mathcal{L} f_1\|^2_{L^2(S)},$$

since in the normalized measure ($Vol(S) = 1$) we have $\|z_1\| = \|z_2\| = 2^{-1/2}$.

Remark 7. The requirement that $\overline{\partial}_{p,n} f_1 \in L^2(S)$ cannot be relaxed. On the unit ball B, the estimate which is obtained from estimate (**) in §4.2 by taking the $L^2(S)$–norm also in the left–hand side is no longer valid (take for example $\phi \in \mathcal{H}_{k-1,1}(S)$). The necessity part of the existence principle gives that there exists $f_1 \in L^2(S)$ for which does not exist any $L^2(S)$ function f_2 such that $f_1 + f_2 j$ is the trace of a regular function on B. This means that the operation of quaternionic regular conjugation is not bounded in the harmonic Hardy space $h^2(B)$.

As it was shown in [29], a function $f_1 \in L^2(S)$ with the required properties is $f_1 = z_2(1-\bar{z}_1)^{-1}$.

This phenomenon is different from what happens for pluriharmonic conjugation (cf. [36]) and in particular from the one–variable situation, which can be obtained by intersecting the domains with the complex plane $\mathbb{C}_j$ spanned by 1 and j. In this case f_1 and f_2 are real–valued and $f = f_1 + f_2 j$ is the trace of a holomorphic function on $\Omega \cap \mathbb{C}_j$ with respect to the variable $\zeta = x_0 + x_2 j$.

6. Application to pluriholomorphic functions

In [29], Theorem 5 was applied in the case of the standard complex structure, to obtain a characterization of the boundary values of *pluriholomorphic* functions. These functions are solutions of the PDE system

$$\frac{\partial^2 g}{\partial \bar{z}_i \partial \bar{z}_j} = 0 \quad \text{on } \Omega \quad (1 \le i, j \le 2).$$

We refer to the works of Detraz[13], Dzhuraev[14, 15] and Begehr[2, 3] for properties of pluriholomorphic functions of two or more variables. The key point is that if $f = f_1 + f_2 j$ is regular, then f_1 is pluriholomorphic (and harmonic) if and only if f_2 is pluriharmonic. i.e. $\partial\overline{\partial} f_2 = 0$ on Ω.

We recall a characterization of the boundary values of pluriharmonic functions, proposed by Fichera in the 1980's and proved in Refs. [11] and [26]. Let

$$\mathrm{Harm}^1_0(\Omega) = \{\phi \in C^1(\overline{\Omega}) \mid \phi \text{ is harmonic on } \Omega,\ \overline{\partial}_n \phi \text{ is real on } \partial\Omega\}.$$

This space can be characterized by means of the Bochner-Martinelli operator of the domain Ω. Cialdea[11] proved the following result for boundary values of class L^2 (and more generally of class L^p).

Let $g \in L^2(\partial\Omega)$ be complex valued. Then g is the trace of a pluriharmonic function on Ω if and only if the following orthogonality condition is satisfied:

$$\int_{\partial\Omega} g * \overline{\partial}\phi = 0 \quad \forall \phi \in \mathrm{Harm}^1_0(\Omega).$$

If $f = f_1 + f_2 j : \partial\Omega \to \mathbb{H}$ is a function of class $L^2(\partial\Omega)$ and it is the trace of a regular function on Ω, then it satisfies the integral condition

$$\int_{\partial\Omega} f_1\, \overline{\partial}\phi \wedge d\zeta = -2 \int_{\partial\Omega} \overline{f_2} * \overline{\partial}\phi \quad \forall \phi \in \mathrm{Harm}^1(\Omega).$$

If $\partial\Omega$ is connected, it can be proved that also the converse is true (cf. §4.2).

We can use this relation and the preceding result on pluriharmonic traces to obtain the following characterization of the traces of pluriholomorphic functions (cf. [29]). It generalizes some results obtained by Detraz [13] and Dzhuraev [14] on the unit ball (cf. also Refs. [2, 3, 4, 5, 15]).

Theorem 8. *Assume that Ω has connected boundary and satisfies the $L^2(\partial\Omega)$–estimate* (2)*. Let $h \in \overline{W}^1_n(\partial\Omega)$. Then h is the trace of a harmonic pluriholomorphic function on Ω if and only if the following orthogonality condition is satisfied:*

$$\int_{\partial\Omega} h\, \overline{\partial}\phi \wedge d\zeta = 0 \quad \forall \phi \in \mathrm{Harm}^1_0(\Omega).$$

7. Directional Hilbert operators

In the complex one–variable case, there is a close connection between harmonic conjugates and the Hilbert transform (see for example the monograph [6, §21]). There are several extensions of this relation to higher dimensional spaces (cf. e.g. [7, 8, 9, 12, 21, 32]), mainly in the framework of Clifford analysis, which can be considered as a generalization of quaternionic (and complex) analysis. In this section we apply the results obtained in §3 in order to introduce quaternionic Hilbert operators which depend on the complex structure J_p.

Let $L^2(\partial\Omega, \mathbb{C}_p^{\perp})$ be the space of functions fq, $f \in L^2(\partial\Omega, \mathbb{C}_p)$, where $q \in \mathbb{S}^2$ is any unit orthogonal to p and let

$$W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p^{\perp}) = \{f \in L^2(\partial\Omega, \mathbb{C}_p^{\perp}) \mid \overline{\partial}_p f \in L^2(\partial\Omega, \mathbb{C}_p^{\perp})\}.$$

Then $W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p^{\perp}) = \{fq \mid f \in W^1_{\overline{\partial}_p}(\partial\Omega)\}$ for any $q \in \mathbb{S}^2$ orthogonal to p. On these spaces we consider the products w.r.t. which the right multiplication by q is an isometry:

$$(f,g)_{L^2(\partial\Omega,\mathbb{C}_p^{\perp})} = (fq, gq)_{L^2(\partial\Omega,\mathbb{C}_p)},$$
$$(f,g)_{W^1_{\overline{\partial}_p}(\partial\Omega,\mathbb{C}_p^{\perp})} = (fq, gq)_{W^1_{\overline{\partial}_p}(\partial\Omega)}.$$

Proposition 9. *The above products are independent of $q \perp p$.*

Proof. Let $q' = aq + bpq \in \mathbb{C}_p^{\perp}$ be another element of $\mathbb{S}^2$ orthogonal to p, with $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$. If $fq = f^0 + f^1 p$, then $fq' = (af^0 + bf^1) + (af^1 - bf^0)p$. Similarly, $gq' = (ag^0 + bg^1) + (ag^1 - bg^0)p$, from which we get

$$\begin{aligned}(fq', gq')_{L^2(\partial\Omega,\mathbb{C}_p)} &= (af^0 + bf^1, ag^0 + bg^1)_{L^2} + (af^1 - bf^0, ag^1 - bg^0)_{L^2} \\ &= (a^2+b^2)(f^0, g^0)_{L^2} + (a^2+b^2)(f^1, g^1)_{L^2} = (fq, gq)_{L^2(\partial\Omega,\mathbb{C}_p)}.\end{aligned}$$

The independence of the second product follows from that of the first. □

We will consider also the space of $\mathbb{H}$–valued functions

$$W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{H}) = \{f \in L^2(\partial\Omega, \mathbb{H}) \mid \overline{\partial}_p f \in L^2(\partial\Omega, \mathbb{H})\}$$

with norm

$$\|f\|_{W^1_{\overline{\partial}_p}(\partial\Omega,\mathbb{H})} = \left(\|f_1\|^2_{W^1_{\overline{\partial}_p}(\partial\Omega,\mathbb{C}_p)} + \|f_2\|^2_{W^1_{\overline{\partial}_p}(\partial\Omega,\mathbb{C}_p)} \right)^{1/2},$$

where $f = f_1 + f_2 q \in W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p) \oplus W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p^\perp)$, $f_i \in W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p)$ and q is any imaginary unit orthogonal to p. It follows from Proposition 9 that this norm does not depends on q.

Now we come to our main result. We show how it is possible, for every fixed direction p, to choose a quaternionic regular harmonic conjugate of a $\mathbb{C}_p$–valued harmonic function in a way independent of the orthogonal direction q. Taking restrictions to the boundary $\partial\Omega$ this construction permits to define a directional, p-dependent, Hilbert operator for regular functions.

Theorem 10. *Assume that $\partial\Omega$ is connected and that Ω satisfies estimate* (1)*. For every $\mathbb{C}_p$–valued function $f_1 \in W^1_{\overline{\partial}_p}(\partial\Omega)$, there exists $H_p(f_1) \in W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p^\perp)$ such that $f = f_1 + H_p(f_1)$ is the trace of a regular function on Ω. Moreover, $H_p(f_1)$ satisfies the estimate*

$$\|H_p(f_1)\|_{W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p^\perp)} \leq \sqrt{C_p^2 + 1}\, \|f_1\|_{W^1_{\overline{\partial}_p}(\partial\Omega)}$$

where C_p is the same constant as in estimate (1)*. The operator $H_p : W^1_{\overline{\partial}_p}(\partial\Omega) \to W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p^\perp)$ is a right $\mathbb{C}_p$–linear bounded operator, with kernel $CR_p(\partial\Omega)$.*

Proof. Let $q, q' \in \mathbb{S}^2$ be two vectors orthogonal to p. We prove that

$$H_{p,q}(f_1)q = H_{p,q'}(f_1)q'.$$

Let $g = H_{p,q}(f_1)q - H_{p,q'}(f_1)q' \in W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p^\perp)$. Then $gq \in W^1_{\overline{\partial}_p}(\partial\Omega)$ is the restriction of a $\mathbb{C}_p$–valued, regular function on Ω. But this implies that gq is a CR_p–function on $\partial\Omega$. On the other hand, gq belongs also to the space $CR_p(\partial\Omega)^\perp$, since $H_{p,q}(f_1) = S_p^\perp(f_2)$ and $H_{p,q'}(f_1)q'q = S_p^\perp(f_2')q'q$, with $q'q \in \mathbb{C}_p$, where f_2 and f_2' are functions given by Theorem 5. This implies that $gq = 0$ and then also g vanishes. Therefore we can put

$$H_p(f_1) = H_{p,q}(f_1)q \quad \text{for any } q \perp p.$$

The estimate is a direct consequence of what stated in Theorem 6. □

From Theorem 7, we immediately get the optimal estimate on the unit sphere:

$$\|H_p(f_1)\|_{W^1_{\overline{\partial}_p}(S, \mathbb{C}_p^\perp)} \leq \left(2\|\overline{\partial}_{p,n} f_1\|^2_{L^2(S)} + \|\mathcal{L}_p f_1\|^2_{L^2(S)}\right)^{1/2}.$$

The operator H_p can be extended by right $\mathbb{H}$–linearity to the space $W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{H})$. If $f \in W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{H})$ and q is any imaginary unit orthogonal to p, let $f = f_1 + f_2 q \in W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p) \oplus W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p^\perp)$, $f_i \in W^1_{\overline{\partial}_p}(\partial\Omega, \mathbb{C}_p)$. We set

$$H_p(f) = H_p(f_1) + H_p(f_2)q.$$

This definition is independent of q, because if $f = f_1 + f_2' q'$, then $(f_2 q - f_2' q')q$ is a CR_p–function and therefore $0 = H_p(-f_2 - f_2' q' q) = -H_p(f_2) - H_p(f_2')q'q \Rightarrow H_p(f_2)q = H_p(f_2')q'$. The operator H_p will be called a *directional Hilbert operator* on $\partial\Omega$.

Corollary 11. *The Hilbert operator* $H_p : W^1_{\bar\partial_p}(\partial\Omega,\mathbb{H}) \to W^1_{\bar\partial_p}(\partial\Omega,\mathbb{H})$ *is right* $\mathbb{C}_p$*–linear and* $\mathbb{H}$*–linear, its kernel is the space of* $\mathbb{H}$*–valued* CR_p*–functions and satisfies the estimate*

$$\|H_p(f)\|_{W^1_{\bar\partial_p}(\partial\Omega,\mathbb{H})} \le \sqrt{C_p^2+1}\,\|f\|_{W^1_{\bar\partial_p}(\partial\Omega,\mathbb{H})}.$$

For every $f \in W^1_{\bar\partial_p}(\partial\Omega,\mathbb{H})$*, the function* $R_p(f) := f + H_p(f)$ *is the trace of a regular function on* Ω*.*

The "*regular signal*" $R_p(f) := f + H_p(f)$ associated with f has a property similar to the one satisfied by analytic signals (cf. [31, Theorem 1.1]).

Corollary 12. *Let* $f \in W^1_{\bar\partial_p}(\partial\Omega,\mathbb{H})$*. Then* f *is the trace of a regular function on* Ω *if and only if* $R_p(f) = 2f$ *(modulo* CR_p*–functions). Moreover,* f *is a* CR_p*–function if and only if* $R_p(f) = f$*.*

Proof. Let q be any imaginary unit orthogonal to p and let $f = f_1 + f_2 q$, $f_1, f_2 \in W^1_{\bar\partial_p}(\partial\Omega,\mathbb{C}_p)$. Then

$$R_p(f) = 2f \pmod{CR_p} \Leftrightarrow \begin{cases} f_1 + H_p(f_2)q = 2f_1 \\ f_2 q + H_p(f_1) = 2f_2 q \end{cases} \pmod{CR_p}$$

$$\Leftrightarrow \begin{cases} f_1 = H_p(f_2)q \\ f_2 = -H_p(f_1)q \end{cases} \pmod{CR_p}$$

Therefore $R_p(f) = 2f \pmod{CR_p} \Leftrightarrow f = f_1 + f_2 q = f_1 + H_p(f_1) = (f_2 + H_p(f_2))q \pmod{CR_p}$, i.e. f is (the trace of) a regular function.

If $f_1, f_2 \in CR_p$, then $H_p(f_1) = H_p(f_2) = 0$ and therefore $R_p(f) = f$. Conversely, if $R_p(f) = f$, then from the first part we get $f = 2f \pmod{CR_p}$ and so f is CR_p. □

We now study the relation between the Hilbert operator and the Szegö projection. When $f_1 \in W^1_{\bar\partial_p}(\partial\Omega)$, then $H_p(f_1) \in W^1_{\bar\partial_p}(\partial\Omega,\mathbb{C}_p^\perp)$ and therefore $H_p(H_p(f_1))$ is again in $W^1_{\bar\partial_p}(\partial\Omega)$.

Theorem 13. *Let* $S_p : W^1_{\bar\partial_p}(\partial\Omega) \to CR_p(\partial\Omega) \subset W^1_{\bar\partial_p}(\partial\Omega)$ *be the Szegö projection. Then* $H_p^2 = id - S_p$*. The same relation holds on the space* $W^1_{\bar\partial_p}(\partial\Omega,\mathbb{H})$ *if* S_p *is extended to* $W^1_{\bar\partial_p}(\partial\Omega,\mathbb{H})$ *in the same way as* H_p*. As a consequence,* $R_p^2(f) = 2R_p(f)$ *(modulo* CR_p*–functions) for every* $f \in W^1_{\bar\partial_p}(\partial\Omega,\mathbb{H})$*.*

Proof. For every $f_1 \in W^1_{\bar\partial_p}(\partial\Omega)$, the harmonic extension of $f = f_1 + H_{p,q}(f_1)q$ is regular. Then also $f' = (H_{p,q}(f_1) + H^2_{p,q}(f_1)q)q$ has regular extension and therefore the $\mathbb{C}_p$–valued function $f - f' = f_1 + H^2_{p,q}(f_1)$ is a CR_p–function. We have the decomposition

$$f_1 = (f_1 + H^2_{p,q}(f_1)) - H^2_{p,q}(f_1) \in CR_p(\partial\Omega) \oplus CR_p(\partial\Omega)^\perp,$$

that gives $f_1 + H^2_{p,q}(f_1) = S_p(f_1)$. But $H_{p,q} = -H_p q$ and then

$$H^2_{p,q}(f_1) = -H_{p,q}(H_p(f_1)q) = H_p(H_p(f_1)q)q.$$

By definition, $H^2_p(f_1) = H_p(H_p(f_1)) = H_p(-H_p(f_1)q)q$. Then $f_1 - H^2_p(f_1) = f_1 + H^2_{p,q}(f_1) = S_p(f_1)$. □

Remark 8. The Hilbert operator H_p can be expressed in terms of H_i using the rotations introduced in §4.2. It can be shown that $H_{p,q}(f_1)^a = H_{i,j}(f_1^a)$, from which it follows that

$$H_p(f_1)^a = H_i(f_1^a).$$

Theorem 10 says that even if the rotation vector a depends on p and q, the function $H_p(f_1) = H_i(f_1^a)^{1/a}$ only depends on p.

7.1. Examples

Let $\Omega = B$, $f = |z_1|^2 - |z_2|^2$.

1. $p = i$. We get

$$H_i(f) = \bar{z}_1\bar{z}_2 j \quad \text{and} \quad R_i(f) = |z_1|^2 - |z_2|^2 + \bar{z}_1\bar{z}_2 j$$

is a regular polynomial. We can check that

$$R^2_i(f) = 2(|z_1|^2 - |z_2|^2 + \bar{z}_1\bar{z}_2 j) = 2R_i(f).$$

2. $p = j$. We have

$$H_j(f) = (\bar{z}_1\bar{z}_2 - z_1z_2)j \quad \text{and} \quad R_j(f) = |z_1|^2 - |z_2|^2 + (\bar{z}_1\bar{z}_2 - z_1z_2)j.$$

Now

$$\begin{aligned} R^2_j(f) &= \frac{3}{2}\left(|z_1|^2 - |z_2|^2\right) + \frac{1}{2}\left(3\bar{z}_1\bar{z}_2 - 5z_1z_2\right)j \\ &= 2R_j(f) + \frac{1}{2}\left(|z_2|^2 - |z_1|^2\right) - \frac{1}{2}\left(z_1z_2 + \bar{z}_1\bar{z}_2\right)j \\ &= 2R_j(f) + CR_j\text{–function} \end{aligned}$$

In fact, the Hilbert operator H_j vanishes on $\frac{1}{2}\left(|z_2|^2 - |z_1|^2\right) - \frac{1}{2}\left(z_1z_2 + \bar{z}_1\bar{z}_2\right)j$.

3. $p = k$. In this case

$$H_k(f) = (\bar{z}_1\bar{z}_2 + z_1z_2)j, \quad R_k(f) = |z_1|^2 - |z_2|^2 + (\bar{z}_1\bar{z}_2 + z_1z_2)j \quad \text{and}$$

$$\begin{aligned} R^2_k(f) &= \frac{3}{2}\left(|z_1|^2 - |z_2|^2\right) + \frac{1}{2}\left(3\bar{z}_1\bar{z}_2 + 5z_1z_2\right)j \\ &= 2R_k(f) + \frac{1}{2}\left(|z_2|^2 - |z_1|^2\right) + \frac{1}{2}\left(z_1z_2 - \bar{z}_1\bar{z}_2\right)j \\ &= 2R_k(f) + CR_k\text{–function} \end{aligned}$$

Another example: let $g = z_1^2 \in \mathrm{Hol}_i(\mathbb{H})$.

1. $p = i$. Since g is holomorphic, we get $H_i(g) = 0$, $R_i(g) = g$.

2. $p = j$. We have

$$\begin{aligned} H_j(g) &= H_j(x_0^2 - x_1^2) + H_j(2x_0x_1)i \\ &= \frac{1}{8}\left(3z_1^2 - z_2^2 - 3\bar{z}_1^2 + \bar{z}_2^2\right) + \frac{1}{4}(z_1\bar{z}_2 - \bar{z}_1 z_2)j \\ &+ \frac{1}{8}\left(3z_1^2 + z_2^2 + 3\bar{z}_1^2 + \bar{z}_2^2\right) - \frac{1}{4}(z_1\bar{z}_2 + \bar{z}_1 z_2)j \\ &= \frac{3}{4}z_1^2 + \frac{1}{4}\bar{z}_2^2 - \frac{1}{2}\bar{z}_1 z_2 j. \end{aligned}$$

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Alessandro Perotti
Department of Mathematics
University of Trento
Via Sommarive, 14
I–38050 Povo Trento ITALY
e-mail: `perotti@science.unitn.it`