5

History of Linear Algebra

Linear algebra is a very useful subject, and its basic concepts arose and were used in different areas of mathematics and its applications. It is therefore not surprising that the subject had its roots in such diverse fields as number theory (both elementary and algebraic), geometry, abstract algebra (groups, rings, fields, Galois theory), analysis (differential equations, integral equations, and functional analysis), and physics. Among the elementary concepts of linear algebra are linear equations, matrices, determinants, linear transformations, linear independence, dimension, bilinear forms, quadratic forms, and vector spaces. Since these concepts are closely interconnected, several usually appear in a given context (e.g., linear equations and matrices) and it is often impossible to disengage them.

By 1880, many of the basic results of linear algebra had been established, but they were not part of a general theory. In particular, the fundamental notion of vector space, within which such a theory would be framed, was absent. This was introduced only in 1888 by Peano. Even then it was largely ignored (as was the earlier pioneering work of Grassmann), and it took off as the essential element of a fully-fledged theory in the early decades of the twentieth century. So the historical development of the subject is the reverse of its logical order.

We will describe the elementary aspects of the evolution of linear algebra under the following headings: linear equations; determinants; matrices and linear transformations; linear independence, basis and dimension; and vector spaces. Along the way, we will comment on some of the other concepts mentioned above.

5.1 Linear equations

About 4000 years ago the Babylonians knew how to solve a system of two linear equations in two unknowns (a 2×2 system). In their famous *Nine Chapters of the Mathematical Art* (c. 200 BC) the Chinese solved 3×3 systems by working solely with their (numerical) coefficients. These were prototypes of matrix methods, not unlike the "elimination methods" introduced by Gauss and others some 2000 years later. See [20].

The modern study of systems of linear equations can be said to have originated with Leibniz, who in 1693 invented the notion of a determinant for this purpose. But his investigations remained unknown at the time. In his *Introduction to the Analysis of Algebraic Curves* of 1750, Cramer published the rule named after him for the solution of an $n \times n$ system, but he provided no proofs. He was led to study systems of linear equations while attempting to solve a geometric problem, determining an algebraic curve of degree n passing through $(1/2)n^2 + (3/2)n$ fixed points. See [1], [20].

Gottfried Wilhelm Leibniz (1646–1716)

Euler was perhaps the first to observe that a system of n equations in n unknowns does not necessarily have a unique solution, noting that to obtain uniqueness it is necessary to add conditions. He had in mind the idea of dependence of one equation on the others, although he did not give precise conditions. In the eighteenth century the study of linear equations was usually subsumed under that of determinants, so no consideration was given to systems in which the number of equations differed from the number of unknowns. See [8], [9].

In connection with his invention of the method of least squares (published in a paper in 1811 dealing with the determination of the orbit of an asteroid), Gauss introduced a systematic procedure, now called *Gaussian elimination*, for the solution of systems of linear equations, though he did not use the matrix notation. He dealt with the cases in which the number of equations and unknowns may differ [20]. The theoretical properties of systems of linear equations, including the issue of their consistency, were treated in the second half of the nineteenth century, and were at least partly motivated by questions of the reduction of quadratic and bilinear forms to “simple” (canonical) ones. See [16], [18].

5.2 Determinants

Although one speaks nowadays of the determinant of a matrix, the two concepts had different origins. In particular, determinants appeared before matrices, and the early stages in their history were closely tied to linear equations. Subsequent problems that gave rise to new uses of determinants included *elimination theory* (finding conditions under which two polynomials have a common root), transformation of coordinates to simplify algebraic expressions (e.g., quadratic forms), change of variables in multiple integrals, solution of systems of differential equations, and celestial mechanics. See [24].

As we have noted in the previous section on linear equations, Leibniz invented determinants. He "knew in substance the[ir] modern combinatorial definition" [21], and he used them in solving linear equations and in elimination theory. He wrote many papers on determinants, but they remained unpublished till recently. See [21], [22].

The first publication to contain some elementary information on determinants was Maclaurin's *Treatise of Algebra*, in which they were used to solve 2×2 and 3×3 systems. This was soon followed by Cramer's significant use of determinants (cf. the previous section). See [1], [20], [21].

An exposition of the theory of determinants independent of their relation to the solvability of linear equations was first given by Vandermonde in his "Memoir on elimination theory" of 1772. (The word "determinant" was used for the first time by Gauss, in 1801, to stand for the discriminant of a quadratic form, where the discriminant of the form $ax^2 + bxy + cy^2$ is $b^2 - 4ac$.) Laplace extended some of Vandermonde's work in his *Researches on the Integral Calculus and the System of the World* (1772), showing how to expand $n \times n$ determinants by cofactors. See [24].

The first to give a systematic treatment of determinants was Cauchy in an 1815 paper entitled "On functions which can assume but two equal values of opposite sign by means of transformations carried out on their variables." He can be said to be the founder of the theory of determinants as we know it today. Many of the results on determinants found in a first textbook on linear algebra are due to him. For example, he proved the important product rule $\det(AB) = (\det A)(\det B)$. His work provided mathematicians with a powerful algebraic apparatus for dealing with n-dimensional algebra, geometry, and analysis. For instance, in 1843 Cayley developed the analytic geometry of n dimensions using determinants as a basic tool, and in the 1870s Dedekind used them to prove the important result that sums and products of algebraic integers are algebraic integers. See [18], [21], [22], [24].

Weierstrass and Kronecker introduced a definition of the determinant in terms of axioms, probably in the 1860s. (Rigorous thinking was characteristic of both mathematicians.) For example, Weierstrass defined the determinant as a normed, linear, homogeneous function. Their work became known in 1903, when Weierstrass' *On Determinant Theory* and Kronecker's *Lectures on Determinant Theory* were published posthumously. Determinant theory was a vigorous and independent subject of research in the nineteenth century, with over 2000 published papers. But it became largely unfashionable for much of the twentieth century, when determinants were no longer needed to prove the main results of linear algebra. See [21], [22], [24], [25].

5.3 Matrices and linear transformations

Matrices are "natural" mathematical objects: they appear in connection with linear equations, linear transformations, and also in conjunction with bilinear and quadratic forms, which were important in geometry, analysis, number theory, and physics.

Matrices as rectangular arrays of numbers appeared around 200 BC in Chinese mathematics, but there they were merely abbreviations for systems of linear equations. Matrices become important only when they are operated on—added, subtracted, and especially multiplied; more important, when it is shown what use they are to be put to.

Matrices were introduced implicitly as abbreviations of linear transformations by Gauss in his *Disquisitiones* mentioned earlier, but now in a significant way. Gauss undertook a deep study of the arithmetic theory of binary quadratic forms, $f(x, y) = ax^2 + bxy + cy^2$. He called two forms $f(x, y)$ and $F(X, Y) = AX^2 + BXY + CY^2$ "equivalent" if they yield the same set of integers, as x, y, X, and Y range over all the integers (a, b, c and A, B, C are integers). He showed that this is the same as saying that there exists a linear transformation T of the coordinates (x, y) to (X, Y) with determinant $= 1$ that transforms $f(x, y)$ into $F(X, Y)$. The linear transformations were represented as rectangular arrays of numbers—matrices, although Gauss did not use matrix terminology. He also defined implicitly the product of matrices (for the 2×2 and 3×3 cases only); he had in mind the composition of the corresponding linear transformations. See [1], [7], [16].

Linear transformations of coordinates, $y_j = \sum_{k=1}^{n} a_{jk}x_k (1 \leq j \leq m)$, appear prominently in the analytic geometry of the seventeenth and eighteenth centuries (mainly for $m = n \leq 3$). This led naturally to computations done on rectangular arrays of numbers (a_{jk}). Linear transformations also show up in projective geometry, founded in the seventeenth century and described analytically in the early nineteenth. See [2], [9].

In attempts to extend Gauss' work on quadratic forms, Eisenstein and Hermite tried to construct a general arithmetic theory of forms $f(x_1, x_2, \ldots, x_n)$ of any degree in any number of variables. In this connection they too introduced linear transformations, denoted them by single letters—an important idea—and studied them as independent entities, defining their addition and multiplication (composition). See [16].

Cayley formally introduced $m \times n$ matrices in two papers in 1850 and 1858 (the *term* "matrix" was coined by Sylvester in 1850). He noted that they "comport themselves as single entities" and recognized their usefulness in simplifying systems of linear equations and composition of linear transformations. He defined the sum and product of matrices for suitable pairs of rectangular matrices, and the product of a matrix by a scalar, a real or complex number. He also introduced the identity matrix and the inverse of a square matrix, and showed how the latter can be used in solving $n \times n$ linear systems under certain conditions.

In his 1858 paper "A memoir on the theory of matrices" Cayley proved the important Cayley–Hamilton theorem that a square matrix satisfies its characteristic polynomial. The proof consisted of computations with 2×2 matrices, and the observation that he had verified the result for 3×3 matrices. He noted that the result applies more widely. But he added: "I have not thought it necessary to undertake the labour of a formal proof of the theorem in the general case of a matrix of any degree." Hamilton proved the theorem independently (for $n = 4$, but without using the matrix notation) in his work on quaternions. Cayley used matrices in another paper to solve a significant problem, the so-called Cayley–Hermite problem, which asks for the determination of all linear transformations leaving a quadratic form in n variables invariant. See [16], [18].

Cayley advanced considerably the important idea of viewing matrices as constituting a symbolic algebra. In particular, his use of a single letter to represent a matrix was a significant step in the evolution of matrix algebra. But his papers of the 1850s were little noticed outside England until the 1880s. See [3], [4], [12], [16], [20], and Chapter 8.1.3.

During the intervening years (roughly the 1820s–1870s) deep work on matrices (in one guise or another) was done on the continent, by Cauchy, Jacobi, Jordan, Weierstrass, and others. They created what may be called the *spectral theory* of matrices: their classification into types such as symmetric, orthogonal, and unitary; results on the nature of the eigenvalues of the various types of matrices; and, above all, the theory of canonical forms for matrices—the determination, among all matrices of a certain type, of those that are canonical in some sense. An important example is the *Jordan canonical form*, introduced by Weierstrass (and independently by Jordan), who showed that two matrices are similar if and only if they have the same Jordan canonical form.

Spectral theory originated in the eighteenth century in the study of physical problems. This led to the investigation of differential equations and eigenvalue problems. In the nineteenth century these ideas gave rise to a purely mathematical theory. Hawkins gives an excellent account of this development in several articles [15], [16], [17], [18]; see also [11].

In a seminal paper in 1878 titled "On linear substitutions and bilinear forms" Frobenius developed substantial elements of the theory of matrices in the language of bilinear forms. (The theory of bilinear and quadratic forms was created by Weierstrass and Kronecker.) The forms, he said, can be viewed "as a system of n^2 quantities which are ordered in n rows and n columns." He was inspired by his teacher Weierstrass, and his paper is in the Weierstrass tradition of stressing a rigorous approach and seeking the fundamental ideas underlying the theories. For example, he made a thorough study of the general problem of canonical forms for bilinear forms, attributing special cases to Kronecker and Weierstrass. "Frobenius' paper … represents an important landmark in the history of the theory of matrices, for it brought together for the first time the work on spectral theory of Cauchy, Jacobi, Weierstrass and Kronecker with the symbolical tradition of Eisenstein, Hermite and Cayley" [18]. See also [15], [16].

Frobenius applied his theory of matrices to group representations and to quaternions, showing for the latter that the only n-tuples of real numbers which are division algebras are the real numbers, the complex numbers, and the quaternions, a result proved independently by C. S. Peirce. (Cayley, in his 1858 paper, also related matrices to quaternions by showing that the quaternions are isomorphic to a subalgebra of the algebra of 2×2 matrices over the complex numbers.) The relationship

Georg Ferdinand Frobenius (1849–1917)

between (associative) algebras and matrices was to be of fundamental importance for subsequent developments of noncommutative ring theory. See Chapter 3.1.

5.4 Linear independence, basis, and dimension

The notions of linear independence, basis, and dimension appear, not necessarily with formal definitions, in various contexts, among them algebraic number theory, fields and Galois theory, hypercomplex number systems (algebras), differential equations, and analytic geometry.

In algebraic number theory the objects of study are algebraic number fields $Q(\alpha)$, where Q denotes the rationals and α is an algebraic number. If the minimal polynomial of α has degree n, then every element of $Q(\alpha)$ can be expressed uniquely in the form $a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1}$, where $a_i \in Q$. Thus $1, \alpha, \alpha^2, \ldots \alpha^{n-1}$ form a basis of $Q(\alpha)$, considered as a vector space over Q. This is precisely the line of thought pursued by Dedekind in his Supplement X of 1871 to Dirichlet's book on number theory (and with more clarity and detail in Supplements to subsequent editions of Dirichlet's book), although there is no formal definition of a vector space. See Chapter 3.2 and especially Chapter 8.2.4.

In connection with his work in algebraic number theory Dedekind introduced the notion of a field. He defined it as a subset of the complex numbers satisfying certain axioms. He included important concepts and results on fields, some related to ideas of linear algebra. For example, if E is a subfield of K, he defined the "degree" of

K over E as the dimension of K considered as a vector space over E, and showed that if the degree is finite, then every element of K is algebraic over E. The notions of linear independence, basis, and dimension appear here in a transparent way; the notion of vector space also appears, but only implicitly. See Chapter 4.2 and Chapter 8.2.

In 1893 Weber gave an axiomatic definition of finite groups and fields, with the objective of giving an abstract formulation of Galois theory. Among the results on fields is the following: If F is a subfield of E, which in turn is a subfield of K, then $(K : F) = (K : E)(E : F)$, where for any subfield S of T, $(T : S)$ denotes the dimension of T as a vector space over S (we assume that all dimensions in question are finite). See Chapter 4.6.

Many of the ideas of Dedekind and Weber on field extensions were brought to perfection in Steinitz's groundbreaking paper of 1910, "Algebraic theory of fields," in which he presented an abstract development of field theory. In the 1920s Artin "linearized" Galois theory—a most important idea. Contemporary treatment of the subject usually follows his. See [9] and Chapter 4.8.

An algebra (it is both a ring and a vector space over a field) was called in the nineteenth century a hypercomplex number system. The first such example was Hamilton's quaternions. This inspired generalizations to higher dimensions. For example, Cayley and Graves independently introduced octonions (in 1844), elements of the form $a_1e_1 + a_2e_2 + \cdots + a_8e_8$, where a_i are real numbers and e_i are "basis" elements subject to laws of multiplication. In 1854, in a paper in which he defined finite groups, Cayley introduced the group algebra of such a group—a linear combination of the group elements with real or complex coefficients. He called it a system of "complex quantities" and observed that it is analogous in many ways to the quaternions. See Chapter 3.1.

In a groundbreaking paper in 1870 entitled "Linear associative algebra," B. Peirce gave a definition of a finite-dimensional associative algebra as the totality of formal expressions of the form $\sum_{i=1}^{n} a_i e_i$, where the a_i are complex numbers and the e_i are "basis" elements, subject to associative and distributive laws. See Chapter 3.1.

Euler began to lay the framework for the solution of linear homogeneous differential equations. He observed that the general solution of such an equation with constant coefficients could be expressed as a linear combination of linearly independent particular solutions. Later in the eighteenth century Lagrange extended his result to equations with nonconstant coefficients. Demidov [6] discusses the analogy between linear algebraic equations and linear differential equations, focusing on the early nineteenth century.

The interaction of algebra and geometry is fundamental in linear algebra (for example, n-dimensional Euclidean geometry can be viewed as an n-dimensional vector space over the reals together with a symmetric bilinear form $B(x, y) = \sum a_{ij}x_i y_j$ that serves to define the length of a vector and the angle between two vectors). It began with the introduction of analytic geometry by Descartes and Fermat in the early seventeenth century, was extended by Euler in the eighteenth to 3 dimensions, and was put in modern form (the way we see it today) by Monge in the early nineteenth. The notions of linear combination, coordinate system, and basis were fundamental,

Leonhard Euler (1707–1783)

as were other basic notions of linear algebra such as matrix, determinant, and linear transformation. See [2], [8], [9], [13].

Further details on linear independence, basis, and dimension appear in the following section.

5.5 Vector spaces

As we mentioned, by 1880 many of the fundamental results of linear algebra had been established, but they were not considered as parts of a general theory. In particular, the fundamental notion of vector space, within which such a theory would be framed, was absent. It was introduced by Peano in 1888.

The earliest notion of *vector* comes from physics, where it means a quantity having both magnitude and direction (e.g., velocity or force). This idea was well established by the end of the seventeenth century, as was that of the parallelogram of vectors, a parallelogram determined by two vectors as its adjacent sides. In this setting the addition of vectors and their multiplication by scalars had clear physical meanings. See [5].

The *mathematical* notion of vector originated in the geometric representation of complex numbers, introduced independently by several authors in the late eighteenth and early nineteenth centuries, starting with Wessel in 1797 and culminating with Gauss in 1831. The representation of the complex numbers in these works was geometric, as points or as directed line segments in the plane. In 1835 Hamilton defined the complex numbers algebraically as ordered pairs of reals, with the usual operations of addition and multiplication, as well as multiplication by real numbers (the term "scalar" originated in his work on quaternions). He noted that these pairs satisfy the closure laws and the commutative and distributive laws, have a zero element, and have additive and multiplicative inverses. (The associative laws were mentioned in his 1843 work on quaternions.) See [5], [14].

An important development was the extension of vector ideas to three-dimensional space. Hamilton constructed an algebra of vectors within his system of quaternions. (Josiah Willard Gibbs and Oliver Heaviside introduced a competing system—their vector analysis—in the 1880s.) These were represented in the form $ai + bj + ck$, where a, b, c are real numbers and i, j, k the quaternion units—a clear precursor of a basis for three-dimensional Euclidean space. It was here that he introduced the term "vector" for these objects. See [5], [14].

A crucial development in vector-space theory was the further extension of the notions on three-dimensional space to spaces of higher dimension, advanced independently in the early 1840s by Cayley, Hamilton, and Grassmann. Hamilton called the extension of 3-space to four dimensions a "leap of the imagination." He had in mind, of course, his quaternions, a four-dimensional vector space (also a division algebra). He introduced them in dramatic fashion in 1843, and spent the next twenty years in their exploration and applications. Cayley's ideas on dimensionality appeared in his 1843 paper "Chapters of analytic geometry of n-dimensions." See [2], [5], [14], [19], and Chapter 8.5.

The pioneering ideas were expounded by Grassmann in his *Doctrine of Linear Extension* (1844). This was a brilliant work whose aim was to construct a coordinate-free algebra of n-dimensional space. It contained many of the basic ideas of linear algebra, including the notion of an n-dimensional vector space, subspace, spanning set, independence, basis, dimension, and linear transformation.

The definition of vector space was given as the set of linear combinations $\sum a_i e_i (i = 1, 2, \ldots, n)$, where a_i are real numbers and e_i "units," assumed to be linearly independent. Addition, subtraction, and multiplication by real numbers of such sums were defined in the usual manner, followed by a list of "fundamental properties." Among these are the commutative and associative laws of addition, the subtraction laws $a+b-b = a$ and $a-b+b = a$, and several laws dealing with multiplication by scalars. From these, Grassmann claimed, all the usual laws of addition, subtraction, and multiplication by scalars follow. He proved various results about vector spaces, including the fundamental relation $\dim V + \dim W = \dim(V + W) + \dim(V \cap W)$ for subspaces V and W of a vector space. See [5], [9], [10].

Grassmann's work was difficult to understand, containing many new ideas couched in philosophical language. It was thus ignored by the mathematical community. An 1862 edition was better received. It motivated Peano to give an abstract formulation of some of Grassmann's ideas in his *Geometric Calculus* (1888).

In the last chapter of this work, entitled "Transformations of linear systems," Peano gave an axiomatic definition of a vector space over the reals. He called it a "linear system." It was in the modern spirit of axiomatics, more or less as we have it today. He postulated the closure operations, associativity, distributivity, and the existence of a zero element. This was defined to have the property $0 \times a = 0$ for every element a in the vector space. He defined $a - b$ to mean $a + (-1)b$ and showed that $a - a = 0$ and $a + 0 = a$. Another of his axioms was that $a = b$ implies $a + c = b + c$ for every c. As examples of vector spaces he gave the real numbers, the complex numbers, vectors in the plane or in 3-space, the set of linear transformations from one vector space to another, and the set of polynomials in a single variable. See [23].

Peano also defined other concepts of linear algebra, including dimension and linear transformation, and proved a number of theorems. For example, he defined the dimension of a vector space as the maximum number of linearly independent elements (but did not prove that it is the same for every choice of such elements), and showed that any set of n linearly independent elements in an n-dimensional vector space constitutes a basis. He noted that if the set of polynomials in a single variable is at most of degree n, the dimension of the resulting vector space is $n + 1$, but if there is no restriction on the degree, the resulting vector space is infinite-dimensional. See [9], [23].

Peano's work was ignored for the most part, probably because axiomatics was in its infancy, and perhaps because the work was tied so closely to geometry, setting aside other important contexts of the ideas of linear algebra, which we have described above.

A word about the axiomatic method. Although it became well established only in the early decades of the twentieth century, it was "in the air" in the last two decades of the nineteenth. For example, there appeared axiomatic definitions of groups and fields, the positive integers (cf. the Peano axioms), and projective geometry.

In 1918, in his book *Space, Time, Matter*, which dealt with general relativity, Weyl axiomatized *finite-dimensional* real vector spaces, apparently unaware of Peano's work. The definition appears in the first chapter of the book, entitled *Foundations of Affine Geometry*. As in Peano's case, this was not quite the modern definition. It took time to get it just right! See [23].

In his doctoral dissertation of 1920 Banach axiomatized complete normed vector spaces (now called Banach spaces) over the reals. The first thirteen axioms are those of a vector space, in very much a modern spirit. He put it thus:

> I consider sets of elements about which I postulate certain properties; I deduce from them certain theorems, and I then prove for each particular functional domain that the postulates adopted are true for it.

In her 1921 paper "Ideal theory in rings" Emmy Noether introduced modules, and viewed vector spaces as special cases (see Chapter 6.2 and 6.3). We thus see vector spaces arising in three distinct contexts: geometry, analysis, and algebra. In his 1930 classic text *Modern Algebra* van der Waerden has a chapter entitled Linear Algebra [25]. Here for the first time the term is used in the sense in which we employ it today. Following in Noether's footsteps, he begins with the definition of a module over a (not necessarily commutative) ring. Only on the following page does he define a vector space!

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